Lie theory and infinitesimal extensions in algebraic geometry

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Abstract

- We apply the Lie theoretic formalism of Calaque-Caldararu-Tu, to some extension problems of vector bundles to the first infinitesimal neighborhood of a subvariety in the complex projective space. (joint work with N. Buruiana and D. Cheptea)

Outline

- In recent years, Lie algebraic type constructions, received many applications going from low dimensional topology to pure algebraic geometry.
- We want to focus on one aspect of these applications, namely a result of Arinkin-Caldararu and its connection to the classical Serre's construction from the world of vector bundles on projective varieties.
- In the first part we shall introduce the relevant definitions concerning Lie-type phenomenons in the algebraic geometric world together with their original Lie-algebraic counterpart, following mainly Calaque.
- The second part is devoted to Serre's construction and to the main result of this paper. The very last section contain a questions which could be considered for future research.

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LIE ALGEBRAS AND DUFLO'S THEOREM

In what follows, we fix a commutative field k of characteristic 0, and g a Lie algebra over k. It is well known the following fact:

PBW-ISOMORPHISM

The natural map

$$S(g) \to U(g)$$

is an isomorphism of vector spaces and not of algebras.

Also, the restriction of the PBW-map to the g-invariant parts is also a linear isomorphism:

$$S(g)^g \to U(g)^g.$$

Neither this restriction is an algebra isomorphism, but the deep insight of Duflo was that a certain modification of it become an isomorphism of algebras.

- More precisely let's consider the following magic element in $(\hat{S})(g^\star)^g$:

$$J = det(\frac{1 - e^{-ad}}{ad}).$$

- Formally it is expressed as a series in the elements $tr(ad^k)$.
- An important fact is that the above elements are g-invariants wrt the co-adjoint action extended by derivations.
- Moreover, $g^\star$ acts on $S(g)$ by derivations and this action preserve the g-invariant parts.

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- Putting all the above together Duflo's theorem can be stated as follows:

DUFLO'S THEOREM

The composition

$$PBW \circ J^{\frac{1}{2}} : S(g)^g \to U(g)^g$$

is an isomorphism of algebras.

- The above result could be extended at least at a conjectural level as follows.
- Let $h \subseteq g$ an inclusion of Lie algebras, such that $g = h \oplus m$ with $[h, m] \subseteq m$.
- In this setting, Duflo conjectured the following:

Duflo's Conjecture

The Poisson center of $S(m)^h$ and the center of $(U(g)/hU(g))^h$ are isomorphic as algebras.

- Despite many efforts, this conjecture is widely open even today.

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Duflo's Conjecture II

- In the same spirit, Calaque posed the following weaker question:

Calaque Question

Under which assumptions $S(g/h)$ and $(U(g)/hU(g))$ are isomorphic as h-modules.

- The answer obtained by Calaque-Caldararu-Tu concerns the splitting of the natural filtration on $(U(g)/hU(g))$, a problem which is also relevant in some deformation quantisation problems.
- In the particular case of the embedding $h \subset h \oplus h$ their result recover the usual PBW isomorphism.

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Duflo's Conjecture III

- The interesting part in the C-C-T solution to the above question is the fact that the ideas are inspired from an analogous algebraic geometric problem through a surprisingly dictionary:
- let's fix two algebraic varieties X and Y, and an embedding $X \hookrightarrow Y$;
- by $T_X[-1]$ and $T_Y[-1]$ we denote the shifted tangent bundles of X and Y, as objects in the derived categories of coherent sheaves $D(X)$ and D_Y;
- the Lie algebras h and g are the counterpart of the Lie algebra objects $T_X[-1]$ and $T_Y[-1]$.
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- The path to the derived category of coherent sheaves goes through two well known procedures in category theory:
- in the first place we construct a category of fractions and secondly we perform a localization of that category.
- Let's fix X a smooth projective algebraic variey over $\mathbb{C}$. We denote by $Coh(X)$, the category of coherent sheaves on X. (if unfamiliar with the subject you can think simply at holomorphic vector bundles over X)
- $Coh(X)$ is an abelian category and one can take the usual category of complexes $Cpx(X)$:
- here the objects are complexes of coherent sheaves

$$... \to F_{i-1} \to F_i \to F_{i+1} \to ...$$

and the morphisms are morphisms of complexes.

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The Derived Category of coherent sheaves II

- The first operation is to take $Hot(X)$, the homotopy category of $Cpx(X)$:
- the objects are still complexes but the morphisms are modified by taking the quotient wrt homotopy equivalence of morphisms between complexes.
- The second operation, which produces finally $D(X)$ from $Hot(X)$ is the localisation wrt the class of quasi-isomorphisms. For this step recall that a quasi-isomorphism is a map between two complexes which induces isomorphisms at the level of homology groups.
- Morphisms in $D(X)$ are more complicated than the usual maps between complexes; in fact they can be represented by triangles $F^\bullet \leftarrow G^\bullet \to E^\bullet$ of usual morphisms such that the left hand side one is a quasi-isomorphism.
- Also, $D(X)$ inherits from $Cpx(X)$ the translation functor [1] which act as follows:

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The Atiyah class and the Lie algebra object $T_X[-1]$

- The Atiyah class can be constructed for an arbitrary object in $D(X)$.
- It can be interpreted in various ways, as an extension class, as a morphism in the derived category or in the simplest case of a vector bundle as the obstruction to have a holomorphic connection.
- Let's consider the simplest situation: take E an holomorphic vector bundle over the smooth projective variety X. The bundle of 1-jets JE is the sheaf $E \oplus E \otimes \Omega^1$ with the following $\mathcal{O}_X$-module structure:

$$f(s, t \otimes \theta) = (f \cdot s, f \cdot t \otimes \theta + s \otimes df)$$

- With this structure, JE is described as the following extension:

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The Atiyah class and the Lie algebra object $T_X[-1]$ II

- From the sequence above, the extension class $[JE]$ can be seen in either of the following groups:

$$Ext^1(E, E \otimes \Omega^1) \simeq Ext^1(E \otimes T, E)$$

- The Atiyah class defines a structure of a Lie algebra object on $T[-1]$ through the following result:

Proposition

$$Hom_{D(X)}(E, F[i]) \simeq Ext^i(E, F).$$

- The above Proposition express the Atiyah class $[JE]$ as a morphism in the derived category $D(X)$:

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$D(X)$ as representation category of the Lie algebra object $T_X[-1]$

- In the particular case $E = T[-1]$ the above morphism in $D(X)$, endow $T[-1]$ with a "bracket" such that we have the following:

Proposition

The bracket induced on $T[-1]$ by the Atiyah class verify the axioms for a Lie algebra object in $D(X)$.

- Moreover, the above morphism induced by the Atiyah class for an arbitrary object E in $D(X)$, endow E with a $T[-1]$-action such that we have:

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- Viewing the above proposition, one can ask the following:

Question

For an arbitrary morphism $E \to F$ in $D(X)$ is it a morphism of $T[-1]$-modules ?

- The result below gives a positive answer and express the fact that the derived category is the representation category of the Lie algebra object $T[-1]$.

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Any morphism $E \to F$ in $D(X)$ is it a morphism of $T[-1]$-modules ?

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The HKR isomorphism as an analogue of Duflo's

- In the sequel, we will translate almost all the Lie algebraic facts into their algebraic geometric counterpart using the dictionary between Lie algebras and the Lie algebra object $T[-1]$ from the derived category $D(X)$.
- Concerning the Duflo's isomorphism, we remark firstly that on the algebraic geometric side one have:

Proposition

The symmetric and universal enveloping algebra of $T[-1]$, are: $S = \oplus(\Lambda^k T)[-k]$ and $U = p_\star(\mathbb{R}\mathcal{H}om(\mathcal{O}_\Delta, \mathcal{O}_\Delta))$.

- Concerning the invariant parts, they must be defined in a categorical setting as $Hom(\mathcal{O}_X, -)$.

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The HKR isomorphism as an analogue of Duflo's II

- For S and U, one obtains: $\oplus H^\star(\Lambda^\star T)$ and $Ext^\star_{\mathcal{O}_{X\times X}}(\mathcal{O}_\Delta, \mathcal{O}_\Delta)$.
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An algebraic geometric analogue of the Duflo's Conjecture

- In the algebraic geometris setting, the Lie algebras h and g are replaced by the shifted tangent bundles $T_X[-1]$ and $T_Y[-1]$ for an embedding of algebraic varieties $i : X \hookrightarrow Y$.
- The invariant part $S(m)^h$ is replaced by $\oplus\Lambda^k(N_{X,Y})[-k]$ where N is the normal bundle of the embedding.
- On the other hand, the invariant part $(U(g)/hU(g))^h$, is replaced by $\mathbb{R}\mathcal{HOM}_X(i^\star i_\star \mathcal{O}_X, \mathcal{O}_X)$.
- In this setting, the Duflo's Conjecture translates into the question of finding conditions that ensures the above invariant parts to be isomorphic.

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An algebraic geometric analogue of the Duflo's Conjecture

- The answer is given in terms of the composition of the extension

$$0 \to T_X \to T_{Y|X} \to N_{X,Y} \to 0,$$

  viewed as the morphism $N_{X,Y}^{\otimes 2} \to N_{X,Y} \otimes T_X[1]$ in $D(X)$, and the Atiyah class of $N_{X,Y}$.
- Let's denote by $a \in Ext^2(N_{X,Y}^{\otimes 2}, N_{X,Y})$ the resulting extension class.

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The Arinkin's-Caldararu extendability criteria

- With the above notation, the following result was proved by Arinkin and Caldararu:

Theorem Arinkin-Caldararu

The following are equivalent: 1. the extension class $a = 0$.
2. $N_{X,Y}$ admits an extension to the first infinitesimal neighborhood of X in Y.
3. the algebraic geometric Duflo's Conjecture holds true in this situation.

- We can interpret the above theorem as a criteria for the algebraic geometric Duflo's theorem to hold, in terms of the extendability of the normal bundle, at least to the first infinitesimal neighborhood.
- Fortunately, this geometric condition is known to be true in a variety of special cases. Among these, we wish to restrict to the case of co-dimension two smooth sub-manifolds $X \subset Y$. In this case, Serre's construction asserts the following:

Serre's Construction

Let I_X the ideal sheaf of X, $L = (detN)^{\vee}$. Then: $\mathcal{EXT}_{\mathcal{O}_X}(I_X, L)$ is a trivial line bundle and its generator defines an extension

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Moreover, E is locally free and $E^{\vee}$ extends N to all of Y.

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Moreover, E is locally free and $E^\vee$ extends N to all of Y.

- Putting together the Arinkin-Caldararu theorem and Serre's construction, we arrive at our main result:

Theorem -. B. C.

For a co-dimension 2 smooth submanifold, the algebraic geometric Duflo's Conjecture holds true.

Conclusions and future directions

- We combined Serre construction and the Arinkin-Caldararu theorem, to argue that in the co-dimension 2 case, the algebraic geometric Duflo Conjecture holds true.
- The question of the extendability of vector bundles over a sub-manifold X up to a certain infinitesimal neighborhood is a very old one. In recent years, for example Badescu obtained new results in this direction in the small co-dimension situation and a vector bundle of rank 2 on X. A natural question in this direction could be the following:

Question

To find cohomological obstructions of the type of the class a in the Arinkin-Caldararu theorem that prevent the extension of an arbitrary rank 2 vector bundle on X, to its first infinitesimal neighborhood.

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THANK YOU!