A stable version of Terao conjecture

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Abstract

- Terao conjecture (1981), asserts that the freeness of a hyperplane arrangement depends only of its combinatorics. The freeness is equivalent with the fact that the associated bundle splits completely as direct sum of line bundles. This last property, thanks to Horrocks criterion, is equivalent with the vanishing of certain cohomology modules of the bundle in question. Also, using the famous Barth-Van de Ven-Sato-Tyurin result, the freeness of an arrangement is equivalent with the indefinitely extendability of the associated bundle. In the first part we shall describe the above circle of ideas.
- The second part will be devoted to the notion of stable extendability, introduced by Horrocks in 1966, and its connection with the above results, thanks to a theorem of Coanda (2009), which gives a characterization of indefinitely stable extendable vector bundles in terms of some cohomology modules of the bundle. Finally, we shall formulate a problem with the same flavor as Terao conjecture, using the Coanda notion of indefinitely stable extendabiliy.

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- The second part will be devoted to the notion of stable extendability, introduced by Horrocks in 1966, and its connection with the above results, thanks to a theorem of Coanda (2009), which gives a characterization of indefinitely stable extendable vector bundles in terms of some cohomology modules of the bundle. Finally, we shall formulate a problem with the same flavor as Terao conjecture, using the Coanda notion of indefinitely stable extendabiliy.

Outline

Arrangements and their lattices I

- An arrangement in the complex projective space $\mathbb{P}^n(\mathbb{C})$ is a finite collection of hyperplanes $\mathcal{A} = \{H_1, ..., H_k\}$.
- For a fixed arrangement $\mathcal{A}$, its intersection lattice $L_{\mathcal{A}}$ is the poset with elements the finite intersections between the $H_i's$, ordered by reverse inclusion: for $L_1, L_2 \in L_{\mathcal{A}}$, $L_1 \leq L_2$ *iff* $L_1 \supseteq L_2$.
- Using the interection lattice, we have a first equivalence relation for arrangements:

Definition

Two arrangements $\mathcal{A}_1, \mathcal{A}_2$ have the same combinatorics if their lattices $L_{\mathcal{A}_1}, L_{\mathcal{A}_2}$ are isomorphic.

- For example, if $\mathcal{A}_1$ is defined by three concurrent lines in $\mathbb{P}^2(\mathbb{C})$ and $\mathcal{A}_2$ by three lines without a common point, then $\mathcal{A}_1$ and $\mathcal{A}_2$ have different combinatorics.

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Arrangements and their lattices II

- A fundamental question in the theory of hyperplanes arrangements is to find which properties of the arrangement depends only on its lattice i.e. only of its combinatorics.
- For example, concerning the cohomology algebra of the complement we have the following celebrated result:

Arnold-Brieskorn-Orlik-Solomon

The cohomology ring $H^*(\mathbb{P}^n(\mathbb{C}) \setminus \bigcup_{i=1}^{k} H_i)$ of the complement of $\mathcal{A} = \{H_1, ..., H_k\}$ is combinatorially determined by $L_{\mathcal{A}}$.

- Also, a negative result in this direction, concern the homotopy type of the complement: for example $\pi_1(\mathbb{P}^n(\mathbb{C}) \setminus \bigcup_{i=1}^{k} H_i)$ is not combinatorially determined.
- In fact, Rybnikov (1998) constructed two arrangements in $\mathbb{P}^2(\mathbb{C})$ with the same combinatorics but different π_1 for the complements.

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Bundles associated with arrangements I

- Apart the lattice and homological or homotopical invariants associated to an arrangement $\mathcal{A}$ another interesting object is the sheaf $\mathcal{T}_{\mathcal{A}}$ of vector fields with logarithmic poles along $\mathcal{A}$. It was introduced for the first time by Saito and Deligne in the '80s and used in the context of hyperplane arrangements by Dolgachev, Kapranov, Terao and others. Its construction goes as follows:
- denote by f_i an homogenous equation of the hyperplane H_i and by f the product $\prod_{i=1}^{k} f_i$.
- Then $\mathcal{T}_{\mathcal{A}}$ is defined as the kernel of the map

$$\mathcal{O}_{\mathbb{P}^n}^{\oplus(n+1)} \to \mathcal{O}_{\mathbb{P}^n}(k-1),$$

defined by the partial derivatives of f: $(\partial_{x_0} f, ..., \partial_{x_n} f)$.
- The sheaf $\mathcal{T}_{\mathcal{A}}$ will be the principal object of study in the sequel. In general it is a rank-n sheaf on $\mathbb{P}^n$, but we will be interested mainly in the case where it is locally free. For example, due to a result of Dolgachev this is the case for the normal crossing arrangements.

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Bundles associated with arrangements II

- An important problem concerning $\mathcal{T}_{\mathcal{A}}$, in the case when it is locally free, is its splittability:

Definition

A vector bundle on $\mathbb{P}^n$ is splittable if it is direct sum of line bundles.

- With the definition above, an arrangement $\mathcal{A}$ is called free if the associated sheaf $\mathcal{T}_{\mathcal{A}}$ is splittable.
- Of course if $\mathcal{A}$ is free, then $\mathcal{T}_{\mathcal{A}}$ is locally free and consequently, concerning the freeness one can consider only arrangements with locally free $\mathcal{T}_{\mathcal{A}}$.
- In the above terms, one can enounce the:

Terao's conjecture

The freeness of an arrangement is combinatorially determined. Namely, for two arrangements $\mathcal{A}_1, \mathcal{A}_2$ with isomorphic lattices, if $\mathcal{A}_1$ is free then $\mathcal{A}_2$ is also free.

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The Horrocks criterion

- As long as the freeness of $\mathcal{A}$ means the splittability of $\mathcal{T}_{\mathcal{A}}$, a good starting point in the study of free arrangements could be a criterion which ensure the splittability of a vector bundle on $\mathbb{P}^n$.
- In this direction, the fundamental result is Horrocks theorem. Let F a vector bundle on $\mathbb{P}^n$. For any $1 \leq i \leq n-1$ we denote by $H^i_*(F)$ the cohomology module

$$\bigoplus_{k \in \mathbb{Z}} H^i(\mathbb{P}^n, F \otimes \mathcal{O}_{\mathbb{P}^n}(k)).$$

- With the above notations we have:

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A vector bundle F on $\mathbb{P}^n$ splitts completely as direct sum of line bundles iff for any $1 \leq i \leq n-1$ the cohomology module $H^i_*(F)$ is zero.

- Consequently, the freeness of an arrangement $\mathcal{A}$ is equivalent with the vanishing of all intermediate cohomology modules $H^i_*(\mathcal{T}_{\mathcal{A}})$ of its bundle of logarithmic vector fields.

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The Barth-Van de Ven-Sato-Tyurin theorem

- A second viewpoint concerning the splittability of bundles on $\mathbb{P}^n$ is connected with the following phenomenon: a vector bundle F on $\mathbb{P}^n$ is indefinitely extendable if for any $m \geq n$ there exist a bundle F_m on $\mathbb{P}^m$ such that

$$F_{m|\mathbb{P}^n} \simeq F.$$

- The following result, due to Barth-Van de Ven-Sato-Tyurin, asserts that in fact the indefinitely extendability is equivalent with the complete splittability of the bundle in question:

The babylonian tower theorem

For a vector bundle F on $\mathbb{P}^n$, the following are equivalent:
1. F splitts completely as direct sum of line bundles,
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- As consequence, one obtain another characterization of the freeness of an arrangement $\mathcal{A}$, namely the indefinitely extendability of $\mathcal{T}_{\mathcal{A}}$.

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Motivation for the second part

- The conclusion of the previous results is that freeness of an arrangement, which is the main property in the statement of the Terao conjecture, admits at least two equivalent formulations:

vanishing of all the intermediate cohomology of $\mathcal{T}_{\mathcal{A}}$ $\Leftrightarrow$ freeness $\Leftrightarrow$ indefinitely extendability of $\mathcal{T}_{\mathcal{A}}$.

- The main question we will discus in the second part is the following:

Question

Is there a weaker (than freeness) property with a similar cohomological and geometrical flavor which could be used in a modified form of Terao conjecture?

- The answer is yes and is connected with the notion of indefinitely stable extendability, characterized by Coanda in 2009.

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Horrocks theory of stable extendability

- In analogy with the previous notion of extendability, Horrocks (1966) introduced the following weaker concept:

Definition

A vector bundle F on $\mathbb{P}^n$ is stable extendable on a larger space $\mathbb{P}^m$ if there exists a bundle F_m on $\mathbb{P}^m$ whose restriction to $\mathbb{P}^n$ is the direct sum between F and certain line bundles.

- A first remark is that an extendable bundle is obviously stably extendable, but the converse is not true. For example the tangent bundle of $\mathbb{P}^n$, $T_{\mathbb{P}^n}$ is stable extendable but not extendable.
- Also, one should note that the above notion is connected with the complete splittability of a bundle by the following result:

Horrocks theorem

If the bundle F on $\mathbb{P}^n$ extends stably to $\mathbb{P}^{2n-3}$ and the cohomology modules $H^1_*(F)$, $H^{n-1}_*(F)$ vanishes, then F splits completely on $\mathbb{P}^n$.

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Indefinitely stable extendability

- The result above, show that the condition of stable extendability of a bundle F, has a subtle connection with the property of complete splittability and is also a good motivation for the following definition introduced (and as we shall see, characterized) by Coanda in 2009:

Definition

A vector bundle F on $\mathbb{P}^n$ is indefinitely stable extendable, if for any $m \geq n$ it extends stably on $\mathbb{P}^m$.

- As in the case of stable extendability, the above property is strictly weaker than indefinitely extendability, as long as again, the example of the tangent bundle of $\mathbb{P}^n$ shows that there are bundles indefinitely stable extendable which are not splittable and therefore (using the babylonian tower theorem of Barth-Van de Ven-Sato-Tyurin) are not indefinitely extendable.

Indefinitely stable extendability

- The result above, show that the condition of stable extendability of a bundle F, has a subtle connection with the property of complete splittability and is also a good motivation for the following definition introduced (and as we shall see, characterized) by Coanda in 2009:

Definition

A vector bundle F on $\mathbb{P}^n$ is indefinitely stable extendable, if for any $m \geq n$ it extends stably on $\mathbb{P}^m$.

- As in the case of stable extendability, the above property is strictly weaker than indefinitely extendability, as long as again, the example of the tangent bundle of $\mathbb{P}^n$ shows that there are bundles indefinitely stable extendable which are not splittable and therefore (using the babylonian tower theorem of Barth-Van de Ven-Sato-Tyurin) are not indefinitely extendable.

The Coanda criterion

- The main point concerning the above property is that, like the complete splittability and therefore -via the babylonian tower theorem- like the indefinitely extendability, it admits an analogous cohomological characterization in terms of some intermediate cohomology modules. This one, was obtained by Coanda in 2009:

Coanda theorem

A vector bundle F on $\mathbb{P}^n$ is indefinitely stable extendable iff for any $2 \leq i \leq n-2$ the intermediate cohomology module $H^i_*(F)$ vanishes.

- On should remark that in fact, the original theorem of Coanda, contains also a third characterization of the indefinitely stable extendability, namely as the condition for F of being the cohomology of a free monad.

- Also, one should note that the condition in the theorem is empty for $n \leq 3$ and so any bundle on $\mathbb{P}^{\leq 3}$ is indefinitely stable extendable.

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A "stable" Terao conjecture

- Inspired by the above result we consider de following:

Definition

An arrangement $\mathcal{A}$ is stably free if its associated bundle of vector fields with logarithmic poles $\mathcal{T}_{\mathcal{A}}$ is indefinitely stable extendable.

- Obviously, a free arrangement is stably free, the converse is not true and we have a similar picture as in the case of freeness:

vanishing of the intermediate cohomology of $\mathcal{T}_{\mathcal{A}}$ for $2 \leq i \leq n-2$ $\Leftrightarrow$ stable freeness $\Leftrightarrow$ indefinitely stable extendability of $\mathcal{T}_{\mathcal{A}}$.

- Consequently, we introduce the following:

The stable Terao conjecture

The stable freeness of an arrangement is combinatorially determined. Namely, for two arrangements $\mathcal{A}_1, \mathcal{A}_2$ with isomorphic lattices, if $\mathcal{A}_1$ is stable free then $\mathcal{A}_2$ is also stable free.

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Connections with other notions of freeness

- A first point to note is that the two conjectures are not comparable: no one implies the other.
- Also, it could be interesting to compare this notion of stable freeness which can be obviously be extended from arrangements to arbitrary union of hyper-surfaces to other weaker notions of freeness existing in literature.
- For convenience we mention only two:
  - the nearly free and almost free divisors introduced by Dimca and Sticlaru,
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THANK YOU!