Geometry of the Sasakura bundle

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- In fact, it is connected with some surfaces in $\mathbb{P}^4$ which missed in early classification papers.
- The aim of my talk is to present various, scattered in the literature, aspects concerning the geometry of this bundle.
- The last part will be devoted to the place of this bundle in the classification of globally generated locally free sheaves with $c_1 \leq 4$ on $\mathbb{P}^n$ in a joint paper with I. Coanda and N. Manolache.

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Introduction: some interesting bundles on $\mathbb{P}^n$

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- The $r = n - 1$ nullcorrelation bundle on $\mathbb{P}^n$ for odd n ($'77$).
- The $r = 3$ Horrocks bundle on $\mathbb{P}^5$ ($'78$).
- The $r = 3$ Sasakura bundle on $\mathbb{P}^4$ ($'86$).
- More recent examples are the weighted Tango bundles in arbitrary dimension introduced by Cascini ($'01$), and the bundles of Kumar-Peterson-Rao in low dimension for various characteristics ($'02$).

The Abo Decker construction

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- A result of Okonek asserts that:

Okonek's Theorem

A smooth surface $X \subset \mathbb{P}^4$ of degree 8 and sectional genus 5 and irregularity 1 is an elliptic conic bundle with exactly 8 singular fibers composed by pairs of -1 lines.

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- Okonek claimed that such surfaces does not exists.

The Beilinson monad

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- For a sheaf $\mathcal{F}$ on $\mathbb{P}^4$ denote by $F^i := \oplus H^{i+j}(\mathcal{F}(-j)) \otimes \Omega^j(j)$, the direct sum being over all j's.

Beilinson Theorem

The F^i's forms an increasing complex, exact except in dimension 0, where the cohomology is $\mathcal{F}$.

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$$
\begin{array}{c|cccccc}
 & \uparrow i & & & & & \\
4 & & & & & & \\
 & 1 & 1 & & & & \\
 & & & 1 & 1 & & \\
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 & & & & & & \longrightarrow m
\end{array}
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where in the (m, i) box is depicted $h^i(\mathcal{J}_X(m))$.

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- $0 \to 4\mathcal{O}(-1) \to \mathcal{G} \to \mathcal{J}_X(3) \to 0$. It is the rank 5 version of the Sasakura bundle.

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- Moreover, if α, β are the maps in the monad, and $e_0, \ldots, e_4$ is a basis in V-the underlying vector space of $\mathbb{P}^4$, using the identification $Hom(\Omega^i(i), \Omega^j(j)) \simeq \Lambda^{i-j} V$ it can be proved that:

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- $\alpha = \begin{pmatrix} e_4 \\ e_0 \wedge e_2 + e_1 \wedge e_3 \end{pmatrix}$ and $\beta = \begin{pmatrix} e_0 \wedge e_2 + e_1 \wedge e_3 & -e_4 \end{pmatrix}$

The Beilinson monad

Theorem (Conclusion)

An elliptic conic bundle in $\mathbb{P}^4$ determines an unique, up to isomorphism and linear change of coordinates, 5-bundle $\mathcal{G}$ given by the monad above.

- Conversely, starting with such a $\mathcal{G}$, $\mathcal{G}(1)$ is globally generated.

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- Conversely, starting with such a $\mathcal{G}$, $\mathcal{G}(1)$ is globally generated.
- By a result of Banica, the dependancy locus is a smooth surface and has the desired invariants.

The Sasakura method

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- $\mathcal{E}$ is a rank r vector bundle on $\mathbb{P}^n$ with first Chern class c_1. $s_1, ..., s_l$ and $\sigma_1, ..., \sigma_k$ are generators of $H^0{}_*(\mathcal{E})$ and $H^0{}_*(\mathcal{E}^\vee)$.

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- They determines the morphisms (the first epi and the second mono) $\mathcal{L} \to \mathcal{E} \to \mathcal{K}$, where $\mathcal{L}$ and $\mathcal{K}$ are direct sums of line bundles.
- In particular, the composition $S : \mathcal{L} \to \mathcal{K}$ is a matrix of homogenous polynomials.

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- The ideal $(I : f)$ define the empty set in $\mathbb{P}^n$.

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- An useful choice is $f = f'^{r-1}$ with $f' = x_1 \ldots . x_{c_1}$ and $T = (T', T'')$ with $T' = f' \cdot Id$. Using this idea can be produced many known bundles, eg. nullcorelation bundle on $\mathbb{P}^3$, the Horrocks-Mumford on $\mathbb{P}^4$ and the Sasakura rank 3 bundle on $\mathbb{P}^4$.

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- The last one, is constructed with $f = f' = x_0...x_4$ and a convenient but complicated $(3x8)$ matrix T of forms of degree 4.

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- **Step 2** The restriction of φ to X_5 is defined by the system $\mid 2H - L \mid$ with 8 base points: those where G meet the scroll and are not on L.

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- **Step 3** This restrction is an embedding of the blow-up at the 8 points as soon as the 10 points in $G \cap X_5$ are distinct and no other secant of G is a fiber of the scroll. The image $\varphi(\hat{X}_5)$ will be of course an elliptic conic bundle.

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- For the converse Cremona transformation, a first intricate construction, produce a quadratic surface X_2.
- Secondly, a cubic scroll X_3 is produced using the secant variety of G.
- Finally, the converse Cremona is defined by the cubic hypersurfaces through $X_2 \cup X_3$.

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- The general construction starts with a sort of Maruyama's elementary transformation:
  let X projective variety (later it will be $\mathbb{P}^4$) and Y the divisor of a section s in $\mathcal{O}_X(Y)$ (later it will be the thickening of order t of a hyperplane in $\mathbb{P}^4$).

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- let's consider on Y an exact sequence
  $0 \to A \to F \to B \to 0$ of vector bundles, such that F extends to an $\mathcal{F}$ on X.
- Let $\mathcal{G}$ the kernel of the induced surjection $\mathcal{F} \to B$.

The Kumar-Peterson-Rao construction

- The next ingredients are two bundles $\mathcal{L}_1$ and $\mathcal{L}_2$ on the ambient X, their restrictions to Y, L_1 and L_2 with a surjection $L_1 \to A$ and an injection as vector bundle $B \to L_2$.

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- Suppose that the induced $\phi : L_1 \to F$ and $\psi : F \to L_2$ also extends to $\Phi : \mathcal{L}_1 \to \mathcal{F}$ and $\Psi : \mathcal{F} \to \mathcal{L}_2$.

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- Suppose that the induced $\phi : L_1 \to F$ and $\psi : F \to L_2$ also extends to $\Phi : \mathcal{L}_1 \to \mathcal{F}$ and $\Psi : \mathcal{F} \to \mathcal{L}_2$.
- Then $\Psi\Phi$ vanishes on Y and one can construct a map

$$\Delta : \mathcal{F}(-Y) \oplus \mathcal{L}_1 \to \mathcal{F} \oplus \mathcal{L}_2(-Y)$$

given by the matrix below:

$$\begin{pmatrix} s \cdot I & \Phi \\ \Psi & s^{-1} \cdot \Psi\Phi \end{pmatrix}$$

where I is the identity of $\mathcal{F}$ (and s the section which determines Y).

The Kumar-Peterson-Rao construction

- The role of the above map Δ whose image is in fact $\mathcal{G}$ is tied with the fact that if $\mathcal{F}$ splits, under some aditional hypothesis, this will produce sub or quotient bundles of $\mathcal{G}$, producing lower rank bundles.

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- The role of the above map Δ whose image is in fact $\mathcal{G}$ is tied with the fact that if $\mathcal{F}$ splits, under some aditional hypothesis, this will produce sub or quotient bundles of $\mathcal{G}$, producing lower rank bundles.
- For example, concerning the sub-bundle case, the authors have the following

Proposition

Suppose:
1. $\mathcal{F}$ split as $\mathcal{N} \oplus \mathcal{N}'$ with induced splitting $F = N \oplus N'$
2. there is $\theta : N(-Y) \to A$ with lift $\Theta : \mathcal{N}(-Y) \to \mathcal{L}_1$ such that $\Phi\Theta$ $(: \mathcal{N}(-Y) \to \mathcal{F})$ has image in $\mathcal{N}'$.

Then, there is an induced map $\mathcal{N}(-Y) \to \mathcal{G}$ which is a bundle inclusion iff the restriction $N(-Y) \to \mathcal{G}\mid_Y$ is.

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The Kumar-Peterson-Rao construction

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- This goal is achieved through the so called four generated rank two bundles. A vector bundle B on a scheme Y is four generated if there is a completely split rank 4 bundle F and a surjection $F \to B$.
- The result below construct plenty (but of course not all) of four generated rank 2 bundles on $\mathbb{P}^3$:

Proposition

Let T, U, V, W a regular sequence of forms of positive degrees t, u, v, w such that $t + w = u + v$ and $r \geq 2$ an integer. Then there is an exact sequence $L_1 \to F \to L_2$, with maps ϕ and ψ such that:
1. the bundles above are completely split,
2. the images A and B of ϕ and ψ are four generated rank 2 bundles on $\mathbb{P}^3$.

The Kumar-Peterson-Rao construction

- Finally, one can put together the ideas above in the case of $\mathbb{P}^4$:
  - one starts with the above F and four generated rank 2 bundles A, B on $\mathbb{P}^3$
  - one consider a t'-thickening Y of $\mathbb{P}^3 \subset \mathbb{P}^4$ and one pull back F, A and B on Y
  - one apply the Maruyama type construction obtaining a rank 4 bundle $\mathcal{G}$ on $\mathbb{P}^4$
  - for convenient values of the parameters (the forms T, U, V, W and integers $t', r \geq 2$) the bundle $\mathcal{G}$ has line sub or quotient bundles
  - by taking the quotient or the kernel one arrive at rank 3 bundles on $\mathbb{P}^4$.

The Coanda-Manolache method

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- -, Coanda and Manolache in arxiv:1305.3464 classified globally generated vector bundles with $c_1 \leq 4$ on $\mathbb{P}^n$.

The Coanda-Manolache method

- For example, for $c_1 \leq 3$ the main result from the joint paper with Manolache, can be formulated as:

Theorem

Let E an indecomposable globally generated vector bundle on $\mathbb{P}^n$, with $n \geq 2$, $1 \leq c_1 \leq 3$ and $H^i(E^*) = 0$ for $i = 0, 1$. Then one of the following holds:
- $E = \mathcal{O}(a)$
- $E = P(\mathcal{O}(a))$
- $n = 3$ *and* $E = \Omega(2)$
- $n = 4$ *and* $E = \Omega(2)$
- $n = 4$ *and* $E = \Omega^2(3)$

where the P-operation above means the dual of the kernel of the evaluation map.

The Coanda-Manolache method

- The technical condition $H^i(E^*) = 0$ for $i = 0, 1$ is irrelevant. In fact any globally generated bundle can be obtained by one which verify this condition by taking the quotient with a trivial sub-bundle and then adding a trivial summand:

Proposition

For any E there is an F which satisfy $H^i(F^*) = 0$ for $i = 0, 1$ such that if $t = h^0(E^*)$ and $s = h^1(E^*)$ then $E \simeq F/s\mathcal{O} \oplus t\mathcal{O}$.

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- Another important observation is that for a globally generated bundle one has $c_2 \leq {c_1}^2$. This show via the standard sequence $0 \to (r-1)\mathcal{O} \to E \to \mathcal{J}_Y(c_1)$ that globally generated bundles with $c_1 \leq 3$ are related with sub-varieties of degree at most 9 in $\mathbb{P}^n$.

The Coanda-Manolache method

- The main result in the joint paper with Coanda and Manolache is more complicated. In the $c_1 = 4$ case there are 16 indecomposable bundles, the last one being the Sasakura's rank 5 bundle once twisted $\mathcal{G}(1)$.

The Coanda-Manolache method

- The main result in the joint paper with Coanda and Manolache is more complicated. In the $c_1 = 4$ case there are 16 indecomposable bundles, the last one being the Sasakura's rank 5 bundle once twisted $\mathcal{G}(1)$.
- However one can formulate the following consequence:

Corollary

Let E an indecomposable globally generated vector bundle on $\mathbb{P}^n$, with $n \geq 4$, $c_1 = 4$, $r \geq 2$ and $H^i(E^*) = 0$ for $i = 0, 1$. Then E is:
-$P(\mathcal{O}(4))$
-$\Omega(2)$ or $\Omega^3(4)$ on $\mathbb{P}^5$
-$\mathcal{G}(1)$.

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- First we classify the bundles on $\mathbb{P}^2$ (easy) and $\mathbb{P}^3$ (hard)
- Next we try to decide which bundles can be extended to higher dimensional projective spaces using Horrocks method of killing cohomology.
- The problem is tied to the theory of varieties of small degree as we have seen, and also to the theory of rank 2 reflexive sheaves on $\mathbb{P}^3$ via an exact sequence as below:

$$0 \to (r-2)\mathcal{O} \to E \to \mathcal{E}' \to 0.$$

The Coanda-Manolache method

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- Consider the surjection $4\mathcal{O}(-1) \oplus \mathcal{O}(-2) \to \mathcal{O}$ defined by $x_0, ..., x_3, {x_4}^2$, and the Koszul complex (C_p, δ_p) for it. Denote by E' the co-kernel of $\delta_4(4)$:

$$\mathcal{O} \oplus 4\mathcal{O}(-1) \to 4\mathcal{O}(1) \oplus 6\mathcal{O}.$$

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- then E is the kernel of a surjection $E' \to \mathcal{O}(2)$ such that $H^0(E'(-1)) \to H^0(\mathcal{O}(1))$ is injective.

Final remarks

- The Koszul complex appear also in another case, $n = 3$ $c_2 = 8$ where one consider the complex associated with $x_0, x_1, {x_2}^2, {x_3}^2$ and the E is the cohomology of the monad

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- Other different types of Koszul complexes were used by Kumar-Peterson-Rao to produce interesting deformations of known bundles.

Open question

- Recall that Kumar construction use the existence of many four generated rank 2 bundles on $\mathbb{P}^3$.

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- Recall that Kumar construction use the existence of many four generated rank 2 bundles on $\mathbb{P}^3$.
- It would be interesting to produce, with Kumar method, other examples of bundles on $\mathbb{P}^4$ using four generated bundles on divisors of higher degree in $\mathbb{P}^4$.

THANK YOU!