Freeness, extendability and arrangements

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Abstract

The subject of the talk is the freeness of hyperplane arrangements, from the viewpoint of Horrocks-Coanda's concept of (infinitely) stably extendability of vector bundles on the projective space. In the last part, I will speak on a stable version of the Terao conjecture, which appear in the above context.

Outline

- Terao conjecture (1981), asserts that the freeness of a hyperplane arrangement depends only of its combinatorics.
- The freeness is equivalent with the fact that the associated bundle splits completely as direct sum of line bundles.
- This last property, thanks to Horrocks criterion, is equivalent with the vanishing of certain cohomology modules of the bundle in question.
- Also, using the famous Barth-Van de Ven-Sato-Tyurin result, the freeness of an arrangement is equivalent with the infinitely extendability of the associated bundle. In the first part we shall describe the above circle of ideas.

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- The second part will be devoted to the notion of stably extendability of bundles, introduced by Horrocks in 1966, and its connection with the above results, thanks to a theorem of Coanda (2009).
- This one, gives a characterization of infinitely stably extendable vector bundles in terms of the vanishing of some intermediate cohomology modules of the bundle.
- Finally, we shall formulate a problem with the same flavor as Terao conjecture, using the Coanda notion of infinitely stably extendability.

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Arrangements and their lattices I

- An arrangement in the complex projective space $\mathbb{P}^n(\mathbb{C})$ is a finite collection of hyperplanes $\mathcal{A} = \{H_1, ..., H_k\}$.
- For a fixed arrangement $\mathcal{A}$, its intersection lattice $L_{\mathcal{A}}$ is the poset with elements the finite intersections between the $H_i's$, ordered by reverse inclusion: for $L_1, L_2 \in L_{\mathcal{A}}$, $L_1 \leq L_2$ *iff* $L_1 \supseteq L_2$.
- Using the intersection lattice, we have a first equivalence relation for arrangements:

Definition

Two arrangements $\mathcal{A}_1, \mathcal{A}_2$ have the same combinatorics if their lattices $L_{\mathcal{A}_1}, L_{\mathcal{A}_2}$ are isomorphic.

- For example, if $\mathcal{A}_1$ is defined by three concurrent lines in $\mathbb{P}^2(\mathbb{C})$ and $\mathcal{A}_2$ by three lines without a common point, then $\mathcal{A}_1$ and $\mathcal{A}_2$ have different combinatorics.

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- A fundamental question in the theory of hyperplanes arrangements is to find which properties of the arrangement depends only on its lattice i.e. only of its combinatorics.
- For example, concerning the cohomology algebra of the complement we have the following celebrated result:

ARNOLD-BRIESKORN-ORLIK-SOLOMON

The cohomology ring $H^*(\mathbb{P}^n(\mathbb{C}) \setminus \bigcup_{i=1}^{k} H_i)$ of the complement of $\mathcal{A} = \{H_1, ..., H_k\}$ is combinatorially determined by $L_{\mathcal{A}}$.

- Also, a negative result in this direction, concern the homotopy type of the complement: for example $\pi_1(\mathbb{P}^n(\mathbb{C}) \setminus \bigcup_{i=1}^{k} H_i)$ is not combinatorially determined.
- In fact, Rybnikov (1998) constructed two arrangements in $\mathbb{P}^2(\mathbb{C})$ with the same combinatorics but different π_1 for the complements.

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Bundles associated with arrangements I

- Apart the lattice and homological or homotopical invariants associated to an arrangement $\mathcal{A}$ another interesting object is the sheaf $\mathcal{T}_{\mathcal{A}}$ of vector fields with logarithmic poles along $\mathcal{A}$. It was introduced for the first time by Saito and Deligne in the '80s and used in the context of hyperplane arrangements by Dolgachev, Kapranov, Terao and others. Its construction goes as follows:
- denote by f_i an homogenous equation of the hyperplane H_i and by f the product $\prod_{i=1}^{k} f_i$.
- Then $\mathcal{T}_{\mathcal{A}}$ is defined as the kernel of the map

$$\mathcal{O}_{\mathbb{P}^n}^{\oplus(n+1)} \to \mathcal{O}_{\mathbb{P}^n}(k-1),$$

  defined by the partial derivatives of f: $(\partial_{x_0} f, ..., \partial_{x_n} f)$.
- The sheaf $\mathcal{T}_{\mathcal{A}}$ will be the principal object of study in the sequel. In general it is a rank-n sheaf on $\mathbb{P}^n$, but we will be interested mainly in the case where it is locally free. For example, due to a result of Dolgachev this is the case for the normal crossing arrangements.

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Bundles associated with arrangements II

- An important problem concerning $\mathcal{T}_{\mathcal{A}}$, in the case when it is locally free, is its splitability:

Definition

A vector bundle on $\mathbb{P}^n$ is splittable if it is direct sum of line bundles.

- With the definition above, an arrangement $\mathcal{A}$ is called free if the associated sheaf $\mathcal{T}_{\mathcal{A}}$ is splittable.
- Of course if $\mathcal{A}$ is free, then $\mathcal{T}_{\mathcal{A}}$ is locally free and consequently, concerning the freeness one can consider only arrangements with locally free $\mathcal{T}_{\mathcal{A}}$.
- From the work of Dolgachev and Kapranov, we have a first class of examples:

Example

Normal crossing arrangements of at most $n+1$ hyperplanes in $\mathbb{P}^n$ are free.

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Terao's conjecture

The freeness of an arrangement is combinatorially determined. Namely, for two arrangements $\mathcal{A}_1, \mathcal{A}_2$ with isomorphic lattices, if $\mathcal{A}_1$ is free then $\mathcal{A}_2$ is also free.

- One must remark that despite the simplicity of the statement, it was proved only in very few particular cases. For example, Faenzi and Valles proved that Terao conjecture holds true for arrangements in $\mathbb{P}^2$ with at most 12 lines.

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The Horrocks criterion

- As long as the freeness of $\mathcal{A}$ means the splittability of $\mathcal{T}_{\mathcal{A}}$, a good starting point in the study of free arrangements could be a criterion which ensure the splitability of a vector bundle on $\mathbb{P}^n$.
- In this direction, the fundamental result is Horrocks theorem. Let F a vector bundle on $\mathbb{P}^n$. For any $1 \leq i \leq n-1$ we denote by $H^i_*(F)$ the cohomology module

$$\bigoplus_{k\in\mathbb{Z}} H^i(\mathbb{P}^n, F \otimes \mathcal{O}_{\mathbb{P}^n}(k)).$$

- With the above notations we have:

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A vector bundle F on $\mathbb{P}^n$ splits completely as direct sum of line bundles iff for any $1 \leq i \leq n-1$ the cohomology module $H^i_*(F)$ is zero.

- Consequently, the freeness of an arrangement $\mathcal{A}$ is equivalent with the vanishing of all intermediate cohomology modules $H^i_*(\mathcal{T}_{\mathcal{A}})$ of its bundle of logarithmic vector fields.

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- A second viewpoint concerning the splitability of bundles on $\mathbb{P}^n$ is connected with the following phenomenon: a vector bundle F on $\mathbb{P}^n$ is infinitely extendable if for any $m \geq n$ there exist a bundle F_m on $\mathbb{P}^m$ such that

$$F_{m|\mathbb{P}^n} \simeq F.$$

- As well known examples we have:

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All the line bundles on $\mathbb{P}^n$, and their direct sums, are infinitely extendable.

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- The following result, due to Barth-Van de Ven-Sato-Tyurin, asserts that in fact the infinitely extendability is equivalent with the complete splitability of the bundle in question:

The babylonian tower theorem

For a vector bundle F on $\mathbb{P}^n$, the following are equivalent:
1. F splits completely as direct sum of line bundles,
2. F is infinitely extendable.

- As consequence, one obtain another characterization of the freeness of an arrangement $\mathcal{A}$, namely the infinitely extendability of $\mathcal{T}_{\mathcal{A}}$.

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Motivation for the second part

- The conclusion of the previous results is that freeness of an arrangement, which is the main property in the statement of the Terao conjecture, admits at least two equivalent formulations:

vanishing of all the intermediate cohomology of $\mathcal{T}_{\mathcal{A}}$	$\Leftrightarrow$ freeness $\Leftrightarrow$	infinitely extendability of $\mathcal{T}_{\mathcal{A}}$.

- The main question we will discus in the second part is the following:

Question

Is there a weaker (than freeness) property with a similar cohomological and geometrical flavor which could be used in a modified form of Terao conjecture?

- The answer is yes and is connected with the notion of infinitely stably extendability, characterized by Coanda in 2009.

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- In analogy with the previous notion of extendability, Horrocks (1966) introduced the following weaker concept:

Definition

A vector bundle F on $\mathbb{P}^n$ is stably extendable on a larger space $\mathbb{P}^m$ if there exists a bundle F_m on $\mathbb{P}^m$ whose restriction to $\mathbb{P}^n$ is the direct sum between F and certain line bundles.

- A first remark is that an extendable bundle is obviously stably extendable, but the converse is not true. For example the tangent bundle of $\mathbb{P}^n$, $T_{\mathbb{P}^n}$ is stably extendable but not extendable.
- Also, one should note that the above notion is connected with the complete splitability of a bundle by the following result:

Horrocks theorem

If the bundle F on $\mathbb{P}^n$ extends stably to $\mathbb{P}^{2n-3}$ and the cohomology modules $H^1_*(F)$, $H^{n-1}_*(F)$ vanishes, then F splits completely on $\mathbb{P}^n$.

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- A first remark is that an extendable bundle is obviously stably extendable, but the converse is not true. For example the tangent bundle of $\mathbb{P}^n$, $T_{\mathbb{P}^n}$ is stably extendable but not extendable.
- Also, one should note that the above notion is connected with the complete splitability of a bundle by the following result:

Horrocks theorem

If the bundle F on $\mathbb{P}^n$ extends stably to $\mathbb{P}^{2n-3}$ and the cohomology modules $H^1_*(F)$, $H^{n-1}_*(F)$ vanishes, then F splits completely on $\mathbb{P}^n$.

- The result above, show that the condition of stably extendability of a bundle F, has a subtle connection with the property of complete splitability and is also a good motivation for the following definition introduced (and as we shall see, characterized) by Coanda in 2009:

DEFINITION

A vector bundle F on $\mathbb{P}^n$ is infinitely stably extendable, if for any $m \geq n$ it extends stably on $\mathbb{P}^m$.

- As in the case of stably extendability, the above property is strictly weaker than infinitely extendability, as long as again, the example of the tangent bundle of $\mathbb{P}^n$ shows that there are bundles infinitely stably extendable which are not splittable and therefore (using the babylonian tower theorem of Barth-Van de Ven-Sato-Tyurin) are not infinitely extendable.

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The Coanda criterion

- The main point concerning the above property is that, like the complete splitability and therefore -via the babylonian tower theorem- like the infinitely extendability, it admits an analogous cohomological characterization in terms of some intermediate cohomology modules. This one, was obtained by Coanda in 2009:

Coanda theorem

A vector bundle F on $\mathbb{P}^n$ is infinitely stably extendable iff for any $2 \leq i \leq n-2$ the intermediate cohomology module $H^i_*(F)$ vanishes.

- On should remark that in fact, the original theorem of Coanda, contains also a third characterization of the infinitely stably extendability, namely as the condition for F of being the cohomology of a free monad.
- Also, one should note that the condition in the theorem is empty for $n \leq 3$ and so any bundle on $\mathbb{P}^{\leq 3}$ is infinitely stably extendable.

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A "stable" Terao conjecture

- Inspired by the above result we consider de following:

Definition

An arrangement $\mathcal{A}$ is stably free if its associated bundle of vector fields with logarithmic poles $\mathcal{T}_{\mathcal{A}}$ is infinitely stably extendable.

- Obviously, a free arrangement is stably free, the converse is not true and we have a similar picture as in the case of freeness:

vanishing of the intermediate cohomology of $\mathcal{T}_{\mathcal{A}}$ for $2 \leq i \leq n-2$	$\Leftrightarrow$ stably freeness $\Leftrightarrow$	infinitely stably extendability of $\mathcal{T}_{\mathcal{A}}$.

- Consequently, we introduce the following:

The stable Terao conjecture

The stably freeness of an arrangement is combinatorially determined. Namely, for two arrangements $\mathcal{A}_1, \mathcal{A}_2$ with isomorphic lattices, if $\mathcal{A}_1$ is stably free then $\mathcal{A}_2$ is also stably free.

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- A first point to note is that the two conjectures are not comparable: no one implies the other.
- Also, it could be interesting to compare this notion of stably freeness which can be obviously be extended from arrangements to arbitrary union of hyper-surfaces to other weaker notions of freeness existing in literature.
- For convenience we mention only two:
  - the nearly free and almost free divisors introduced by Dimca and Sticlaru,
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Connections with other notions of freeness

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A final question

- Another point to note is the following. Due to Horrocks and Coanda criteria, both conjectures can be expressed as the combinatorial invariance of the vanishing in a certain range of the intermediate cohomology modules of $\mathcal{T}_{\mathcal{A}}$.
- From this viewpoint, one can ask the following question which was already formulated by Yoshinaga in relation with the lattice cohomolgy introduced by Yuzvinsky:

Question

For an arrangement $\mathcal{A}$ in $\mathbb{P}^n$, and a fixed $1 \leq i \leq n-1$, can be the cohomology module $H^i_*(\mathcal{T}_{\mathcal{A}})$ expressed/computed only in terms of the lattice $L_{\mathcal{A}}$ of $\mathcal{A}$?

- Related to the above Question, one should note that using a previous remark, the first nontrivial case for the Stable Terao conjecture is on $\mathbb{P}^4$: in this case it asserts the combinatorial invariance of the vanishing of $H^2_*(\mathcal{T}_{\mathcal{A}})$.

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THANK YOU!