Horrocks theory and applications

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- I intend to review it briefly and to present some applications obtained by Coanda, Coanda-Trautmann and Malaspina-Rao.
- The last part will be devoted to the role of Horrocks ideas in a joint paper with Coanda and Manolache on globally generated vector bundles on $\mathbb{P}^n$.

Outline

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- In fact, even on lower dimensional projective spaces the situation is not much better: on $\mathbb{P}^4$ the only known example is the Horrocks-Mumford bundle and at least in characteristic 0, there are no examples on $\mathbb{P}^5$ or $\mathbb{P}^6$.

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- The Ellingsrud Peskine theorem asserts that there is a d_0 such that a smooth surface in $\mathbb{P}^4$ of degree $d \geq d_0$ must be of general type. The best known bound is 53 due to Decker-Schreyer in (2000) and the conjectural one is 16; for $d = 15$ there are surfaces of non-general type due to Aure et all in (1997).

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- In this setting we don't have yet an equivalence of categories: isomorphic modules in high degree, defines isomorphic sheaves; but using a convenient category of fractions, Serre's correspondence can be viewed as an equivalence between $Coh(\mathbb{P}^n)$ and an algebraic category.

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- Also, as a byproduct, one has the following :

Theorem (Serre)

A sheaf $\mathcal{F}$ on $\mathbb{P}^n$ is locally free if and only if:
a) $\Gamma_*(\mathcal{F})$ is finitely generated and
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What kind of information is encoded in the intermediate cohomology modules $H^i{}_*$ for $0 < i < n$?

- The answer is given by Horrocks theory!

What is Horrocks theory?

- A first result which is a good introduction to this subject is the following:

Theorem (Horrocks)

A coherent sheaf $\mathcal{F}$ on $\mathbb{P}^n$ split as a direct sum of line bundles if and only if
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- In this category, isomorphism means isomorphism up to direct sum with line bundles.

What is Horrocks theory?

- The purpose of Horrocks theory is the description of the stable category SVP in terms of complexes of S-modules. To acheive this, one introduce the full sub-category FL in the derived category $\mathcal{D}^b(S)$, consisting in complexes wich are exact except in the range $1 \leq i \leq n-1$, where the cohomology is supposed to have finite length.

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- Moreover, the cohomology modules of the image are nothing else than the intermediate cohomology modules of F itself.
- With these notations, one can conclude by:

Theorem (Horrocks-Walter)

There is an equivalence of categories $Trunc \circ \mathbb{R}\Gamma_* : SVP \to FL$.

The Coanda-Trautmann result

- Let Λ the exterior algebra in $n+1$ generators. Another description of the stable category of vector bundles on $\mathbb{P}^n$ SVP is obtained by considering certain complexes over Λ. The main tools for the construction are:

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- L is the extension to complexes of the following elementary construction on Λ-modules:
- for $N = \oplus N_p$ a module over Λ, $L(N)$ is the complex on $\mathbb{P}^n$ defined by
$$\ldots \to N_p \otimes \mathcal{O}(p) \to N_{p+1} \otimes \mathcal{O}(p+1) \to \ldots$$
where the differential is induced by the module multiplication: $N_p \otimes V \to N_{p+1}$.

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Theorem (Coanda-Trautmann) (2006)

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- As a conclusion, the original Horrocks's correspondence and the "exterior" one of Coanda-Trautmann describe the stable category of vector bundles by certain complexes over the symmetric or exterior algebra associated with $\mathbb{P}^n$.
- Also, in a paper from (2010) Coanda obtained a generalization of the above correspondence for sheaves on $\mathbb{P}^n$ satisfying certain conditions and a more general class of HT-complexes.

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- The Walter's version, which uses the derived category can be seen as if it works with the "homotopy type" of the corresponding object as a whole and not with the Postnikov decomposition.

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- If M has finite length and $1 \leq j \leq n-1$ then $G^\bullet = R(M)(-j)$ is a HT-complex and the bundles which correspond to it is an Eilenberg-Maclane bundle with M as the only non-zero intermediate cohomology module in the j place.

Horrocks correspondence on the quadric surface

- A natural idea is to see if a Horrocks-type correspondence exists on other varieties than $\mathbb{P}^n$. Such a result was obtained by Malaspina-Rao in (2013). Let $\mathbb{Q} \subset \mathbb{P}^3$ the quadric surface. In general, a vector bundle with all the intermediate cohomology modules 0 is called ACM. Let E a vector bundle on $\mathbb{Q}$ without ACM summands. Let M its first (in fact only) intermediate cohomology module.

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- Consider the two intermediate cohomology modules: $M_i = {H^1}_*(F_M \otimes \Sigma_i)$, where the Σ_i's are the two spinor bundles on the quadric.
- The bundle E determines certain graded vector spaces W_i in M_i.

Horrocks correspondance on the quadric surface

The main result is:

Theorem (Malaspina-Rao)

There is a bijection between isomorphism classes of bundles whitout ACM summands, and isomorphism classes of triples (M, W_1, W_2)

The splitting theorem and babilonean towers

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- A first result in this direction is the following theorem:

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- Also, for the babilonean towers problem, the following is true (cf. Barth-Van de Ven (1974), Tyurin (1975), Sato (1978)):

THEOREM

If E is indefinitely extendable, then it is splittable.

The splitting theorem and babilonean towers

- A natural question is to find an effective version of the babilonian tower theorem. For example, in (2006), Coanda-Trautmann obtained:

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If E extends to $\mathbb{P}^{n+m}$ with $m > \sum_{i>0} dim\, Ext^1(E, E(-i))$, then it is splittable.

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- In particular, the above result explain the difference between indefinitely extendable and indefinitely stable extendable bundles

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- -, Coanda and Manolache in arxiv:1305.3464 classified globally generated vector bundles with $c_1 \leq 4$ on $\mathbb{P}^n$.

Globally generated bundles and Horrocks theory

- The main result in the joint paper with Coanda and Manolache is a list of 16 indecomposable bundles, the last one being the Sasakura's rank 5 bundle on $\mathbb{P}^4$ once twisted.

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THANK YOU!

References

- I. Coanda, G. Trautmann: Horrocks theory and the Bernstein-Gelfand-Gelfand correspondence , Trans. Amer. Math. Soc. 358 , no. 3, 1015-1031, 2006.
- I. Coanda: The Horrocks correspondence for coherent sheaves on projective spaces , Homology, Homotopy Appl. 12 , no. 1, 327-353, 2010.
- I. Coanda: Infinitely stably extendable vector bundles on projective spaces , Arch. Math. 94 , no. 6, 539-545, 2010.
- F. Malaspina, A. P. Rao: Horrocks correspondence on a quadric surface, Geom. Dedicata. 169, 15-31, 2014.
- Ch. Walter: Pfaffian Subschemes, J. Alg. Geometry 5, 671-704, 1996.
- M. Manetti, Ch. Walter: Seminario di geometria algebrica 2009-2010, http://www1.mat.uniroma1.it/ricerca/seminari/geo-alg/dispense/Horrocks.pdf

References

- G. Horrocks: Vector bundles on the punctured spectrum of a local ring, Proc. London Math. Soc. 14, 689-713, 1964.
- G. Horrocks: Construction of bundles on $\mathbb{P}^n$, In: A. Douady and J.-L. Verdier (eds.): Les equations de YangMills, Seminaire E.N.S. (1977-1978), Asterisque 71-72, Soc. Math. de France, 197- 203, 1980.
- F. Catanese: Homological algebra and algebraic surfaces , in "Algebraic Geometry", Santa Cruz 1995, Proc. of AMS Symp. Pure Math. 62 , 1 AMS, 3-56, 1997.
- C. Anghel, I. Coanda, N. Manolache: Globally Generated Vector Bundles on $\mathbb{P}^n$ with $c_1 = 4$, arXiv:1305.3464.
- C. Anghel, N. Manolache:Globally Generated Vector Bundles on $\mathbb{P}^n$ with $c_1 = 3$, Math. Nachr., 1-17 (2013) / DOI 10.1002/mana.201200211