Heegaard-Floer homology and cuspidal multiple planes

Cristian Anghel

IMAR

Workshop for Young Researchers in Mathematics
Ovidius University, Constanta
May 20-24, 2015

Abstract

- Heegaard-Floer homology has been used recently in problems concerning cuspidal plane curves.
- In the first part of the talk we intend to review these results following mainly

  M. Borodzik, M. Hedden, C. Livingston, *Plane algebraic curves of arbitrary genus via Heegaard Floer homology*, arXiv:1409.2111

  and

  J. Bodnar, D. Celoria, M Golla, *Cuspidal curves and Heegaard Floer homology* , arXiv:1409.3282 .
- In the second part we shall present some connections with the theory of ramified coverings of the complex projective plane.
- This is joint work in progress with Cristina A-M. Anghel.

Abstract

- Heegaard-Floer homology has been used recently in problems concerning cuspidal plane curves.
- In the first part of the talk we intend to review these results following mainly
  M. Borodzik, M. Hedden, C. Livingston, *Plane algebraic curves of arbitrary genus via Heegaard Floer homology*, arXiv:1409.2111
  and
  J. Bodnar, D. Celoria, M Golla, *Cuspidal curves and Heegaard Floer homology* , arXiv:1409.3282 .
- In the second part we shall present some connections with the theory of ramified coverings of the complex projective plane.
- This is joint work in progress with Cristina A-M. Anghel.

Abstract

- Heegaard-Floer homology has been used recently in problems concerning cuspidal plane curves.
- In the first part of the talk we intend to review these results following mainly
  M. Borodzik, M. Hedden, C. Livingston, *Plane algebraic curves of arbitrary genus via Heegaard Floer homology*, arXiv:1409.2111
  and
  J. Bodnar, D. Celoria, M Golla, *Cuspidal curves and Heegaard Floer homology* , arXiv:1409.3282 .
- In the second part we shall present some connections with the theory of ramified coverings of the complex projective plane.
- This is joint work in progress with Cristina A-M. Anghel.

Abstract

- Heegaard-Floer homology has been used recently in problems concerning cuspidal plane curves.
- In the first part of the talk we intend to review these results following mainly
  M. Borodzik, M. Hedden, C. Livingston, *Plane algebraic curves of arbitrary genus via Heegaard Floer homology*, arXiv:1409.2111
  and
  J. Bodnar, D. Celoria, M Golla, *Cuspidal curves and Heegaard Floer homology* , arXiv:1409.3282 .
- In the second part we shall present some connections with the theory of ramified coverings of the complex projective plane.
- This is joint work in progress with Cristina A-M. Anghel.

Outline

1. Cuspidal curves and HF-type obstructions
   - Cuspidal curves
   - Heegaard-Floer theory
   - Obstructions for cuspidal curves and some examples

2. Cuspidal curves and ramified coverings of $\mathbb{C}P^2$
   - General coverings of surfaces
   - Cuspidal coverings of the plane
   - Examples of low degree coverings

Cuspidal curves

- The link of a plane curve singularity is a usual link in S^3.
- A cusp is a singular point whose link is a knot i.e. has one component (also called uni-branched)
- In the most part we will be concerned with cusps with only one Puiseaux pair: namely the link is a torus knot or equivalently the cusp has a local parametrization of type $t \to (t^a, t^b)$.
- In general, for a singular point, its semi-group $\Gamma \subset \mathbb{Z}$ consists of the local intersection number with complex curves. For example, for a cusp with one Puiseaux pair (a, b) the semi-group is generated by a and b.
- The gap counting function of the semi-group Γ is

$$I_m = \sharp\{\mathbb{Z} \setminus \Gamma \cap [\ m, \infty)\}$$

Cuspidal curves

- The link of a plane curve singularity is a usual link in S^3.
- A cusp is a singular point whose link is a knot i.e. has one component (also called uni-branched)
- In the most part we will be concerned with cusps with only one Puiseaux pair: namely the link is a torus knot or equivalently the cusp has a local parametrization of type $t \to (t^a, t^b)$.
- In general, for a singular point, its semi-group $\Gamma \subset \mathbb{Z}$ consists of the local intersection number with complex curves. For example, for a cusp with one Puiseaux pair (a, b) the semi-group is generated by a and b.
- The gap counting function of the semi-group Γ is

$$I_m = \sharp\{\mathbb{Z} \setminus \Gamma \cap [\ m, \infty)\}$$

Cuspidal curves

- The link of a plane curve singularity is a usual link in S^3.
- A cusp is a singular point whose link is a knot i.e. has one component (also called uni-branched)
- In the most part we will be concerned with cusps with only one Puiseaux pair: namely the link is a torus knot or equivalently the cusp has a local parametrization of type $t \to (t^a, t^b)$.
- In general, for a singular point, its semi-group $\Gamma \subset \mathbb{Z}$ consists of the local intersection number with complex curves. For example, for a cusp with one Puiseaux pair (a, b) the semi-group is generated by a and b.
- The gap counting function of the semi-group Γ is

$$I_m = \sharp\{\mathbb{Z} \setminus \Gamma \cap [\ m, \infty)\}$$

Cuspidal curves

- The link of a plane curve singularity is a usual link in S^3.
- A cusp is a singular point whose link is a knot i.e. has one component (also called uni-branched)
- In the most part we will be concerned with cusps with only one Puiseaux pair: namely the link is a torus knot or equivalently the cusp has a local parametrization of type $t \to (t^a, t^b)$.
- In general, for a singular point, its semi-group $\Gamma \subset \mathbb{Z}$ consists of the local intersection number with complex curves. For example, for a cusp with one Puiseaux pair (a, b) the semi-group is generated by a and b.
- The gap counting function of the semi-group Γ is

$$I_m = \sharp\{\mathbb{Z} \setminus \Gamma \cap [\, m, \infty)\}$$

Cuspidal curves

- The link of a plane curve singularity is a usual link in S^3.
- A cusp is a singular point whose link is a knot i.e. has one component (also called uni-branched)
- In the most part we will be concerned with cusps with only one Puiseaux pair: namely the link is a torus knot or equivalently the cusp has a local parametrization of type $t \to (t^a, t^b)$.
- In general, for a singular point, its semi-group $\Gamma \subset \mathbb{Z}$ consists of the local intersection number with complex curves. For example, for a cusp with one Puiseaux pair (a, b) the semi-group is generated by a and b.
- The gap counting function of the semi-group Γ is

$$I_m = \sharp\{\mathbb{Z} \setminus \Gamma \cap [\ m, \infty)\}$$

Topology of the complement

- For a cuspidal curve C of degree d, genus g and n cusps $p_1, ..., p_n$ we denote by δ_i the Milnor number of the cusp p_i; it is in fact the genus of K_i-the link of p_i.
- The relation between the above numbers is:

$$\sum \delta_i + g = \frac{(d-1)(d-2)}{2}$$

- An essential step toward the application of HF-theory is the understanding of the topology of the complement: let N a regular neighborhood of C, $W = \mathbb{CP}^2 \setminus Int(N)$ and $Y = W \cap N$. With these notations we have:

Lemma

$H_1(Y,\mathbb{Z}) = \mathbb{Z}_{d^2} \oplus \mathbb{Z}^{2g}$, $b_2{}^{\pm}(W) = 0$. In fact $H_2(W)$ is generated by elements with square 0.

Topology of the complement

- For a cuspidal curve C of degree d, genus g and n cusps $p_1, ..., p_n$ we denote by δ_i the Milnor number of the cusp p_i; it is in fact the genus of K_i-the link of p_i.
- The relation between the above numbers is:

$$\sum \delta_i + g = \frac{(d-1)(d-2)}{2}$$

- An essential step toward the application of HF-theory is the understanding of the topology of the complement: let N a regular neighborhood of C, $W = \mathbb{CP}^2 \setminus Int(N)$ and $Y = W \cap N$. With these notations we have:

Lemma

$H_1(Y, \mathbb{Z}) = \mathbb{Z}_{d^2} \oplus \mathbb{Z}^{2g}$, ${b_2}^{\pm}(W) = 0$. In fact $H_2(W)$ is generated by elements with square 0.

Topology of the complement

- For a cuspidal curve C of degree d, genus g and n cusps $p_1, \ldots, p_n$ we denote by δ_i the Milnor number of the cusp p_i; it is in fact the genus of K_i-the link of p_i.
- The relation between the above numbers is:

$$\sum \delta_i + g = \frac{(d-1)(d-2)}{2}$$

- An essential step toward the application of HF-theory is the understanding of the topology of the complement: let N a regular neighborhood of C, $W = \mathbb{CP}^2 \setminus Int(N)$ and $Y = W \cap N$. With these notations we have:

Lemma

$H_1(Y, \mathbb{Z}) = \mathbb{Z}_{d^2} \oplus \mathbb{Z}^{2g}$, ${b_2}^{\pm}(W) = 0$. In fact $H_2(W)$ is generated by elements with square 0.

Heegaard-Floer theory I

- HF-theory associate to every 3-manifold with a torsion *spin*^{c}-structure $t \in H^2(Y, \mathbb{Z})$ two $\mathbb{Z}_2$ vector spaces denoted by $HF^\infty(Y,t)$ and $HF^+(Y,t)$.
- Roughly speaking, one take a Heegaard diagram $(S, (\alpha_i), (\beta_i))$ and one considers the $\mathbb{Z}_2$ space generated by the intersection of the two tori $\prod \alpha_i$ and $\prod \beta_i$ in the symmetric product of S. On these spaces on introduce a differential taking into account the "holomorphic disks" between different generators. The homology produced by these differential are the above HF-homology groups.
- An important property of these two homology theories is that there is a canonical map

$$\pi : HF^\infty(Y,t) \to HF^+(Y,t)$$

- Fact: the main role of π is to produce two numbers associated to every torsion *spin*^{c}-structure:

$$d(Y,t) \text{ and } d_b(Y,t)$$

- HF-theory associate to every 3-manifold with a torsion *spin*^{c}-structure $t \in H^2(Y, \mathbb{Z})$ two $\mathbb{Z}_2$ vector spaces denoted by $HF^\infty(Y, t)$ and $HF^+(Y, t)$.
- Roughly speaking, one take a Heegaard diagram $(S, (\alpha_i), (\beta_i))$ and one considers the $\mathbb{Z}_2$ space generated by the intersection of the two tori $\prod \alpha_i$ and $\prod \beta_i$ in the symmetric product of S. On these spaces on introduce a differential taking into account the "holomorphic disks" between different generators. The homology produced by these differential are the above HF-homology groups.
- An important property of these two homology theories is that there is a canonical map

$$\pi : HF^\infty(Y, t) \to HF^+(Y, t)$$

- Fact: the main role of π is to produce two numbers associated to every torsion *spin*^{c}-structure:

$$d(Y, t) \text{ and } d_b(Y, t)$$

Heegaard-Floer theory I

- HF-theory associate to every 3-manifold with a torsion *spin*^{c}-structure $t \in H^2(Y, \mathbb{Z})$ two $\mathbb{Z}_2$ vector spaces denoted by $HF^\infty(Y, t)$ and $HF^+(Y, t)$.
- Roughly speaking, one take a Heegaard diagram $(S, (\alpha_i), (\beta_i))$ and one considers the $\mathbb{Z}_2$ space generated by the intersection of the two tori $\prod \alpha_i$ and $\prod \beta_i$ in the symmetric product of S. On these spaces on introduce a differential taking into account the "holomorphic disks" between different generators. The homology produced by these differential are the above HF-homology groups.
- An important property of these two homology theories is that there is a canonical map

$$\pi : HF^\infty(Y, t) \to HF^+(Y, t)$$

- Fact: the main role of π is to produce two numbers associated to every torsion *spin*^{c}-structure:

$$d(Y, t) \text{ and } d_b(Y, t)$$

- HF-theory associate to every 3-manifold with a torsion *spin*^{c}-structure $t \in H^2(Y, \mathbb{Z})$ two $\mathbb{Z}_2$ vector spaces denoted by $HF^\infty(Y, t)$ and $HF^+(Y, t)$.
- Roughly speaking, one take a Heegaard diagram $(S, (\alpha_i), (\beta_i))$ and one considers the $\mathbb{Z}_2$ space generated by the intersection of the two tori $\prod \alpha_i$ and $\prod \beta_i$ in the symmetric product of S. On these spaces on introduce a differential taking into account the "holomorphic disks" between different generators. The homology produced by these differential are the above HF-homology groups.
- An important property of these two homology theories is that there is a canonical map

$$\pi : HF^\infty(Y, t) \to HF^+(Y, t)$$

- Fact: the main role of π is to produce two numbers associated to every torsion *spin*^{c}-structure:

$$d(Y, t) \text{ and } d_b(Y, t)$$

- With the above notations, the following result is the basic tool coming from the HF-world to constrain the invariants of cuspidal curves:

Theorem (Ozsvath-Szabo)

Consider (W, s) a *spin*^{c} 4-manifold with boundary (Y, t) such that:
W is negative semi-definite
the restriction $H^1(W, \mathbb{Z}) \to H^1(Y, \mathbb{Z})$ is trivial
t is torsion
(Y, t) has standard HF^∞ (i.e. $\simeq \Lambda^* H^1(Y, t) \otimes \mathbb{Z}_2[\, U, U^{-1}]$)
Then the following holds:

$$c_1^2(s) + b_2^-(W) \leq 4d_b(Y, t) + 2b_1(Y)$$

Heegaard-Floer theory III

- Apart the above theorem, the following two facts are essential for constraints for cuspidal curves:
  - the numbers $d_b(Y, t)$ are effectively computable
  - if Y is the boundary of a regular neighborhood of C, they are related to the gap function of the cusps.
- From all the previous facts, the first main result of Bodnar, Celoria and Golla is the following:

Theorem (BCG)

Let C a cuspidal curve of degree d, genus g and one singular point with gap function I. Then for any $-1 \leq j \leq d-2$ and any $0 \leq k \leq g$ we have:

$$k - g \leq I_{jd+1-2k} - \frac{(d-j-2)(d-j-1)}{2} \leq k.$$

- Apart the above theorem, the following two facts are essential for constraints for cuspidal curves:
  - the numbers $d_b(Y, t)$ are effectively computable
  - if Y is the boundary of a regular neighborhood of C, they are related to the gap function of the cusps.
- From all the previous facts, the first main result of Bodnar, Celoria and Golla is the following:

THEOREM (BCG)

Let C a cuspidal curve of degree d, genus g and one singular point with gap function I. Then for any $-1 \leq j \leq d-2$ and any $0 \leq k \leq g$ we have:

$$k - g \leq I_{jd+1-2k} - \frac{(d-j-2)(d-j-1)}{2} \leq k.$$

- The main point in the above story is the Ozsvath-Szabo theorem where the topological properties of the pair (W, Y) play an important role.
- The same methods can be applied for cuspidal curves on other surfaces with similar topological properties.
- For example, Borodzik and Moe in arxiv:1410.4464 obtains similar obstructions for cuspidal curves on Hirzebruch surfaces.

Heegaard-Floer theory IV

- The main point in the above story is the Ozsvath-Szabo theorem where the topological properties of the pair (W, Y) play an important role.
- The same methods can be applied for cuspidal curves on other surfaces with similar topological properties.
- For example, Borodzik and Moe in arxiv:1410.4464 obtains similar obstructions for cuspidal curves on Hirzebruch surfaces.

- The main point in the above story is the Ozsvath-Szabo theorem where the topological properties of the pair (W, Y) play an important role.
- The same methods can be applied for cuspidal curves on other surfaces with similar topological properties.
- For example, Borodzik and Moe in arxiv:1410.4464 obtains similar obstructions for cuspidal curves on Hirzebruch surfaces.

Obstructions for cuspidal curves I

- The second main result of BCG refers to cuspidal curves with a single cusp of type (a, b):

Theorem (BCG)

For fixed genus g and sufficiently high degree d, one have:

$$a + b = 3d$$

$$\left(\frac{7b - 2a}{3}\right)^2 - 5b^2 = 8g - 4.$$

- In few words the proof goes as follows:
- the degree-genus relation in the case of one cusp of type (a, b) is

$$(d-1)(d-2) = (a-1)(b-1) + 2g$$

(the genus of the torus knot is $\frac{(a-1)(b-1)}{2}$)

Obstructions for cuspidal curves I

- The second main result of BCG refers to cuspidal curves with a single cusp of type (a, b):

Theorem (BCG)

For fixed genus g and sufficiently high degree d, one have:

$$a + b = 3d$$

$$\left(\frac{7b - 2a}{3}\right)^2 - 5b^2 = 8g - 4.$$

- In few words the proof goes as follows:
- the degree-genus relation in the case of one cusp of type (a, b) is

$$(d-1)(d-2) = (a-1)(b-1) + 2g$$

(the genus of the torus knot is $\frac{(a-1)(b-1)}{2}$)

Obstructions for cuspidal curves I

- The second main result of BCG refers to cuspidal curves with a single cusp of type (a, b):

Theorem (BCG)

For fixed genus g and sufficiently high degree d, one have:

$$a + b = 3d$$

$$\left(\frac{7b - 2a}{3}\right)^2 - 5b^2 = 8g - 4.$$

- In few words the proof goes as follows:
- the degree-genus relation in the case of one cusp of type (a, b) is

$$(d - 1)(d - 2) = (a - 1)(b - 1) + 2g$$

(the genus of the torus knot is $\frac{(a-1)(b-1)}{2}$)

Obstructions for cuspidal curves II

- A pair (a, b) is defined to be admissible if a, b are coprime, there is a degree satisfying the above relation and the gap function of the semi-group generated by a and b verifies the main theorem.
- The proof relies on the following two technical facts:
  - if $g \geq 1$ for almost all admissible pairs we have $6 < \frac{b}{a} < 7$
  - if $g \geq 1$ for almost all admissible pairs with $6 < \frac{b}{a} < 7$ we have $a + b = 3d$
- As consequence, for almost all admissible pairs we have $a + b = 3d$ and using the degree-genus relation one obtain the second identity of the theorem.

Obstructions for cuspidal curves II

- A pair (a, b) is defined to be admissible if a, b are coprime, there is a degree satisfying the above relation and the gap function of the semi-group generated by a and b verifies the main theorem.
- The proof relies on the following two technical facts:
  - if $g \geq 1$ for almost all admissible pairs we have $6 < \frac{b}{a} < 7$
  - if $g \geq 1$ for almost all admissible pairs with $6 < \frac{b}{a} < 7$ we have $a + b = 3d$
- As consequence, for almost all admissible pairs we have $a + b = 3d$ and using the degree-genus relation one obtain the second identity of the theorem.

Obstructions for cuspidal curves II

- A pair (a, b) is defined to be admissible if a, b are coprime, there is a degree satisfying the above relation and the gap function of the semi-group generated by a and b verifies the main theorem.
- The proof relies on the following two technical facts:
  - if $g \geq 1$ for almost all admissible pairs we have $6 < \frac{b}{a} < 7$
  - if $g \geq 1$ for almost all admissible pairs with $6 < \frac{b}{a} < 7$ we have $a + b = 3d$
- As consequence, for almost all admissible pairs we have $a + b = 3d$ and using the degree-genus relation one obtain the second identity of the theorem.

Examples I

- A second objective of the BCG paper is the construction of some families of cuspidal curves with a single cusp with toral link.
- For any $k \in \mathbb{Z}$ define the Lucas sequence L_n^k by the Fibonacci recurrence starting from $k-1$ and 1.
- The first existence result is:

Theorem

Let $k \geq 2$ and $g = \frac{k(k-1)}{2}$. Then:
1. for any $i \geq 2$ there is a cuspidal curve with one toral cusp of degree $d = L_{4i-1}^k$ and Puiseaux pair (L_{4i-3}^k, L_{4i+1}^k).
2. for any $j \geq 1$ there is a cuspidal curve with one toral cusp of degree $d = -L^k_{-4j-1}$ and Puiseaux pair $(-L^k_{-4j+1}, -L^k_{-4j-3})$.
3. if $3 \mid g$ and $2g-1$ is prime, then for $d >> 0$ any such curve is of one of the the above types.

- Remark: the examples from the theorem all verifies $a + b = 3d$.

Examples I

- A second objective of the BCG paper is the construction of some families of cuspidal curves with a single cusp with toral link.
- For any $k \in \mathbb{Z}$ define the Lucas sequence L_n^k by the Fibonacci recurrence starting from $k-1$ and 1.
- The first existence result is:

Theorem

Let $k \geq 2$ and $g = \frac{k(k-1)}{2}$. Then:
1. for any $i \geq 2$ there is a cuspidal curve with one toral cusp of degree $d = L_{4i-1}^k$ and Puiseaux pair (L_{4i-3}^k, L_{4i+1}^k).
2. for any $j \geq 1$ there is a cuspidal curve with one toral cusp of degree $d = -L^k_{-4j-1}$ and Puiseaux pair $(-L^k_{-4j+1}, -L^k_{-4j-3})$.
3. if $3 \mid g$ and $2g-1$ is prime, then for $d >> 0$ any such curve is of one of the the above types.

- Remark: the examples from the theorem all verifies $a + b = 3d$.

Examples I

- A second objective of the BCG paper is the construction of some families of cuspidal curves with a single cusp with toral link.
- For any $k \in \mathbb{Z}$ define the Lucas sequence L_n^k by the Fibonacci recurrence starting from $k-1$ and 1.
- The first existence result is:

Theorem

Let $k \geq 2$ and $g = \frac{k(k-1)}{2}$. Then:
1. for any $i \geq 2$ there is a cuspidal curve with one toral cusp of degree $d = L_{4i-1}^k$ and Puiseaux pair (L_{4i-3}^k, L_{4i+1}^k).
2. for any $j \geq 1$ there is a cuspidal curve with one toral cusp of degree $d = -L^k{}_{-4j-1}$ and Puiseaux pair $(-L^k{}_{-4j+1}, -L^k{}_{-4j-3})$.
3. if $3 \mid g$ and $2g-1$ is prime, then for $d >> 0$ any such curve is of one of the the above types.

- Remark: the examples from the theorem all verifies $a + b = 3d$.

Examples I

- A second objective of the BCG paper is the construction of some families of cuspidal curves with a single cusp with toral link.
- For any $k \in \mathbb{Z}$ define the Lucas sequence L_n^k by the Fibonacci recurrence starting from $k-1$ and 1.
- The first existence result is:

Theorem

Let $k \geq 2$ and $g = \frac{k(k-1)}{2}$. Then:
1. for any $i \geq 2$ there is a cuspidal curve with one toral cusp of degree $d = L_{4i-1}^k$ and Puiseaux pair (L_{4i-3}^k, L_{4i+1}^k).
2. for any $j \geq 1$ there is a cuspidal curve with one toral cusp of degree $d = -L^k_{-4j-1}$ and Puiseaux pair $(-L^k_{-4j+1}, -L^k_{-4j-3})$.
3. if $3 \mid g$ and $2g-1$ is prime, then for $d >> 0$ any such curve is of one of the the above types.

- Remark: the examples from the theorem all verifies $a + b = 3d$.

Examples II

- The following two examples produces curves for which $a + b \neq 3d$:
- Let p non-divisible by 3 and $C_p : x^{p+3} + y^{p+3} + x^p z^3 = 0$ then $d = p + 3$, $g = p + 2$ and the cusp is of toral type $(p, p + 3)$
- Let $p \geq 2$ and $D_p : x^p z^{p-1} + x^{2p-1} + y^{2p-1} = 0$ then $d = 2p - 1$, $g = (p - 1)(p - 2)$ and the cusp is of toral type $(p, 2p - 1)$

Examples II

- The following two examples produces curves for which $a + b \neq 3d$:
- Let p non-divisible by 3 and $C_p : x^{p+3} + y^{p+3} + x^p z^3 = 0$ then $d = p + 3$, $g = p + 2$ and the cusp is of toral type $(p, p + 3)$
- Let $p \geq 2$ and $D_p : x^p z^{p-1} + x^{2p-1} + y^{2p-1} = 0$ then $d = 2p - 1$, $g = (p - 1)(p - 2)$ and the cusp is of toral type $(p, 2p - 1)$

Examples II

- The following two examples produces curves for which $a + b \neq 3d$:
- Let p non-divisible by 3 and $C_p : x^{p+3} + y^{p+3} + x^p z^3 = 0$
  then $d = p + 3$, $g = p + 2$ and the cusp is of toral type $(p, p + 3)$
- Let $p \geq 2$ and $D_p : x^p z^{p-1} + x^{2p-1} + y^{2p-1} = 0$
  then $d = 2p - 1$, $g = (p - 1)(p - 2)$ and the cusp is of toral type $(p, 2p - 1)$

General coverings of surfaces I

The motivation for the sequel is to construct (new/with computable invariants) surfaces starting from singular plane curves.

- The main result concerning general coverings is the following:

Theorem (Grauert-Remmert)

For X normal variety and D a divisor with complement U, an etale cover of U extends uniquely as an normal analytic cover Y of X.

- An important fact is that even if X is smooth, Y will acquire singularities corresponding with the singularities of D.
- A particular class of coverings are constructed from a n-divisible line bundle $L^{\otimes n}$ on X and an holomorphic section with divisor D. The resulting variety Y is the cyclic covering associated with the pair (L, D).

General coverings of surfaces I

The motivation for the sequel is to construct (new/with computable invariants) surfaces starting from singular plane curves.

- The main result concerning general coverings is the following:

Theorem (Grauert-Remmert)

For X normal variety and D a divisor with complement U, an etale cover of U extends uniquely as an normal analytic cover Y of X.

- An important fact is that even if X is smooth, Y will acquire singularities corresponding with the singularities of D.
- A particular class of coverings are constructed from a n-divisible line bundle $L^{\otimes n}$ on X and an holomorphic section with divisor D. The resulting variety Y is the cyclic covering associated with the pair (L, D).

General coverings of surfaces I

The motivation for the sequel is to construct (new/with computable invariants) surfaces starting from singular plane curves.

- The main result concerning general coverings is the following:

Theorem (Grauert-Remmert)

For X normal variety and D a divisor with complement U, an etale cover of U extends uniquely as an normal analytic cover Y of X.

- An important fact is that even if X is smooth, Y will acquire singularities corresponding with the singularities of D.
- A particular class of coverings are constructed from a n-divisible line bundle $L^{\otimes n}$ on X and an holomorphic section with divisor D. The resulting variety Y is the cyclic covering associated with the pair (L, D).

General coverings of surfaces II

Example: double coverings of surfaces

For $n = 2$ and D with only simple singularities (eg. suppose it has only nodes or simple cusps) denote by $\bar{Y}$ the canonical resolution of the double cover $Y \to X$. Then the invariants of X and $\bar{Y}$ are connected by the following relations:

- $\chi(\bar{Y}) = 2\chi(X) + \frac{1}{2}L \cdot K_X + \frac{1}{2}L^2$
- $p_g(\bar{Y}) = p_g(X) + h^0(X, K_X \otimes L)$
- ${c_1}^2(\bar{Y}) = 2{c^2}_1(X) + 4L \cdot K_X + 2L^2$
- $c_2(\bar{Y}) = c_2(X) + 2L \cdot K_X + 4L^2$
- Concerning the Kodaira dimension, for example in the case of $X = \mathbb{CP}^2$ and D of degree $2m$ one have:

$$kod(\bar{Y}) = -1 \ for \ m = 1, 2 \ kod(\bar{Y}) = 0 \ for \ m = 3$$

$$kod(\bar{Y}) = 2 \ for \ m \geq 4$$

General coverings of surfaces II

Example: double coverings of surfaces

For $n = 2$ and D with only simple singularities (eg. suppose it has only nodes or simple cusps) denote by $\bar{Y}$ the canonical resolution of the double cover $Y \to X$. Then the invariants of X and $\bar{Y}$ are connected by the following relations:

- $\chi(\bar{Y}) = 2\chi(X) + \frac{1}{2} L \cdot K_X + \frac{1}{2} L^2$
- $p_g(\bar{Y}) = p_g(X) + h^0(X, K_X \otimes L)$
- ${c_1}^2(\bar{Y}) = 2{c^2}_1(X) + 4L \cdot K_X + 2L^2$
- $c_2(\bar{Y}) = c_2(X) + 2L \cdot K_X + 4L^2$
- Concerning the Kodaira dimension, for example in the case of $X = \mathbb{CP}^2$ and D of degree $2m$ one have:

$$kod(\bar{Y}) = -1 \ for \ m = 1, 2 \ kod(\bar{Y}) = 0 \ for \ m = 3$$

$$kod(\bar{Y}) = 2 \ for \ m \geq 4$$

Example: double coverings of surfaces

For $n = 2$ and D with only simple singularities (eg. suppose it has only nodes or simple cusps) denote by $\bar{Y}$ the canonical resolution of the double cover $Y \to X$. Then the invariants of X and $\bar{Y}$ are connected by the following relations:

- $\chi(\bar{Y}) = 2\chi(X) + \frac{1}{2} L \cdot K_X + \frac{1}{2} L^2$
- $p_g(\bar{Y}) = p_g(X) + h^0(X, K_X \otimes L)$
- ${c_1}^2(\bar{Y}) = 2{c^2}_1(X) + 4L \cdot K_X + 2L^2$
- $c_2(\bar{Y}) = c_2(X) + 2L \cdot K_X + 4L^2$
- Concerning the Kodaira dimension, for example in the case of $X = \mathbb{CP}^2$ and D of degree $2m$ one have:

$$kod(\bar{Y}) = -1 \ for \ m = 1, 2 \ kod(\bar{Y}) = 0 \ for \ m = 3$$

$$kod(\bar{Y}) = 2 \ for \ m \geq 4$$

General coverings of surfaces II

Example: double coverings of surfaces

For $n = 2$ and D with only simple singularities (eg. suppose it has only nodes or simple cusps) denote by $\bar{Y}$ the canonical resolution of the double cover $Y \to X$. Then the invariants of X and $\bar{Y}$ are connected by the following relations:

- $\chi(\bar{Y}) = 2\chi(X) + \frac{1}{2}L \cdot K_X + \frac{1}{2}L^2$
- $p_g(\bar{Y}) = p_g(X) + h^0(X, K_X \otimes L)$
- ${c_1}^2(\bar{Y}) = 2{c^2}_1(X) + 4L \cdot K_X + 2L^2$
- $c_2(\bar{Y}) = c_2(X) + 2L \cdot K_X + 4L^2$
- Concerning the Kodaira dimension, for example in the case of $X = \mathbb{CP}^2$ and D of degree $2m$ one have:

$$kod(\bar{Y}) = -1 \ for \ m = 1, 2 \quad kod(\bar{Y}) = 0 \ for \ m = 3$$

$$kod(\bar{Y}) = 2 \ for \ m \geq 4$$

General coverings of surfaces II

Example: double coverings of surfaces

For $n = 2$ and D with only simple singularities (eg. suppose it has only nodes or simple cusps) denote by $\bar{Y}$ the canonical resolution of the double cover $Y \to X$. Then the invariants of X and $\bar{Y}$ are connected by the following relations:

- $\chi(\bar{Y}) = 2\chi(X) + \frac{1}{2}L \cdot K_X + \frac{1}{2}L^2$
- $p_g(\bar{Y}) = p_g(X) + h^0(X, K_X \otimes L)$
- ${c_1}^2(\bar{Y}) = 2{c^2}_1(X) + 4L \cdot K_X + 2L^2$
- $c_2(\bar{Y}) = c_2(X) + 2L \cdot K_X + 4L^2$
- Concerning the Kodaira dimension, for example in the case of $X = \mathbb{CP}^2$ and D of degree $2m$ one have:

$$kod(\bar{Y}) = -1 \ for \ m = 1, 2 \ kod(\bar{Y}) = 0 \ for \ m = 3$$

$$kod(\bar{Y}) = 2 \ for \ m \geq 4$$

Cuspidal coverings of the plane I

- The study of ramified covers of the plane with cuspidal ramification divisor is a natural question viewing the following (classical) result published in 2008 by Ciliberto and Flamini:

Theorem

The branch curve of a general projection of a surface S on $\mathbb{CP}^2$ is irreducible with only nodes and cusps.

- Also, a natural question is to find the configuration/position of the singularities of a given curve such that it is the branch curve of a generic projection. For example, a classical result in this direction is the following:

Theorem: (Segre)

A plane curve D is the branch curve for a generic projection of a surface in $\mathbb{CP}^3$ iff:

Cuspidal coverings of the plane I

- The study of ramified covers of the plane with cuspidal ramification divisor is a natural question viewing the following (classical) result published in 2008 by Ciliberto and Flamini:

Theorem

The branch curve of a general projection of a surface S on $\mathbb{CP}^2$ is irreducible with only nodes and cusps.

- Also, a natural question is to find the configuration/position of the singularities of a given curve such that it is the branch curve of a generic projection. For example, a classical result in this direction is the following:

Theorem: (Segre)

A plane curve D is the branch curve for a generic projection of a surface in $\mathbb{CP}^3$ iff:

Cuspidal coverings of the plane II

Theorem: (Segre)

1. $deg(D) = n(n-1)$
2. D has $\frac{1}{2}n(n-1)(n-2)(n-3)$ nodes and $n(n-1)(n-2)$ cusps
3. there are curves L_1 and L_2 of degrees $(n-1)(n-2)$ and $(n-1)(n-2)+1$ which contains the 0-cycle of singularities of D and have separate tangents at these points.

The above theorem was extended in 2012 by Friedman, Lehman, Leyenson and Teicher for surfaces in arbitrary projective spaces.

Remark

Not every covering of $\mathbb{CP}^2$ is the restriction of a linear projection !

Examples of low degree coverings I

Apart the degree 2 case where a cover is determined by an even line bundle and a section, a systematic study for low degree coverings was done by Miranda in 1985. His main result can be resumed as:

Theorem (Miranda)

A triple cover $Y \to X$ is determined by a rank two vector bundle E on X and an element

$$\Phi \in Hom(S^3(E) \to \Lambda^2(E))$$

and conversely.

In this correspondence the branch locus in X is a divisor with associated line bundle $(\Lambda^2 E)^{-2}$.

For triple covers of surfaces $Y \to X$ with cuspidal ramification one has the following restriction:

$$\sharp \ of \ cusps \ = 3c_2(E),$$

and the genus of the branch locus is $2{c_1}^2 - c_1 \cdot K_X + 1 - 3c_2(E)$.

Examples of low degree coverings II

In particular, Y has the following invariants:

$$h^i(\mathcal{O}_Y) = h^i(\mathcal{O}_X) + h^i(E)$$

$$\chi(\mathcal{O}_Y) = \chi(\mathcal{O}_X) + \chi(E)$$

$$K^2{}_Y = 3K^2{}_X - 4c_1 \cdot K_X + 2c^2{}_1 - 3c_2$$

$$\chi(Y) = 3\chi(X) - 2c_1 \cdot K_X + 4c^2{}_1 - 9c_2$$

and the above formulas compute the invariants of triple covers of the plane for some explicit bundles (eg. split or twisted tangent bundle). Also, Miranda used the dictionary above to construct a sequence of successive triple covers of general type such that $\frac{{c_1}^2}{c_2}$ tends to 3 which is the Miyaoka upper bound.

THANK YOU!