Geometric convexity properties of coverings of 1-convex surfaces *

Mihnea Colţoiu, Cezar Joiţa

Abstract

We prove that a complex surface that contains an infinite Nori string of rational curves is not p_5-convex and that a covering of a 1-convex complex surface which does not contain an infinite Nori string of rational curves is p_5-convex.

1 Introduction

Let X be a 1-convex complex surface whose exceptional set is the compact complex curve A. In this paper we are interested in studying the geometric convexity properties of unramified coverings $p : \tilde{X} \to X$. In general $\tilde{X}$ is not holomorphically convex and not even weakly pseudoconvex (i.e. it does not carry a plurisubharmonic continuous exhaustion function). In [2] it was proved that $\tilde{X}$ is p_3-convex in the sense of [7], i.e. it can be written as an increasing union of relatively compact strongly pseudoconvex domains.

In this paper we study the p_5-convexity of $\tilde{X}$ in the sense of [7] (see Definition 3 below). Our main result (see Theorem 6) asserts that $\tilde{X}$ is p_5-convex if and only if $\tilde{A} := p^{-1}(A)$ does not contain an infinite Nori string of rational curves.

For arbitrary surfaces (not necessarily coverings of 1-convex surfaces) we are able to show (Theorem 5) that they are not p_5-convex if they contain an infinite Nori string of rational curves (not necessarily exceptional).

We also give an example of a covering $\tilde{X}$ of a 1-convex surface such that $\tilde{X}$ is p_5-convex and p_3-convex but its cohomology group $H^1(\tilde{X}, \mathcal{O}_{\tilde{X}})$ is not separated. In our construction $\tilde{X}$ contains an infinite Nori string of irrational curves.

2 Preliminaries

Definitions 1 and 3 were given in [7].

*Mathematics Subject Classification (2000): 32E05, 32F10, 32C35, 32F32

Key words: 1-convex surface, covering space, holomorphically convex space, Stein space, proper modification

Definition 1. A complex manifold is called p_3-convex if it has an exhaustion with relatively compact strictly pseudoconvex domains.

The following theorem was proved in [2].

Theorem 1. *Suppose that X is a 1-convex manifold and that the exceptional set of X has dimension 1. Then any covering of X is p_3-convex.*

Definition 2. We denote by Δ the unit disk in $\mathbb{C}$, $\Delta = \{z \in \mathbb{C} : |z| < 1\}$. A holomorphic disk in a complex space X is a function $f : \overline{\Delta} \to X$ which is holomorphic on Δ and continuous on $\overline{\Delta}$.

Definition 3. We say that a complex space X is p_5-convex (or that it satisfies the Kontinuitätssatz) if for every sequence of holomorphic disks $\{f_n\}_{n\in\mathbb{N}}$, $f_n : \overline{\Delta} \to X$, if $\bigcup_{n\in\mathbb{N}} f_n(\partial\Delta)$ is relatively compact in X then $\bigcup_{n\in\mathbb{N}} f_n(\overline{\Delta})$ is relatively compact in X.

Definition 4. An infinite Nori string is a connected 1-dimensional complex space which is not compact but all its irreducible components are compact.

In [5] we proved the following.

Theorem 2. *There exists a 1-convex complex surface whose universal covering is not p_5-convex.*

On the other hand in [4] we proved that if X is a 1-convex surface, $p : \tilde{X} \to X$ is a covering and $\tilde{X}$ does not contain an infinite Nori string of rational curves then $\tilde{X}$ satisfies a property which is weaker than p_5-convexity. More precisely we were considering a sequence of holomorphic functions $f_n : U \to \tilde{X}$ defined on the same neighborhood U of $\overline{\Delta}$, we assumed that $\bigcup_{n\geq 1} f_n(U \setminus \Delta)$ is relatively compact in $\tilde{X}$ and that $f_{n|S^1}$ converges uniformly to a continuous function $\gamma : S^1 = \{z \in \mathbb{C} : |z| = 1\} \to \tilde{X}$ and we proved that $\bigcup_{n\geq 1} f_n(\overline{\Delta})$ is relatively compact in $\tilde{X}$. For the study of these two notions of convexity, see [10].

The following theorem was proved in [3].

Theorem 3. *Suppose that X and T are complex spaces and $\pi : X \to T$ is a holomorphic map. Let $t_0 \in T$ and $X_{t_0} := \pi^{-1}(t_0)$. We assume that π is proper and surjective and that $\dim X_{t_0} = 1$. Let $\sigma : \tilde{X} \to X$ be a covering space and let $\tilde{X}_{t_0} = \sigma^{-1}(X_{t_0})$. If $\tilde{X}_{t_0}$ is holomorphically convex, then there exists an open neighbourhood Ω of t_0 such that $(\pi \circ \sigma)^{-1}(\Omega)$ is holomorphically convex.*

The next result was proved in [2].

Proposition 1. *Let X be a 1-convex manifold with exceptional set S and $p : \tilde{X} \to X$ any covering. Then there exists a strongly plurisubharmonic function $\tilde{\phi} : \tilde{X} \to [-\infty, \infty)$ such that $p^{-1}(S) = \{\tilde{\phi} = -\infty\}$ and for any open neighbourhood U of S, the restriction $\tilde{\phi}_{|\tilde{X}\setminus p^{-1}(U)}$ is an exhaustion function on $\tilde{X} \setminus p^{-1}(U)$.*

Definition 5. Suppose that X is a complex surface, $A \subset X$ is a 1-dimensional compact complex subspace, and $A = \bigcup_{j=1}^{k} L_j$ is its decomposition into irreducible components.
a) We say that A is a chain of $\mathbb{P}^1$ if each L_j is isomorphic to $\mathbb{P}^1$, for each $j \in \{1, \cdots, k-1\}$, L_j and L_{j+1} intersect transversely in precisely one point, and $L_i \cap L_j = \emptyset$ for $|i - j| \geq 2$.
b) We say that A is a cycle of $\mathbb{P}^1$ if each L_j is isomorphic to $\mathbb{P}^1$, for each $j \in \{1, \cdots, k-1\}$, L_j and L_{j+1} intersect transversely in precisely one point, L_k and L_1 intersect transversely in precisely one point, and $L_i \cap L_j = \emptyset$ for all other pairs (i, j), $i \neq j$.

For the next result, see [11].

Theorem 4. *Suppose that X and X' are complex surfaces, $A \subset X$ and $A' \subset X'$ are 1-dimensional compact subspaces. Then in either one the following two situations:*

a) A and A' are chains of $\mathbb{P}^1$ of the same length and $(L_j \cdot L_j) = (L'_j \cdot L'_j) \leq -2$ for $j = \overline{1, k}$

b) A and A' are cycles of $\mathbb{P}^1$ of the same length, $(L_j \cdot L_j) = (L'_j \cdot L'_j) \leq -2$ for $j = \overline{1, k}$ and there exists j_0 such that $(L_{j_0} \cdot L_{j_0}) \leq -3$

there exists $U \subset X$ and $U' \subset X'$ biholomorphic neighbourhoods of A and respectively A'.

3 The Results

Theorem 5. *Suppose that X is a smooth complex surface. If X contains an infinite Nori string of rational curves, then X is not p_5-convex.*

Proof. After a locally finite sequence of blow-ups we obtain a complex surface X_1 and a proper surjective morphism $X_1 \to X$such that X_1 contains an infinite Nori string of rational curves as well and, moreover, this Nori string satisfies the following properties:

- all its irreducible components are smooth,
- any two irreducible components intersect in at most one point,
- any two irreducible components intersect transversely.

If we prove that X_1 is not p_5-convex, since the map $X_1 \to X$ is proper, we deduce that X is not p_5-convex as well. Hence we can assume from the beginning that X contains an infinite Nori string of rational curves that satisfies the three properties listed above. It follows then that there exists a sequence $\{F_n\}_{n\geq 0}$ of smooth closed complex curves in X such that each F_j is isomorphic to $\mathbb{P}^1$, F_j and F_{j+1} intersect in precisely one point and the intersection is transversal, $F_j \cap F_k = \emptyset$ if $|j - k| \geq 2$.

Let $K \subset X$ be a compact subset such that $F_0 \subset \overset{\circ}{K}$.

We will prove that there exists a sequence of holomorphic disks $\{g_n\}$, $g_n : \overline{\Delta} \to X$, such that

1. $g_n(\partial\Delta) \subset K$

2. $g_n(\overline{\Delta}) \cap F_n \neq \emptyset$

The second property will guarantee that $\bigcup g_n(\overline{\Delta})$ is not relatively compact in X.

We fix $n \geq 1$.

Let $d = \max\{|F_j \cdot F_j| : j = 0, \ldots, n\} + 2$ where $F_j \cdot F_j$ denotes the self-intersection of F_j. By blowing-up $d + F_j \cdot F_j$ points on each F_j we obtain a surface Y together with a proper map $h : Y \to X$. If $\hat{F}_j \subset Y$, $j = 0, \ldots, n$ are the proper transforms of F_j, then $\hat{F}_j \cdot \hat{F}_j = -d$. If we manage to find $\hat{g}_n : \overline{\Delta} \to Y$ such that $\hat{g}_n(\partial\Delta) \subset h^{-1}(K)$ and $\hat{g}_n(\overline{\Delta}) \cap \hat{F}_n \neq \emptyset$ then $g_n = h \circ \hat{g}_n$ will be the holomorphic disk in X that we are looking for. All these show that we can assume from the beginning that $F_j \cdot F_j = -d$ for $= 0, \ldots, n$ with $d \in \mathbb{N}$, $d \geq 3$.

Now we make a construction that was used in [5]. The main point about this construction is that it allows us to define holomorphic disks in an *explicit* manner.

We consider $\mathbb{C}^2$ with coordinate functions (z_1, z_2). We let $\Omega_0 = \mathbb{C}^2$, the coordinate functions $(z_1^{(0)}, z_2^{(0)}) = (z_1, z_2)$ and $a_0 = (0, 0)$. We consider Ω_1 to be the blow-up of Ω_0 in a_0. Hence $\Omega_1 = \{(z_1^{(0)}, z_2^{(0)}, [\xi_1^{(0)} : \xi_2^{(0)}]) \in \Omega_0 \times \mathbb{P}^1 : z_1^{(0)}\xi_2^{(0)} = z_2^{(0)}\xi_1^{(0)}\}$ and $a_1 = (0, 0, [0 : 1]) \in \Omega_1$. We let Ω_2 to be the blow up of Ω_1 in a_1 and L_0 to be the proper transform of the exceptional set of Ω_1. The subset of Ω_1 given by $\xi_2^{(0)} \neq 0$ is biholomorphic to $\mathbb{C}^2$ and the coordinate functions are $z_1^{(1)} := \frac{\xi_1^{(0)}}{\xi_2^{(0)}}$ and $z_2^{(1)} := z_2^{(0)}$. Moreover, in these coordinates a_1 is defined by $z_1^{(1)} = 0$, $z_2^{(1)} = 0$. We repeat this blowing-up process until we obtain a complex surface Ω_{n+2} and $n+1$ smooth rational curves $L_0, \ldots L_n$, each one of them having self-intersection (-2).

The description of each L_k is the following: we start with $\mathbb{C}^2$ with coordinate functions $(z_1^{(k)}, z_2^{(k)})$ we blow it up at the origin and then we blow it up again at the point $(0, 0, [0 : 1])$. We obtain a surface M and L_k is the proper transform of the exceptional divisor of the first blow-up.

If in $\mathbb{C}^2 \times \mathbb{P}^1 \times \mathbb{P}^1$ we write the coordinate functions as $(z_1^{(k)}, z_2^{(k)}, [\xi_1^{(k)} : \xi_2^{(k)}], [\xi_1^{(k+1)} : \xi_2^{(k+1)}])$ then M is given by

$$z_1^{(k)}\xi_2^{(k)} = z_2^{(k)}\xi_1^{(k)}, \quad \xi_1^{(k)}\xi_2^{(k+1)} = \xi_1^{(k+1)}\xi_2^{(k)}z_2^{(k)}$$

and L_k is given by the equations $z_1^{(k)} = 0$, $\xi_2^{(k+1)} = 0$. This means that $L_k = \{(0, 0, [\xi_1^{(k)} : \xi_2^{(k)}], [1 : 0]) :$ where $[\xi_1^{(k)} : \xi_2^{(k)}] \in \mathbb{P}^1\}$. We will blow-up $p = d - 2$ points on each L_k.

We fix now $b_1, \cdots, b_p$ distinct complex numbers with $|b_j| = 1$ $\forall j \in \{1, \ldots, p\}$.

For each $k \in \{0, \ldots, n\}$ and each $j \in \{1, \ldots, p\}$ we consider the point b_j^k of L_k which in the above description is given by $b_j^k = (0, 0, [1 : b_j], [1 : 0]) \in L_k$, we blow-up Ω_{n+2} at all these points and we obtain $\widetilde{\Omega}_{n+2}$. We let $\tilde{L}_k$ to be the proper transform of L_k. Therefore: each $\tilde{L}_k$ is isomorphic to $\mathbb{P}^1$, $\tilde{L}_k \cdot \tilde{L}_k = -p + 2 = -d$, $\tilde{L}_k$ and $\tilde{L}_{k+1}$ intersect in precisely one point and the intersection is transversal, $\tilde{L}_j \cap \tilde{L}_k = \emptyset$ if $|j - k| \geq 2$.

It follows from Theorem 4, that a neighbourhood of $F_0 \bigcup \cdots \bigcup F_n$ in X is biholomorphic to a neighbourhood of $\tilde{L}_0 \bigcup \cdots \bigcup \tilde{L}_n$ in $\widetilde{\Omega}_{n+2}$. Therefore it suffices to prove the following statement.

Statement: For each neighbourhood W of $\tilde{L}_0 \bigcup \cdots \bigcup \tilde{L}_n$ in $\widetilde{\Omega}_{n+2}$ and for each compact set $K \subset \widetilde{\Omega}_{n+2}$ such that $\tilde{L}_0 \subset \overset{\circ}{K}$ there exist a holomorphic map $g_n : \mathbb{C} \to W$, such that

1. $g_n(\overline{\Delta}) \subset W$
2. $g_n(\partial\Delta) \subset K$,
3. $g_n(\overline{\Delta}) \cap \tilde{L}_n \neq \emptyset$.

We fix W and K. The holomorphic map g_n will be defined as follows: we construct two polynomial functions with convenient properties $f_1 = f_1^{(n)}$ and $f_2 = f_2^{(n)}$ and we will let $g_n : \mathbb{C} \to \widetilde{\Omega}_{n+2}$ to be the proper transform of $(f_1, f_2) : \mathbb{C} \to \Omega_0$ after all the blow-ups we made. We will denote by $\hat{g}_n : \mathbb{C} \to \Omega_{n+2}$ the proper transform of (f_1, f_2) after the first $(n+2)$ blow-ups.

We will construct in fact f_1 and f_2 such that $g_n(\overline{\Delta_2}) \subset W$ where $\Delta_2 = \{z \in \mathbb{C} : |z| < 2\}$.

We have to describe a fundamental system of neighbourhoods for $\tilde{L}_0 \bigcup \cdots \bigcup \tilde{L}_n$.

First we notice that a fundamental system of neighbourhoods for L_k, in the coordinates introduced above is the following: $U_r^{(k)} = \{|\xi_2^{(k+1)}| < r|\xi_1^{(k+1)}|,\ |z_1^{(k)}| < r\}$, $r > 0$. Then $(z_1^{(k)}, \xi_2^{(k)} - b_j)$ are local coordinates around b_j^k and in these coordinates b_j^k is the origin and L_k is given by $z_1^{(k)} = 0$. When we blow-up Ω_{n+2} at b_j^k, locally we obtain $\{(z_1^{(k)}, \xi_2^{(k)} - b_j, [w_1 : w_2]) : w_1(\xi_2^{(k)} - b_j) = w_2 z_1^{(k)}\}$ and the proper transform of L_k is given by $w_1 = 0$. It follows that a fundamental system of neighbourhoods for $\tilde{L}_k$ is given by $\{|w_1| < \rho|w_2|\}$ for $\rho > 0$ which outside $\tilde{L}_k$ is given by

$$\{(z_1^{(k)}, z_2^{(k)}, [\xi_1^{(k)} : \xi_2^{(k)}], [\xi_1^{(k+1)} : \xi_2^{(k+1)}]) \in U_r^{(k)} : |z_1^{(k)}| \cdot |\xi_1^{(k)}| < \rho|\xi_2^{(k)} - b_j \xi_1^{(k)}|,\ \forall j = \overline{1,p}\}.$$

We obtain in this way a fundamental system of neighbourhoods $V_{r,\rho}^{(k)}$, $r > 0, \rho > 0$, for each $\tilde{L}_k$ and hence $\bigcup_{k=0}^n V_{r,\rho}^{(k)}$ is a fundamental system of neighbourhoods for $\tilde{L}_0 \bigcup \cdots \bigcup \tilde{L}_n$. We choose $1 > r > 0, 1 > \rho > 0$ such that $\bigcup_{k=0}^n V_{r,\rho}^{(k)} \subset W$. Moreover we choose them such that

$$r < \frac{\rho}{2}(1 - r).$$

In particular $\frac{\rho}{2} > r$.

If we are working outside $\tilde{L}_0 \bigcup \cdots \bigcup \tilde{L}_n$ and we express $z_1^{(k)}$, $z_2^{(k)}$, $\xi_1^{(k)}$, $\xi_2^{(k)}$ in terms of z_1 and z_2 we obtain:

$$\begin{cases} z_1^{(k)} = \frac{z_1}{z_2^k} \\ z_2^{(k)} = z_2 \\ \frac{\xi_1^{(k)}}{\xi_2^{(k)}} = \frac{z_1^{(k)}}{z_2^{(k)}} = \frac{z_1}{z_2^{k+1}} \end{cases}$$

Hence $U_r^{(k)} \setminus L_0 \cup \cdots \cup L_n$ is given by

$$\{|z_2|^{k+2} < r|z_1|,\ |z_1| < r|z_2|^k\}$$

and $V_{r,\rho}^{(k)} \setminus \tilde{L}_0 \cup \cdots \cup \tilde{L}_n$ is given by

$$\left\{|z_2|^{k+2} < r|z_1|,\ |z_1| < r|z_2|^k,\ |z_1|^2 < \rho|z_2|^k \cdot |z_2^{k+1} - b_j z_1|,\ \forall j = \overline{1,p}\right\}.$$

Note also that if we set $\mathcal{Z} := \{\lambda \in \mathbb{C} : f_1(\lambda) = f_2(\lambda) = 0\}$ then $g_n(\mathcal{Z}) \subset \tilde{L}_0 \bigcup \cdots \bigcup \tilde{L}_n$.

Remark 1.
a) $(U_r^{(k)} \cap U_r^{(k+1)}) \setminus L_0 \cup \cdots \cup L_n$ is given by $\{|z_2|^{k+2} < r|z_1|,\ |z_1| < r|z_2|^{k+1}\}$ and $U_r^{(k)} \cap U_r^{(j)} = \emptyset$ for $|j-k| \geq 2$.
b) $\{|z_1| > r|z_2|\} \cap \left(\bigcup_{k\geq 1} U_r^{(k)}\right) = \emptyset$.

The construction of f_1 and f_2.

• Let $c_1 = 1$. For $k = 1, \ldots, n-1$ we define inductively

$$c_{k+1} = 2k + 1 + p[kc_1 + (k-1)c_2 + \cdots + c_k].$$

• We set $d_n = p$ and we define inductively downward

$$d_k = p(d_{k+1} + 2d_{k+2} + \cdots + (n-k)d_n + n - k + 1).$$

• Let $N = 2(n+1)(d_1 + d_2 + \cdots + d_n + 1)$ and let ε be a positive number such that

$$\varepsilon < \left(\frac{1}{6}\right)^N \frac{r}{n+3}. \qquad (*)$$

• We define the following polynomials:

$$P_{n,b_j}(\lambda) = \varepsilon^{c_n} - b_j\lambda,$$

$$P_n(\lambda) = \prod_{j=1}^{p} P_{n,b_j}(\lambda)$$

and, inductively downward for $k \leq n-1$,

$$P_{k,b_j}(\lambda) = \varepsilon^{c_k} - b_j P_{k+1}(\lambda) P_{k+2}^2(\lambda) \cdots P_n^{n-k}(\lambda)\lambda^{n-k+1},$$

$$P_k(\lambda) = \prod_{j=1}^{p} P_{k,b_j}(\lambda).$$

• f_1 and f_2 are defined by:

$$f_1(\lambda) = \varepsilon P_1(\lambda) P_2^2(\lambda) \cdots P_n^n(\lambda) \cdot \lambda^{n+1}$$

$$f_2(\lambda) = \varepsilon^2 P_1(\lambda) P_2(\lambda) \cdot P_n(\lambda) \cdot \lambda$$

Lemma 1. *The polynomials defined above have the following properties:*

1. $\deg P_k = d_k$ *and the absolute value of its leading coefficient is 1.*
2. $P_k(0) \neq 0$ *and* P_j *and* P_k *have no common zero for* $j \neq k$.
3. *If* $P_k(\lambda) = 0$ *then* $|\lambda| < \frac{1}{2^k}$.
4. $|P_k(\lambda)| < 3^{d_k}$ *for* $|\lambda| \leq 2$.
5. $\left(\frac{1}{2}\right)^{d_k} < |P_k(\lambda)| < 3^{d_k}$ *for* $1 \leq |\lambda| \leq 2$.
6. $|f_1(\lambda)| < \frac{r}{n}$ *and* $|f_2(\lambda)| < \frac{r^2}{n}$ *for* $|\lambda| \leq 2$.
7. $|f_2(\lambda)| < \frac{|f_1(\lambda)|}{n}$ *for* $1 \leq |\lambda| \leq 2$.
8. $|f_2(\lambda)|^k < |f_1(\lambda)|$ *for* $1 \leq |\lambda| \leq 2$ *and* $k \geq 1$.

Proof. *1* and *2* are obvious. For *3* one uses backward induction and Rouché's theorem. Indeed, notice that if all the zeros of P_j, $j \geq k+1$, are inside $\{|\lambda| < \frac{1}{2^j}\}$ then, since the leading coefficient of P_j has the absolute value equal to 1, we get, for $|\lambda| = \frac{1}{2^k}$, that $|P_j(\lambda)| \geq \left(\frac{1}{2^{k+1}}\right)^{d_j}$. Using our choice of ε, we obtain then that $|b_j P_{k+1}(\lambda) P_{k+2}^2(\lambda) \cdots P_n^{n-k}(\lambda)\lambda^{n-k+1}| > \varepsilon^{c_k}$ for $|\lambda| = \frac{1}{2^k}$ and hence P_{k,b_j} and $b_j P_{k+1} \cdot P_{k+2}^2 \cdots P_n^{n-k} \cdot \lambda^{n-k+1}$ have the same number of zeros inside $|\lambda| < \frac{1}{2^k}$. As the two polynomials have the same degree and the latter one has all its zeros in the disk $|\lambda| = \frac{1}{2^k}$, the former has also all its zeros inside $|\lambda| = \frac{1}{2^k}$.

The rest of the relations follow easily from *3*. See also Corollaries 1 and 2 in [5]. □

Lemma 2. $(f_1, f_2)(\overline{\Delta}_2 \setminus \mathcal{Z}) \subset \bigcup_{k\geq 0}^n U_r^{(k)} \setminus (L_0 \cup \cdots \cup L_n)$

Proof. We have that $U_r^{(k)} \setminus (L_0 \cup \cdots \cup L_n)$ is given by $\{|z_2|^{k+2} < r|z_1|,\ |z_1| < r|z_2|^k\} = \{\frac{|z_2|^{k+2}}{r} < |z_1| < r|z_2|^k\}$. If $|z_2| < r^2$ then (because $r < 1$) we have also that $\frac{|z_2|^{k+1}}{r} < r|z_2|^k$. These show that $\bigcup_{k\geq 0}^n U_r^{(k)} \setminus (L_0 \cup \cdots \cup L_n) \supset \{|z_2| < r^2, \frac{|z_2|^{n+2}}{r} < |z_1| < r\}$. By Lemma 1, part *6*, we have $|f_1(\lambda)| < r$ and $|f_2(\lambda)| < r^2$ for $|\lambda| \leq 2$. At the same time we notice that $\frac{f_2(\lambda)^{n+1}}{f_1(\lambda)}$ is a holomorphic function on $\mathbb{C}$. By Lemma 1, part *8*, we have that if $\lambda \in \partial\Delta_2$, then $\frac{r|f_2(\lambda)|^{n+1}}{|f_1(\lambda)|} < 1$. By the maximum modulus principle the same inequality holds for $\lambda \in \Delta_2$. Therefore, if $\lambda \in \Delta_2$, we have that $|f_1(\lambda)| > r|f_2(\lambda)|^{n+1} > \frac{|f_2(\lambda)|^{n+2}}{r}$. □

Next we want to show that if for some $\lambda \in \overline{\Delta}_2 \setminus \mathcal{Z}$ we have that $(f_1, f_2)(\lambda) \in U_r^{(k)} \setminus (L_0 \cup \cdots \cup L_n)$ then, in fact, $(f_1, f_2)(\lambda) \in V_{r,\rho}^k \setminus (\tilde{L}_0 \cup \cdots \cup \tilde{L}_n)$. This is the content of the next proposition.

Proposition 2. *Suppose that* $\lambda \in \overline{\Delta}_2 \setminus \mathcal{Z}$ *and* $k \in \{0, 1, \ldots, n\}$. *If* $|f_1(\lambda)| < r|f_2(\lambda)|^k$ *and* $|f_2(\lambda)|^{k+2} < r|f_1(\lambda)|$ *then*

$$|f_1(\lambda)|^2 < \rho|f_2(\lambda)^{k+1} - b_j f_1(\lambda)| \cdot |f_2(\lambda)|^k \quad \forall j = \overline{1, p}.$$

In order to prove Proposition 2 we need the following lemma which is in fact one of the main motivations for the inductive construction of P_k.

Lemma 3. *a) For every $k \in \{0, 1, \ldots, n-1\}$ we have that*

$$P_{k+1,b_j} \text{ is a divisor of } \varepsilon^{2k+1} P_1^k \cdot P_2^{k-1} \cdots P_k - b_j P_{k+2} \cdot P_{k+3}^2 \cdots P_n^{n-k-1} \cdot \lambda^{n-k}$$

b) $\varepsilon^{2k+1} P_1^k \cdot P_2^{k-1} \cdots P_k - b_j P_{k+2} \cdot P_{k+3}^2 \cdots P_n^{n-k-1} \cdot \lambda^{n-k} = P_{k+1,b_j} \cdot Q$ where Q is a polynomial that does not vanish on $\overline{\Delta}_2$

Proof. a) For $k = 0$ this follows from the definition of P_{1,b_j}. For $k \geq 1$ we notice that for $s \leq k$ we have:
$P_{s,b_j} \equiv \varepsilon^{c_s} (\operatorname{mod} P_{k+1}) \Longrightarrow P_s \equiv \varepsilon^{c_s p} (\operatorname{mod} P_{k+1}) \Longrightarrow$
$\varepsilon^{2k+1} P_1^k \cdots P_k \equiv \varepsilon^{2k+1+p(kc_1+\cdots+c_k)} \equiv \varepsilon^{c_{k+1}} (\operatorname{mod} P_{k+1}) \Longrightarrow P_{k+1,b_j} | \varepsilon^{2k+1} P_1^k \cdots P_k - \varepsilon^{c_{k+1}}$.
However $\varepsilon^{c_{k+1}} = P_{k+1,b_j} + b_j P_{k+2} \cdot P_{k+3} \cdots P_n^{n-k-1} \cdot \lambda^{n-k}$ and the conclusion follows.

b) It follows from Lemma 1 and our choice of ε that $|\varepsilon^{2k+1} P_1^k \cdot P_2^{k-1} \cdots P_k| < |b_j P_{k+2} \cdot P_{k+3}^2 \cdots P_n^{n-k-1} \cdot \lambda^{n-k}|$ for $1 \leq |\lambda| \leq 2$. Hence, by Rouché's theorem, $\varepsilon^{2k+1} P_1^k \cdot P_2^{k-1} \cdots P_k - b_j P_{k+2} \cdot P_{k+3}^2 \cdots P_n^{n-k-1} \cdot \lambda^{n-k}$ and $b_j P_{k+2} \cdot P_{k+3}^2 \cdots P_n^{n-k-1} \cdot \lambda^{n-k}$ have the same number of zeros inside Δ_2. We have seen that all the zeros of each P_k are inside Δ and hence $b_j P_{k+2} \cdot P_{k+3}^2 \cdots P_n^{n-k-1} \cdot \lambda^{n-k}$ has $d_{k+2} + 2d_{k+3} + \cdots + (n-k-1)d_n + n - k = d_{k+1}/p = \deg P_{k+1,b_j}$ zeros inside Δ_2. Therefore $\varepsilon^{2k+1} P_1^k \cdot P_2^{k-1} \cdots P_k - b_j P_{k+2} \cdot P_{k+3}^2 \cdots P_n^{n-k-1} \cdot \lambda^{n-k}$ and P_{k+1,b_j} have the same number of zeros counting multiplicity inside Δ_2 and therefore their quotient does not vanish.

□

Proof of Proposition 2. We fix j.

We deal first with the case $k = 0$. We will to prove that $|f_1(\lambda)|^2 \leq \frac{\varrho}{2} |f_2(\lambda) - b_j f_1(\lambda)|$ for $\lambda \in \overline{\Delta}_2$. This will imply, of course that $|f_1(\lambda)|^2 < \rho |f_2(\lambda) - b_j f_1(\lambda)|$ for $\lambda \in \overline{\Delta}_2 \setminus \mathcal{Z}$.

We notice first that $\frac{f_1(\lambda)^2}{f_2(\lambda) - b_j f_1(\lambda)}$ is holomorphic on a neighbourhood of $\overline{\Delta}_2$. Indeed

$$\frac{f_1(\lambda)^2}{f_2(\lambda) - b_j f_1(\lambda)} = \frac{\varepsilon P_1 \cdot P_2^3 \cdots P_n^{2n-1} \cdot \lambda^{2n+1}}{\varepsilon - b_j P_2 \cdot P_3^2 \cdots P_n^{n-1} \cdot \lambda^n}$$

By the definition of P_{1,b_j}, since $c_1 = 1$, we have that $\varepsilon - b_j P_2 \cdot P_3^2 \cdots P_n^{n-1} \cdot \lambda^n = P_{1,b_j}$. This implies immediately that indeed $\frac{f_1(\lambda)^2}{f_2(\lambda) - b_j f_1(\lambda)}$ is holomorphic on a neighbourhood of $\overline{\Delta}_2$ (in fact on $\mathbb{C}$). Hence, by the maximum modulus principle, it suffices to show that $\frac{|f_1(\lambda)^2|}{|f_2(\lambda) - b_j f_1(\lambda)|} \leq \frac{\varrho}{2}$ on $\partial\Delta_2$. It suffices then to show that $|f_1(\lambda)^2| \leq \frac{\varrho}{2} |f_1(\lambda)| - \frac{\varrho}{2} |f_2(\lambda)|$ which is the same as $|f_1(\lambda)^2| + \frac{\varrho}{2} |f_2(\lambda)| \leq \frac{\varrho}{2} |f_1(\lambda)|$, i.e. $\varepsilon^2 |P_1^2 \cdots P_n^{2n} \lambda^{2n+2}| + \frac{\varrho}{2} \varepsilon^2 |P_1 \cdots P_n \lambda| \leq \frac{\varrho}{2} \varepsilon |P_1 \cdots P_n^n \lambda^{n+1}|$. Hence we want $\varepsilon(|P_1 \cdots P_n^{2n-1} \lambda^{2n+1}| + \frac{\varrho}{2}) < \frac{\varrho}{2} |P_2 \cdots P_n^{n-1} \lambda^n|$ on $\partial\Delta_2$. However this follows from Lemma 1, part *5*, and $(*)$.

Suppose now that $k \geq 1$. In this case we will show in fact that if $|f_2(\lambda)|^{k+2} < r |f_1(\lambda)|$ then $|f_1(\lambda)|^2 \leq \frac{\varrho}{2} |f_2(\lambda)^{k+1} - b_j f_1(\lambda)| \cdot |f_2(\lambda)|^k \quad \forall j$. In order to do this we let $A_k :=$

$\{\lambda \in \Delta_2 : |f_1(\lambda)| < r|f_2(\lambda)|^k\}$. Notice that by Lemma 1, part *8*, $A_k \subset \Delta$ and hence $|f_1(\lambda)| = r|f_2(\lambda)|^k$ on ∂A_k. We also note that, for $l \leq k-1$, P_l does not vanish on $\overline{A}_k$. Indeed the polynomials $P_1, \ldots, P_n$ have no common zero and the order of vanishing of f_1 at a zero of P_l is (strictly) less than the order of vanishing f_2^k at the same zero.

Then part b) of Lemma 3 and a direct computation shows that

$$\frac{f_1^2(\lambda)}{(f_2^{k+1}(\lambda) - b_j f_1(\lambda)) \cdot f_2^k(\lambda)}$$

is holomorphic on a neighbourhood of $\overline{A}_k$.

By the maximum modulus theorem, it suffices to show that

$$\frac{|f_1^2(\lambda)|}{|(f_2^{k+1}(\lambda) - b_j f_1(\lambda)) \cdot f_2^k(\lambda)|} \leq \frac{\rho}{2}$$

on ∂A_k, hence for $|f_1(\lambda)| = r|f_2(\lambda)|^k$. But then it suffices to prove that $|f_1^2(\lambda)| \leq \frac{\rho}{2}(|b_j f_1(\lambda)| - |f_2^{k+1}(\lambda)|)|f_2(\lambda)|^k$ and hence that $r^2|f_2(\lambda)|^{2k} \leq \frac{\rho}{2}(r|f_2(\lambda)|^k - |f_2(\lambda)|^{k+1}) \cdot |f_2(\lambda)|^k$, this means that it suffices to show that $r^2 \leq \frac{\rho}{2}(r - |f_2(\lambda)|)$ and, since $|f_2(\lambda)| \leq r^2$, it suffices to have $r \leq \frac{\rho}{2}(1-r)$ and this exactly the condition that we have imposed on r and ρ. □

All together, from Lemma 2 and Proposition 2 we deduce that $g_n(\overline{\Delta} \setminus \mathcal{Z}) \subset \bigcup_{k=0}^n V_{r,\rho}^k \setminus \left(\tilde{L}_0 \bigcup \cdots \bigcup \tilde{L}_n\right)$. As we have already mentioned, $g_n(\mathcal{Z}) \subset \tilde{L}_0 \bigcup \cdots \bigcup \tilde{L}_n$. Therefore $g_n(\overline{\Delta}) \subset \bigcup_{k=0}^n V_{r,\rho}^k \subset W$.

We prove now that we can choose r and ρ such that $g_n(\partial\Delta) \subset K$.

Since $\{V_{r,\rho}^0\}$ is a fundamental system of neighbourhoods for $\tilde{L}_0$ it follows that there exists r and ρ such that $V_{r,\rho}^0 \subset K$. Hence it suffices to show that $g_n(\partial\Delta) \subset V_{r,\rho}^0$. We have seen that $\mathcal{Z} \subset \Delta$. Therefore it suffices to show that for $|\lambda| = 1$ the following inequalities are satisfied: $|f_2|^2 < r|f_1|$, $|f_1| < r$, $|f_1|^2 < \rho|f_2 - b_j f_1|$ for every j. That $|f_1| < r$ follows from Lemma 1, part *6*. The inequality $|f_1|^2 < \rho|f_2 - b_j f_1|$ for every j was already proved. It remains to deal with the first inequality. For $|\lambda| = 1$ we have that:

$$|f_2|^2 < r|f_1| \iff \varepsilon^4|P_1^2 \cdots P_n^2 \lambda^2| < r\varepsilon|P_1 P_2^2 \cdots P_n^n \lambda^{n+1}| \iff \varepsilon^3 < \frac{r|P_3 P_4^2 \cdots P_n^{n-2}|}{|P_1|}$$

This last inequality follows from Lemma 1, part *5*, and $(*)$.

It remains to check that $g_n(\overline{\Delta}) \cap \tilde{L}_n \neq \emptyset$. Note that since $\lambda = 0$ is a zero of order 1 for f_2 and order $n+1$ for f_1 then $\hat{g}_n(0) \in L_n$ ($\hat{g}_n$ was the proper transform of (f_1, f_2) after the first $(n+2)$ blow-ups). Moreover $\hat{g}_n(0) = (0, 0, [f_1(0) : f_2^{n+1}(0)], [1 : 0])$. Now $\frac{f_2^{n+1}(0)}{f_1(0)} = \varepsilon^{2n+1} P_1^n(0) \cdots P_n(0)$ and $(*)$ and Lemma 1, part *4*, imply that $\frac{|f_2^{n+1}(0)|}{|f_1(0)|} \neq 1$. In particular $\hat{g}_n(0) \neq b_j^n$ and this implies that $g_n(0) \in \tilde{L}_n$.

This finishes the proof of Theorem 5.

□

Proposition 3. *Let X be a 1-convex manifold and let A be its exceptional set. Let also $p : \tilde{X} \to X$ be a covering and $\tilde{A} := p^{-1}(A)$. If $\dim A = 1$ and $\tilde{A}$ is holomorphically convex then $\tilde{X}$ is holomorphically convex.*

Proof. Let U be a neighbourhood of A such that $p^{-1}(U)$ is holomorphically convex. Such an U exists by Theorem 3 since $\dim A = 1$ and $\tilde{A}$ is holomorphically convex. Let $\hat{U}$ be the Remmert reduction of $p^{-1}(U)$ and $\psi : \hat{U} \to \mathbb{R}$ a strongly plurisubharmonic exhaustion function. The same contractions of connected compact subspaces of $p^{-1}(U)$ (hence of $\tilde{A}$) used to obtain $\hat{U}$ can be viewed as taking place in $\tilde{X}$ and we obtain a complex space $\hat{X}$ and a proper modification $\rho : \tilde{X} \to \hat{X}$. We have also that $\hat{U}$ is an open subset of $\hat{X}$.

We let $\tilde{\phi} : \tilde{X} \to [-\infty, \infty)$ be the plurisubharmonic function given by Proposition 1 and $V \subset X$ be an open neighbourhood of A such that $V \Subset U$. Since $\tilde{\phi}_{|\tilde{X} \setminus p^{-1}(V)}$ is an exhaustion we can find a strictly convex and increasing function $\chi : \mathbb{R} \to \mathbb{R}$ such that $\chi \circ \tilde{\phi} > \psi$ on $p^{-1}(\partial V)$. Then $\hat{\phi} : \hat{X} \to \mathbb{R}$ defined as $\chi \circ \tilde{\phi}$ on $\tilde{X} \setminus p^{-1}(V) = \hat{X} \setminus p^{-1}(V)$ and as $\max\{\chi \circ \tilde{\phi}, \psi\}$ on $\rho(p^{-1}(V))$ is a well-defined strictly plurisubharmonic exhaustion function. Therefore $\hat{X}$ is Stein and hence $\tilde{X}$ is holomorphically convex. □

Lemma 4. *Let A be a complex space of dimension 1 such that A does not contain as a closed subspace an infinite Nori string of rational curves. If $L_1, \ldots, L_k$ are finitely many irreducible components of A, then there exists a holomorphically convex covering of A such that $L_1 \bigcup \cdots \bigcup L_k$ is evenly covered.*

Proof. We let $A = \bigcup_{i \in I} L_i$ be the decomposition of A into irreducible components. Hence $\{1, \ldots, k\} \subset I$. We let $I_0 = \{1, \ldots, k\} \cup \{i \in I : A_i \text{ is rational}\}$ and $I_1 = I \setminus I_0$. We set $A_0 = \bigcup_{i \in I_0} L_i$, $A_1 = \bigcup_{i \in I_1} L_i$. Because A does not contain an infinite Nori string of rational curves we have that all connected components of A_0 are compact. We let $p : \tilde{A}_1 \to A_1$ be the universal covering of A_1 (or any other Stein covering), $\{b_j, j \in J\} = A_0 \cap A_1$ and $\{b_{j,n} : n \in \mathbb{N}\} = p^{-1}(b_j) \subset \tilde{A}_1$. We consider A_0^n countably many disjoint copies of A_0 and $b_j^n \in A_0^n$ the points corresponding to b_j.

Now we define $\tilde{A} := (\tilde{A}_1 \bigsqcup_{n \in \mathbb{N}} A_0^n)/\sim$ where $\sim$ identifies $b_{j,n}$ and b_j^n. Also we define $\tilde{p} : \tilde{A} \to A$ by $\tilde{p} = p$ on $\tilde{A}_1$ and $\tilde{p}$ is the identity on A_0^n. It is not difficult to see that $\tilde{p}$ is a covering and $\tilde{A}$ is holomorphically convex. Also A_0 is evenly covered and therefore $L_1 \bigcup \cdots \bigcup L_k$ is evenly covered.

□

Theorem 6. *Let X be a 1-convex complex surface and $p : \tilde{X} \to X$ be a covering. Then $\tilde{X}$ is p_5-convex if and only if $\tilde{X}$ does not contain an infinite Nori string of rational curves.*

Proof. The only if part follows from Theorem 5. We prove the if part.

Let $f_n : \overline{\Delta} \to \tilde{X}$ be a sequence of holomorphic disks such that $f_n(\partial\Delta) \subset K$ where K is a compact subset of $\tilde{X}$.

Let A be the exceptional set of X. Let W be a neighbourhood of A such that there exists a continuous strong deformation retract $W \to A$. It follows that there exists a strong deformation retract $\rho : p^{-1}(W) \to p^{-1}(A)$.

We choose $\psi : X \to \mathbb{R}$ a plurisubharmonic function and $0 < b < a$ real numbers such that $\psi_{|X\setminus A}$ is strictly plurisubharmonic, $\psi_{|A} = 0$, and $A \subset \{\psi < b\} \Subset \{\psi < a\} \Subset W$. We set $U = \{\psi < a\}$ and $V = \{\psi < b\}$.

We apply Proposition 1 and we choose $\phi : \tilde{X} \to [-\infty, \infty)$ a strictly plurisubharmonic function such that $\{\phi = -\infty\} = p^{-1}(A)$ and for every open neighbourhood Ω of A, $\phi_{|\tilde{X}\setminus\phi^{-1}(\Omega)}$ is an exhaustion. Let $M = \max_{x\in K}\phi(x)$. By the maximum principle we have that $\phi \circ f_n \le M$ on Δ.

Since $\phi_{|\tilde{X}\setminus\phi^{-1}(V)}$ is an exhaustion it follows that $\{\phi \le M\} \setminus p^{-1}(V)$ is compact. Let $K_1 = K \bigcup (\{\phi \le M\} \setminus p^{-1}(V))$ which is also compact. Let K_2 be another compact subset such that $K_2 \subset p^{-1}(W)$ and the interior of K_2 contains $K_1 \cap p^{-1}(\overline{U})$. We have that $\rho(K_2)$ is a compact subset of $p^{-1}(A)$. We choose $L_1, \dots, L_k$ finitely many irreducible components of $p^{-1}(A)$ such that $L_1 \cup \dots \cup L_k \supset \rho(K_2)$. We apply Lemma 4 to obtain a holomorphically convex covering $\hat{A} \to p^{-1}(A)$ such that $L_1 \cup \dots \cup L_k$ is evenly covered. We consider the fiber product of this covering map and ρ and we obtain a covering $\hat{p} : \hat{W} \to p^{-1}(W)$ which extends the covering $\hat{A} \to p^{-1}(A)$. It follows that $\rho^{-1}(L_1 \cup \dots \cup L_k)$ is evenly covered for $\hat{p}$. In particular K_2 is also evenly covered. We choose $\hat{K}_2$ a compact subset of $\hat{W}$ such that $\hat{p} : \hat{K}_2 \to K_2$ is a homeomorphism. Since U is strictly pseudoconvex, Proposition 3 implies that $\hat{p}^{-1}(p^{-1}(U))$ is holomorphically convex. Also since U is given by $\{\psi < a\}$ it follows that $f_n^{-1}(p^{-1}(U)) \cap \Delta$ is Runge in Δ. Let $\Omega_{n,j}$ be its connected components. Hence $\Omega_{n,j}$ are all simply connected.

We notice now that $\partial(f_n^{-1}(p^{-1}(U)) \cap \Delta) \subset (\overline{\Delta} \setminus f_n^{-1}(p^{-1}(V))) \cup \partial\Delta$ and hence $f_n(\partial\Omega_{n,j})$ is contained in the interior of K_2. We let $\Omega'_{n,j} \Subset \Omega_{n,j}$ such that they have smooth boundary and they are diffeomorphic to a disk, and, moreover $f_n(\overline{\Omega}_{n,j} \setminus \Omega'_{n,j})$ is still contained in the interior of K_2. Let $\hat{f}_{n,j} : \overline{\Omega}'_{n,j} \to \hat{p}^{-1}(p^{-1}(U))$ be liftings of $f_{n|\overline{\Omega}'_{n,j}}$ such that $\hat{f}_{n,j}(\partial\Omega'_{n,j}) \subset \hat{K}_2$. Because $\hat{p}^{-1}(p^{-1}(U))$ is holomorphically convex, it follows that $\bigcup \hat{f}_{n,j}(\overline{\Omega}'_{n,j})$ is contained in a compact subset K_3 of $\hat{W}$. Hence $f_n(\overline{\Delta}) \subset K_1 \cup K_2 \cup K_3$ and the proof of the theorem is complete. □

4 Some remarks regarding separation of cohomology

In [6] we proved that there exists a 1-convex complex surface X and a covering $\tilde{X}$ of X such that $H^1(\tilde{X}, \mathcal{O}_{\tilde{X}})$ is not separated. The main ingredients where:
• the p_3-convexity of $\tilde{X}$,
• our construction from [5] of a 1-convex surface X such that for its universal covering $\tilde{X}$ there exists a sequence of holomorphic disks $g_n : \overline{\Delta} \to \tilde{X}$ such that

a) $\bigcup g_n(\partial\Delta)$ is relatively compact and $\bigcup g_n(\overline{\Delta})$ is not.

b) there exist closed 1-dimensional analytic subsets A_n of $\tilde{X}$ such that $g_n(\overline{\Delta}) \subset A_n$.

It turns out that the following more general statement holds:

Proposition 4. *Let X be a 1-convex surface, A its exceptional set and $p : \tilde{X} \to X$ a covering map. We assume that A has a closed subspace A_1 which is a cycle of $\mathbb{P}^1$ such that*

$p^{-1}(A_1)$ has a noncompact connected component. Then $H^1(\tilde{X}, \mathcal{O}_{\tilde{X}})$ is not separated.

Proof. Let $\varphi : X \to [0, \infty)$ be a plurisubharmonic exhaustion function on X such that $A = \{\varphi = 0\}$ and φ is strictly plurisubharmonic outside A.

We prove first that there exist:
- a strictly pseudoconvex neighbourhood $\Omega \subset X$ of A, $\Omega = \{\varphi < \varepsilon_0\}$ for some $\varepsilon_0 > 0$,
- a sequence $\{B_n\}$ of closed 1-dimensional analytic subsets of $p^{-1}(\Omega)$, and
- a sequence of holomorphic disks $h_n : \overline{\Delta} \to p^{-1}(\Omega)$,

such that $h_n(\overline{\Delta}) \subset B_n$, $\bigcup h_n(\partial\Delta)$ is relatively compact in $p^{-1}(\Omega)$, and $\bigcup h_n(\overline{\Delta})$ is not relatively compact. This will imply, as in [6], that $H^1(p^{-1}(\Omega), \mathcal{O}_{\tilde{X}})$ is not separated.

Let $\bigcup_{j\in\mathbb{Z}} L_j$ be the noncompact connected component of $p^{-1}(A_1)$. This is an infinite chain of $\mathbb{P}^1$. Let $\bigcup_{j=0}^{q} F_j$ be the decomposition of A_1 into irreducible components and let A_2 be the union of all irreducible components of A that are not included in A_1. After a finite number of blow-ups we can assume that $(F_0 \cup F_q) \cap A_2 = \emptyset$ and $F_i \cdot F_i = F_j \cdot F_j \leq -3$ for $i, j \in \{0, \dots, q\}$. Of course, all these blow-ups can be performed in $\tilde{X}$ as well and still we obtain a covering.

Exactly as in [5], using the construction from the proof of Theorem 5, we can construct a complex surface X' containing a cycle of $\mathbb{P}^1$, A' , $A' = \bigcup_{j=0}^{q} F'_j$ with $F'_j \cdot F'_j = F_j \cdot F_j$ for each j and a covering $p' : \tilde{X}' \to X'$ such that $\tilde{X}'$ is not p_5-convex. In fact $\tilde{X}' = \bigcup_{k\in\mathbb{Z}} V^{(k)}_{r_2,\rho_2}$ and contains an infinite chain of $\mathbb{P}^1$, $\bigcup_{j\in\mathbb{Z}} L'_j$. Here $0 < r_2 < 1$ and $0 < \rho_2 < 1$. By Theorem 4, there exist $U \subset X$ and $U' \subset X'$ biholomorphic neighbourhoods of A_1 and respectively A'. Let $\chi : U \to U'$ be a biholomorphism. We let $W \subset U$ be an open neighborhood of A_1 that has a continuous deformation retract onto A_1 and $W' = \chi(W) \subset U'$. We let X_0 be the connected component of $p^{-1}(W)$ that contains $\bigcup_{j\in\mathbb{Z}} L_j$ and X'_0 be the connected component of $p'^{-1}(W')$ that contains $\bigcup_{j\in\mathbb{Z}} L'_j$. Then X_0 is in fact the universal covering of W and X'_0 is the universal covering of W'. Let $\tilde{\chi} : X_0 \to X'_0$ be the lifting of χ. It follows that $\tilde{\chi}$ is a biholomorphism.

We choose $0 < r_1 < r_2$ and $0 < \rho_1 < \rho_2$ such that $\overline{\bigcup_{k\in\mathbb{Z}} V^{(k)}_{r_1,\rho_1}} \subset X'_0$. Then $\bigcup_{k\in\mathbb{Z}} V^{(k)}_{r_1,\rho_1}$ will cover a neighbourhood of A'. The indices are chosen such that $p'(V^{(k)}_{r_1,\rho_1}) = p'(V^{(k+q+1)}_{r_1,\rho_1}) \supset F'_j$ if $j \equiv k \pmod{q+1}$. For $j \in \{0, \dots, q\}$ we set $\mathcal{V}'_j = p'(V^{(j)}_{r_1,\rho_1}) \subset W'$ and $\mathcal{V}_j = \chi^{-1}(\mathcal{V}'_j)$.

We choose $\varepsilon_0 > 0$ such that $\Omega := \{\varphi < \varepsilon_0\}$, which is a strictly pseudoconvex neighbourhood of A, satisfies $\Omega \cap \partial(\mathcal{V}_0 \cup \mathcal{V}_q) \subset \mathcal{V}_1 \cup \mathcal{V}_{q-1}$. This is possible because $(F_0 \cup F_q) \cap A_2 = \emptyset$.

Finally, we choose $0 < r < r_1$ and $0 < \rho < \rho_1$ such that $r < \frac{\rho}{2}(1-r)$ and $\bigcup_{k\in\mathbb{Z}} \overline{V}^{(k)}_{r,\rho} \subset p'^{-1}(W' \cap \chi(\Omega))$.

As in the proof of Theorem 5, we can construct a sequence of holomorphic disks, $g_n : \overline{\Delta} \to \bigcup_{k\in\mathbb{Z}} V^{(k)}_{r,\rho}$ such that $\bigcup g_n(\partial\Delta)$ is relatively compact in $\bigcup_{k\in\mathbb{Z}} V^{(k)}_{r,\rho}$ and $\bigcup g_n(\overline{\Delta})$ is not. We let $h_n = g_n \circ \tilde{\chi}^{-1}$ and we regard them as holomorphic disks in $p^{-1}(\Omega)$. Then $\bigcup h_n(\partial\Delta)$ is relatively compact in $p^{-1}(\Omega)$ and $\bigcup h_n(\overline{\Delta})$ is not.

At the same time there exist 1-dimensional analytic subsets B'_n which are closed in X'_0 such that $g_n(\overline{\Delta}) \subset B'_n$. These analytic sets are nothing else than the intersection of X'_0

with $g_n(\mathbb{C})$, g_n being the proper transform of (f_1, f_2) where $f_1 = f_1^{(n)}$, $f_2 = f_2^{(n)}$ are the polynomials defined in the proof of Theorem 5. They are closed analytic subsets because f_1 and f_2, being nonconstant polynomials, are proper maps from $\mathbb{C}$ to $\mathbb{C}$. At the same time, by construction, $g_n(\mathbb{C}) \cap X_0' \subset \bigcup_{k=0}^n V_{r_2,\rho_2}^{(k)}$.

Let $B_n = \tilde{\chi}^{-1}(B_n' \cap (\cup_{k\in\mathbb{Z}} V_{r_1,\rho_1}^{(k)})) \cap p^{-1}(\Omega)$. Clearly $h_n(\overline{\Delta}) \subset B_n$. We claim that the sets B_n are closed analytic subsets of $p^{-1}(\Omega)$. That they are analytic is obvious. We have to check that they are closed.

Because $g_n(\overline{\Delta})$ is a compact subset of X_0' and $g_n(\overline{\Delta}) \subset \cup_{k\in\mathbb{Z}} V_{r_1,\rho_1}^{(k)}$, it follows that we have to deal only with $g_n(\mathbb{C} \setminus \Delta)$. That means that it suffices to show that $\tilde{\chi}^{-1}(g_n(\mathbb{C} \setminus \Delta) \cap (\cup_{k\in\mathbb{Z}} V_{r_1,\rho_1}^{(k)})) \cap p^{-1}(\Omega)$ is a closed subset of $p^{-1}(\Omega)$.

We note now that Lemma 1 and our choice of ε imply that for $|\lambda| \geq 1$ we have that $|f_1(\lambda)| > r|f_2(\lambda)|$. This inequality and Remark 1, b) imply that $(f_1, f_2)(\mathbb{C} \setminus \Delta) \cap (\cup_{k\geq 1} V_{r_2,\rho_2}^{(k)}) = \emptyset$, i.e. $g_n(\mathbb{C} \setminus \Delta) \cap (\cup_{k\geq 1} V_{r_2,\rho_2}^{(k)}) = \emptyset$. At the same time, since $g_n(\mathbb{C} \setminus \Delta) \cap X_0' \subset \bigcup_{k=0}^n V_{r_2,\rho_2}^{(k)}$ and $V_{r_2,\rho_2}^{(j)} \cap (\bigcup_{k=0}^n V_{r_2,\rho_2}^{(k)}) = \emptyset$ for $j \leq -2$ (see Remark 1, a)), we deduce that $g_n(\mathbb{C} \setminus \Delta) \cap X_0' \cap \left[\cup_{k\in\mathbb{Z}\setminus\{-1,0\}} V_{r_2,\rho_2}^{(k)})\right] = \emptyset$.

The inclusion $\Omega \cap \partial(\mathcal{V}_0 \cup \mathcal{V}_q) \subset \mathcal{V}_1 \cup \mathcal{V}_{q-1}$ implies that $p^{-1}(\Omega) \cap \partial\tilde{\chi}^{-1}(V_{r_1,\rho_1}^{(-1)} \cup V_{r_1,\rho_1}^{(0)}) \subset \tilde{\chi}^{-1}(V_{r_1,\rho_1}^{(-2)} \cup V_{r_1,\rho_1}^{(1)})$. We deduce that

$$\tilde{\chi}^{-1}(g_n(\mathbb{C} \setminus \Delta) \cap X_0') \cap \partial\tilde{\chi}^{-1}(V_{r_1,\rho_1}^{(-1)} \cup V_{r_1,\rho_1}^{(0)}) \cap p^{-1}(\Omega) = \emptyset.$$

The following simple remark implies then that the sets B_n are closed in $p^{-1}(\Omega)$.

Remark 2. Suppose that D, D_1, D_2 are open sets in a topological space such that $\overline{D}_1 \subset D_2$. Let A be a closed subset of D_2. If $A \cap \partial D_1 \cap D = \emptyset$ then $A \cap D_1 \cap D$ is closed in D.

Indeed, we apply this remark for $D_2 = X_0'$, $D_1 = \tilde{\chi}^{-1}(V_{r_1,\rho_1}^{(-1)} \cup V_{r_1,\rho_1}^{(0)})$, $D = p^{-1}(\Omega)$ and $A = \tilde{\chi}^{-1}(g_n(\mathbb{C} \setminus \Delta) \cap X_0')$.

In order to finish the proof of the proposition we need the following:

Lemma 5. *Suppose that X is a 1-convex manifold with exceptional set A and $p : \tilde{X} \to X$ is a covering. Let $\varphi : X \to [0, \infty)$ be a plurisubharmonic exhaustion function on X such that $A = \{\varphi = 0\}$ and φ is strictly plurisubharmonic outside A. If the cohomology group $H^1(p^{-1}\{\varphi < \varepsilon_0\}, \mathcal{O}_{\tilde{X}})$ is non-separated for some $\varepsilon_0 > 0$ then $H^1(\tilde{X}, \mathcal{O}_{\tilde{X}})$ is non-separated.*

Proof. (Sketch) Using "bumpings" (see [8]) we have that the morphism induced by restriction $H^1(X, \mathcal{O}_X) \to H^1(\{\varphi < \varepsilon_0\}, \mathcal{O}_X)$ is surjective and becomes injective when passing to separates (see Proposition 1.3, page 346 in [1]). The bumpings on X induce bumpings on $\tilde{X}$ which gives easily that $H^1(\tilde{X}, \mathcal{O}_{\tilde{X}}) \to H^1(p^{-1}\{\varphi < \varepsilon_0\}, \mathcal{O}_{\tilde{X}})$ is surjective and becomes injective when passing to separates. This implies, of course, the conclusion of the lemma. □

Example 1. We give an example of a 1-convex surface X and a covering $\tilde{X}$ of X such that even though $\tilde{X}$ does not contain an infinite Nori string of rational curves, $H^1(\tilde{X}, \mathcal{O}_{\tilde{X}})$ is not separated. Note that by Theorem 1 $\tilde{X}$ is p_3-convex and by Theorem 6 it is p_5-convex.

Let us start with a 1-convex complex surface Y with exceptional set A and a covering $p_Y : \tilde{Y} \to Y$ such that
• $A = F_1 \cup F_2$ where F_1 and F_2 are isomorphic with $\mathbb{P}^1$,
• F_1 and F_2 intersect in precisely two points a and b and the intersection is transversal,
• $F_1 \cdot F_1 = F_2 \cdot F_2 = -3$,
• $p_Y^{-1}(A) = \bigcup_{k \in \mathbb{Z}} L_k$ is an infinite chain of $\mathbb{P}^1$, $p_Y^{-1}(F_1) = \bigcup_{j \in \mathbb{Z}} L_{2k+1}$, $p_Y^{-1}(F_2) = \bigcup_{k \in \mathbb{Z}} L_{2k}$.

We have seen that $H^1(\tilde{Y}, \mathcal{O}_{\tilde{Y}})$ is not separated.

For $j \in \{1, 2\}$, let $\mathcal{F}_j \to F_j$ be the normal bundle of F_j in Y. We choose simply connected neighbourhoods U_j of F_j in Y such that U_j is biholomorphic to a neighbourhood V_j of the zero section of $\mathcal{F}_j$ and $U_1 \cap U_2$ has two connected components $U^a \ni a$ and $U^b \ni b$. The existence of U_j follows from Theorem 4. See also [9]. Let $\phi_j : U_j \to V_j$ be biholomorphisms. We have that $p_Y^{-1}(U_j) = \bigcup_{k \in \mathbb{Z}} U_{j,k}$ where $U_{1,k}$ is a neighborhood of L_{2k+1} isomorphic via p_Y with U_1 and $U_{2,k}$ is a neighbourhood of L_{2k} isomorphic via p_Y with U_2.

Let S_1 and S_2 be two compact complex curves of genus ≥ 1 and $\pi_1 : S_1 \to F_1$, $\pi_2 : S_2 \to F_2$ be ramified coverings that have the same number, p, of points in the generic fiber. This is possible if p is large enough. Moreover, we assume that a and b are not ramification points for π_j. We pull-back $\mathcal{F}_j$ to S_j and we let $\psi_j : \pi_j^* \mathcal{F}_j \to \mathcal{F}_j$ be the canonical maps. We have that ψ_j are also ramified coverings and by shrinking U_1 and U_2 we can assume that $\phi_j(U_1 \cap U_2)$ is evenly covered by ψ_j. We let $\psi_j^{-1}(\phi_j(U^a)) = \bigcup_{l=1}^p V_{l,j}^a$ and $\psi_j^{-1}(\phi_j(U^b)) = \bigcup_{l=1}^p V_{l,j}^b$.

Now we let $X = \psi_1^{-1}(V_1) \bigsqcup \psi_2^{-1}(V_2)/\sim$, where, for $l = 1, \ldots, p$, $\sim$ identifies, $V_{l,1}^a$ with $V_{l,2}^a$ and $V_{l,1}^b$ with $V_{l,2}^b$ using the automorphisms induced by ϕ_j and ψ_j. It follows that X is a 1-convex surface and there exists a finite map $f : X \to Y$. The exceptional set of X is the union of two complex curves of genus g that intersect transversely in $2p$ points.

In a completely similar manner, by gluing ramified coverings of $U_{j,k}$, $j \in \{1, 2\}$, $k \in \mathbb{Z}$, we can construct a covering $\tilde{X}$ of X and a finite map $\tilde{f} : \tilde{X} \to \tilde{Y}$.

Since $H^1(\tilde{Y}, \mathcal{O}_{\tilde{Y}})$ is not separated, we get that $H^1(\tilde{X}, \mathcal{O}_{\tilde{X}})$ is not separated: simply consider the canonical map $\mathcal{O}_{\tilde{Y}} \to \tilde{f}_* \mathcal{O}_{\tilde{X}}$ and the trace map $\tilde{f}_* \mathcal{O}_{\tilde{X}} \to \mathcal{O}_{\tilde{Y}}$. Their composition $\mathcal{O}_{\tilde{Y}} \to \tilde{f}_* \mathcal{O}_{\tilde{X}} \to \mathcal{O}_{\tilde{Y}}$ is an isomorphism. By passing to cohomology we deduce that $H^1(\tilde{Y}, \mathcal{O}_{\tilde{Y}}) \to H^1(\tilde{Y}, \tilde{f}_* \mathcal{O}_{\tilde{X}})$ is injective. Since f is finite, $H^1(\tilde{Y}, \tilde{f}_* \mathcal{O}_{\tilde{X}})$ is isomorphic to $H^1(\tilde{X}, \mathcal{O}_{\tilde{X}})$ and the conclusion follows.

Given this example, a natural question is the following.

Problem. Suppose that X is a 1-convex surface and $\tilde{X}$ is a covering of X. If $\tilde{X}$ is not holomorphically convex, does it follow that $H^1(\tilde{X}, \mathcal{O}_{\tilde{X}})$ is not separated?

Remark 3. The complex surface $\tilde{X}$ constructed in the above example is satisfies the Kontinuitätssatz with respect to holomorphic disks but not with respect with 1-dimensional analytic sets with boundary.

References

[1] C. Bănică; O. Stănăşilă: Méthodes algébriques dans la théorie globale des espaces complexes. Troisième édition. Gauthier-Villars, Paris, 1977.

[2] M. Colţoiu: Coverings of 1-convex manifolds with 1-dimensional exceptional set. *Comment. Math. Helv.* **68** (1993), 469–479.

[3] M. Colţoiu; K. Diederich: On the coverings of proper families of 1-dimensional complex spaces. *Michigan Math. J.* **47** (2000), 369–375.

[4] M. Colţoiu; C. Joiţa: The disk property of coverings of 1-convex surfaces. *Proc. Amer. Math. Soc.* **140** (2012), 575–580.

[5] M. Colţoiu; C. Joiţa: Convexity properties of coverings of 1-convex surfaces. *Math. Z.* **275** (2013), 781–792.

[6] M. Colţoiu; C. Joiţa: On the separation of the cohomology of universal coverings of 1-convex surfaces. *Adv. Math.* **265** (2014), 362–370.

[7] F. Docquier; H. Grauert: Levisches Problem und Rungescher Satz für Teilgebiete Steinscher Mannigfaltigkeiten. *Math. Ann.* **140** (1960), 94–123.

[8] H. Grauert: On Levi's problem and the imbedding of real-analytic manifolds. *Ann. of Math. (2)* **68** (1958), 460–472.

[9] H. Grauert: Über Modifikationen und exzeptionelle analytische Mengen. *Math. Ann.* **146** (1962), 331–368.

[10] C. Joiţa: The disk property. A short survey. *An. Ştiinţ. Univ. "Ovidius" Constanţa Ser. Mat.* **20** (2012), 35–42.

[11] H. B. Laufer: Taut two-dimensional singularities. *Math. Ann.* **205** (1973), 131–164.

Mihnea Colţoiu
Simion Stoilow Institute of Mathematics of the Romanian Academy
P.O. Box 1-764, Bucharest 014700, ROMANIA
E-mail address: Mihnea.Coltoiu@imar.ro

Cezar Joiţa
Simion Stoilow Institute of Mathematics of the Romanian Academy
P.O. Box 1-764, Bucharest 014700, ROMANIA
E-mail address: Cezar.Joita@imar.ro