On the image of an algebraic projective space *

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Abstract

We prove that if X is a projective algebraic space, Y is a normal compact complex space and $p : X \to Y$ is a surjective morphism with equidimensional fibers then Y is also projective algebraic.

1   Introduction

If X is a Kähler manifold, Y is a complex manifold, $p : X \to Y$ is a proper holomorphic map, and p is equidimensional, then it follows by a result of Varouchas [10] that Y is also Kähler. This result was extended in [11] to complex spaces with singularities (under the flatness assumption of p). On the other hand it is known that the image of a Moishezon space by a holomorphic map is itself Moishezon (see, for example [1]). The well-known result of Moishezon [7] asserts that a compact complex manifold is projective if and only if it is Kähler and Moishezon. A similar result does not hold for spaces with singularities even for normal complex surfaces (see, e.g. [3]). In this paper we consider morphisms $p : X \to Y$ of compact complex spaces with equidimensional fibers such that X is a projective algebraic space. If X and Y are assumed to be smooth it follows from the results mentioned above that Y is projective algebraic.

Our main result (Theorem 2) states that if X is a projective algebraic space, Y is a normal compact complex space and $p : X \to Y$ is a morphism with equidimensional fibers then Y is also projective algebraic. When the

fibers of p are 0-dimensional (i.e. p is a ramified covering map) this result was obtained by Remmert and Van de Ven in [8]. For fibers of positive dimension Theorem 2 was proved by C. Horst in [6], under the additional assumption that Y has isolated singularities, using an analytic version of Chevalley's criterion of projectivity (see [2]). Note that the normality assumption is essential (see e.g. [4] page 171, ex. 7.13 and [6]).

2   The Results

We denote by $\mathbb{P}_{\nu,n}$ the projective space that parametrizes homogeneous polynomials $F \in \mathbb{C}[z_0, \dots, z_n]$ of degree ν. For $F \in \mathbb{P}_{\nu,n}$ we set $Z(F) := \{[z_0 : \cdots : z_n] \in \mathbb{P}^n : F(z_0, \dots, z_n) = 0\}$.

Lemma 1. *Given a subvariety C, $\dim C \geq 1$, of $\mathbb{P}^n$ then $\{F \in \mathbb{P}_{\nu,n} : \dim(Z(F) \cap C) = \dim(C)\}$ is a finite union of linear subspaces of $\mathbb{P}_{\nu,n}$ of codimension at least $\nu + 1$.*

Proof. Let C_j, $j = 1, \dots, k$, be the irreducible components of C with $\dim C_j = \dim C$. Then we have

$$\{F \in \mathbb{P}_{\nu,n} : \dim(Z(F) \cap C) = \dim(C)\} = \cup_{j=1}^{k}\{F \in \mathbb{P}_{\nu,n} : Z(F) \supset C_j\}.$$

Obviously each $\{F \in \mathbb{P}_{\nu,n} : Z(F) \supset C_j\}$ is a linear subspace of $\mathbb{P}_{\nu,n}$.

For the codimension inequality notice that if $A_1, \cdots, A_{\nu+1}$ are distinct points on C_j for any fixed j then $\{F \in \mathbb{P}_{\nu,n} : F(A_l) = 0, \ l = \overline{1, \nu+1}\}$ has codimension $\nu + 1$ in $\mathbb{P}_{\nu,n}$. Indeed, as $F(A_l) = 0$ is just a linear equation in $\mathbb{P}_{\nu,n}$, it suffices to show that for each $k \leq \nu$ there exists a homogeneous polynomial, F, of degree ν such that $F(A_1) = 0, \dots, F(A_k) = 0$ and $F(A_{k+1}) \neq 0$. To see this let G_l be homogeneous polynomials of degree 1 in $z_0, \dots, z_n$, $l = 1, \dots, k$, such that $G_l(A_l) = 0$ and $G_l(A_i) \neq 0$ for $i \neq l$, $i = 1, \dots, k+1$. We set $F = G_1^{i_1} \cdots G_k^{i_k}$ where $i_1, \dots, i_k \geq 1$ are integers such that $i_1 + \cdots + i_k = \nu$. $\square$

We denote by $C_{n,k,d}$ the Chow variety that parametrizes subvarieties of degree d and dimension k of $\mathbb{P}^n$. We have then that $C_{n,k,d}$ is a quasi-projective variety and that the incidence set $\{(X, z) \in C_{n,k,d} \times \mathbb{P}^n : z \in X\}$ is an algebraic subset of $C_{n,k,d} \times \mathbb{P}^n$ (see, for example [9]).

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Theorem 1. *Suppose that n, k and d are integers, $n, k, d \geq 1$. Then there exists $\nu_0 \in \mathbb{Z}$, $\nu_0 \geq 1$, such that for every $\nu \in \mathbb{Z}$, $\nu \geq \nu_0$ there exists a homogeneous polynomial $F \in \mathbb{C}[z_0, z_1, \ldots, z_n]$ of degree ν with the property that $Z(F) \subset \mathbb{P}^n$ contains no subvariety of $\mathbb{P}^n$ of dimension k and degree at most d.*

Proof. For $1 \leq j \leq d$ let $H_j := \{(X, F) \in C_{n,k,j} \times \mathbb{P}_{\nu,n} : \dim(Z(F) \cap X) = k\}$. We prove that H_j is a closed algebraic subset of $C_{n,k,j} \times \mathbb{P}_{\nu,n}$. Let $\tilde{H}_j = \{(X, z, F) \in C_{n,k,j} \times \mathbb{P}^n \times \mathbb{P}_{\nu,n} : z \in X, \ F(z) = 0\}$. Notice that $\tilde{H}_j = \tilde{H}'_j \cap \tilde{H}''_j$ where $\tilde{H}'_j = \{(X, z) \in C_{n,k,j} \times \mathbb{P}^n : z \in X\} \times \mathbb{P}_{\nu,n}$ and $\tilde{H}''_j = C_{n,k,j} \times \{(z, F) \in \mathbb{P}^n \times \mathbb{P}_{\nu,n} : F(z) = 0\}$. As both $\tilde{H}'_j$ and $\tilde{H}''_j$ are closed algebraic subsets of $C_{n,k,j} \times \mathbb{P}^n \times \mathbb{P}_{\nu,n}$ it follows that $\tilde{H}_j$ is a closed algebraic subsets of $C_{n,k,j} \times \mathbb{P}^n \times \mathbb{P}_{\nu,n}$. Let $\pi_j : C_{n,k,j} \times \mathbb{P}^n \times \mathbb{P}_{\nu,n} \to C_{n,k,j} \times \mathbb{P}_{\nu,n}$ be the canonical projection. As $\mathbb{P}^n$ is compact it follows that π_j is proper. If we denote by $\tilde{\pi}_j : \tilde{H}_j \to C_{n,k,j} \times \mathbb{P}_{\nu,n}$ the restriction of π_j to $\tilde{H}_j$ we have that $\tilde{\pi}_j$ is also proper. It follows that $\{(X, F) \in C_{n,k,j} \times \mathbb{P}_{\nu,n} : \dim \tilde{\pi}_j^{-1}(X, F) \geq k\}$ is an analytic subset of $C_{n,k,j} \times \mathbb{P}_{\nu,n}$ (by the semi-continuity of the dimension of the fibers in the Zariski topology, see e.g. [12], page 240). However $\pi_j^{-1}(X, F) = \{X\} \times (Z(F) \cap X) \times \{F\}$ and therefore $\dim \tilde{\pi}_j^{-1}(X, F) \leq k$. We deduce that $\{(X, F) \in C_{n,k,j} \times \mathbb{P}_{\nu,n} : \dim \tilde{\pi}_j^{-1}(X, F) \geq k\} = H_j$ and hence H_j is a closed analytic subset of $C_{n,k,j} \times \mathbb{P}_{\nu,n}$ as claimed.

Let $p_{1,j} : H_j \to C_{n,k,j}$, $p_{2,j} : H_j \to \mathbb{P}_{\nu,n}$ be the canonical projections. From Lemma 1 it follows that the fibers of $p_{1,j}$ have dimension at most $\dim(\mathbb{P}_{\nu,n}) - \nu - 1$ and therefore $\dim(H_j) \leq \dim C_{n,k,j} + \dim(\mathbb{P}_{\nu,n}) - \nu - 1$. If we choose $\nu \geq \max\{\dim C_{n,k,j} : j = 1 \ldots, d\}$ then $\dim(H_j) < \dim \mathbb{P}_{\nu,n}$. Here the projections p_{2j} are not necessarily proper, but we can nevertheless conclude that $\bigcup_{j=1}^{d} p_{2,j}(H_j)$ is of Hausdorff dimension $\leq 2n - 2$ in $\mathbb{P}_{\nu,n}$. Therefore for almost every polynomial $F \in \mathbb{P}_{\nu,n}$ we will have that $Z(F)$ does not contain any irreducible component of X of dimension k for any $X \in C_{n,k,j}$, $j \leq d$. $\qquad \square$

For the next Lemma we assume that X is a closed subvariety of $\mathbb{P}^n$, Y is a reduced compact complex space and $p : X \to Y$ is a surjective morphism. For $y \in Y$ we set $X_y := p^{-1}(y)$. If $\dim X_y = m$ we denote by $X_y^{(m)}$ the union of all irreducible components of X_y of dimension m.

Lemma 2. *If the fibers of p, X_y, $y \in Y$, have all the same dimension m then there exists an integer d such that $\deg X_y^{(m)} \leq d$ for every $y \in Y$*

Proof. We prove first that if $y_0 \in Y$ is any point then there exists U a neighborhood of y_0 and an integer d_U such that $\deg X_y^{(m)} \leq d_U$ for every $y \in U$. Indeed, let L be a linear subspace of $\mathbb{P}^n$ such that $\dim L = n - m - 1$ and $L \cap X_{y_0} = \emptyset$. For some small connected neighborhood U of y_0 we have that $L \cap X_y = \emptyset$ for every $y \in U$. Let $X(U) = p^{-1}(U)$. Note that $\mathbb{P}^n \setminus L$ has the structure of a holomorphic vector bundle $\pi : \mathbb{P}^n \setminus L \to \mathbb{P}^m$ and, for any $y \in U$, $\pi_{|X_y^{(m)}} : X_y^{(m)} \to \mathbb{P}^m$ is a branched covering of degree $d_y = \deg X_y^{(m)}$. We consider the analytic map $G : X(U) \to \mathbb{P}^m \times U$, $G(x) = (\pi(x), p(x))$. Then G is a proper finite surjective morphism. It follows that there exists an integer d_U such that $d_y \leq d_U$ for every $y \in U$.

The conclusion of the Lemma follows now from the compacity of Y. $\qquad\square$

Theorem 2. *Suppose that X and Y are reduced compact complex spaces and $p : X \to Y$ is a surjective holomorphic mapping. We assume that X is projective algebraic, Y is normal and the fibers of p have all the same dimension. Then Y is projective algebraic.*

Proof. We will prove the theorem by induction on the dimensions of the fibers of p. If p has discrete fibers this was proved in [8].

We assume now that we proved our theorem for every morphism such that each fiber has dimension $k - 1$, $k \geq 1$, and we consider a proper surjective holomorphic mapping $p : X \to Y$ such that $X_y := p^{-1}(y)$ has dimension k for each $y \in Y$. Let $X \hookrightarrow \mathbb{P}^n$ be an embedding of X. It follows from Lemma 2 that there exists a positive integer d such that $\deg X_y^{(k)}$ is at most d for every $y \in Y$. We apply then Theorem 1 and we deduce that there exists $F \in \mathbb{C}[z_0, z_1, \ldots, z_n]$, a homogeneous polynomial of sufficiently large degree, such that $Z(F)$ contains no irreducible component of dimension k of the fibers of p. Then for every $y \in Y$ we have that $Z(F) \cap X_y \neq \emptyset$ and $\dim Z(F) \cap X_y = k - 1$. If we let $X_1 := Z(F) \cap X$ and $p_1 : X_1 \to Y$ to the restriction of p we can apply the induction hypothesis and deduce that Y is projective algebraic. $\qquad\square$

Remark: In [5], Theorem 2.6, Hironaka studied proper mappings $\pi : X \to Y$ in the algebraic category and and showed that a generic hypersection of X does not contain any fiber of Y. However his arguments rely heavily on the algebraicity of Y.

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