Finite coverings of complex spaces by connected Stein open sets*

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Dedicated to the memory of our friend Adrian Constantinescu

Abstract

We prove that every reduced, second countable, connected complex space X can be written as a finite union of connected Stein open subsets. If X is irreducible, we show that these Stein open subsets can be chosen to be contractible. We also prove that there exist a connected Stein space $\tilde{X}$ and a surjective holomorphic, locally biholomorphic map $p : \tilde{X} \to X$ with finite fiber.

1 Introduction

Let X be a second countable reduced complex space. Clearly X has locally finite coverings with Stein open subsets and any open covering of X has a locally finite refinement with Stein open subsets. On the other hand, any compact complex space admits finite Stein open coverings. Then, the natural question one may ask is whether any connected complex space, of finite dimension, has a covering with finitely many Stein open subsets. A positive answer to this question was given in [3]. However, the open sets that appear in [3] have in general infinitely many connected components. At the same time, in the smooth case, Fornaess and Stout [6] proved that any connected complex manifold can be covered by finitely many polydisks.

The following question was raised in [3]: if X is a connected complex space of finite dimension, is it true that X can be covered by finitely many connected Stein open subsets? In this paper we prove (Theorem 11) that the answer is "yes". Moreover, if X is irreducible we obtain a stronger result, namely we prove (see Theorem 10) that X can be covered by finitely many contractible Stein open subsets. The proof of Theorem 11 can be used to obtain the following result (Theorem 12): given a connected complex space, there exist a connected Stein complex space $\tilde{X}$ and a holomorphic map $p : \tilde{X} \to X$ such that p is surjective, locally biholomorphic and the cardinality of the fiber is bounded by a constant depending on the dimension of X. When X is irreducible and locally irreducible this result was proved in [6].

2 Preliminaries

All complex spaces that appear in this paper are assumed to be reduced and countable at infinity.

The following theorem was proved by M. Colţoiu in [3].

Theorem 1. *For every $n \in \mathbb{N}$ there exists $d(n) \in \mathbb{N}$ such that every complex space of dimension n can be written as the union of $d(n)$ Stein open subsets.*

Theorem 2 was proved by J. E. Fornaess and E. L. Stout in [6].

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Theorem 2. *For every $n \in \mathbb{N}$ there exists $\lambda(n) \in \mathbb{N}$ such that every connected complex manifold of dimension n can be written as the union of $\lambda(n)$ open subsets, each one of them biholomorphic to a polydisk.*

More precisely, $\lambda(n)$ is defined as follows: $\lambda(1) = 3$ and $\lambda(n) = 2(3^{4n} - 1)4^n$ for $n \geq 2$.

A well known theorem of Y. T. Siu [14] says that if X is a complex space and Y is a Stein closed analytic subspace of X then Y has a Stein open neighborhood. The following theorem, proved by N. Mihalache in [11], is an extension of Siu's theorem.

Theorem 3. *Suppose that X is a complex space and Y closed analytic subspace. If Y is Stein then Y has a Stein open neighborhood U such that Y is a strong deformation retract of U.*

The theorem below is also an extension of Siu's theorem. It was proved by F. Forstnerič in [7]. For the form presented here see [8], Theorem 3.2.1, page 62.

Theorem. *Let X be a complex space, $K \subset X$ be a compact subset, and $Z \subset X$ be a closed complex subspace. We assume that the following conditions are satisfied:*

i) Z is Stein and $Z \cap K$ is holomorphically convex in Z,

ii) K has a Stein neighborhood U such that K is holomorphically convex in U.

Then $Z \cup K$ has a Stein neighborhood Ω such that K is holomorphically convex in Ω.

Remark 1. Let X, K, U, and Z as in the statement of the above theorem. We assume also that $\overset{\circ}{K}$ is Stein and Runge in U and let Ω be a Stein neighborhood of $K \cup Z$ such that K is holomorphically convex in Ω. Then $\overset{\circ}{K}$ is Runge in Ω. Indeed, as K is holomorphically convex in U, it has a fundamental system of open neighborhoods that are Runge in U. Let U_1 be a Runge open subset of Ω such that $K \subset U_1 \subset U$ (it exists because K is holomorphically convex in Ω). Because $\overset{\circ}{K}$ is Runge in U, it follows that it is Runge in U_1 and hence it is Runge in Ω.

This implies that we can rewrite Forstnerič's theorem as follows:

Theorem 4. *Let X be a complex space, $K \subset X$ be a compact subset, and $Z \subset X$ be a closed complex subspace. We assume that the following conditions are satisfied:*

i) $\overset{\circ}{K}$ is Stein,

ii) Z is Stein and $Z \cap K$ is holomorphically convex in Z,

iii) K has a Stein neighborhood U such that K is holomorphically convex in U and $\overset{\circ}{K}$ is Runge in U.

Then $Z \cup K$ has a Stein neighborhood Ω such that K is holomorphically convex in Ω and $\overset{\circ}{K}$ is Runge in Ω.

The next theorem was proved by M. Colţoiu in [4].

Theorem 5. *Suppose that X is a Stein space, $Y \subset X$ is a closed complex subspace, $D \subset Y$ is a Runge open subset, $K \subset X$ a holomorphically convex compact subset such that $K \cap Y \subset D$ and $V \subset X$ is an open subset such that $D \cup K \subset V$. Then there exists a Runge open subset $\widetilde{D}$ in X with $\widetilde{D} \cap Y = D$ and $K \subset \widetilde{D} \subset V$.*

Remark 2. The statement of Theorem 3 in [4] does not require that $\widetilde{D} \subset V$, however it follows easily from the proof presented there that this condition can be achieved as well. Also the concluding remark of [4] refers to Stein spaces of finite embedding dimension. If X is any Stein space one can choose an exhaustion of X with relatively compact Runge domains and reduce the problem to the finite embedding dimension case.

Definition 1. A sequence, $\{D_n\}_{n\geq 1}$, of disjoint Runge open sets in a Stein space X is called uniformly Runge if for every sequence of positive real numbers $\{\varepsilon_n\}_{n\geq 1}$, every sequence of compact sets $\{K_n\}_{n\geq 1}$, $K_n \subset D_n$, and every sequence of holomorphic functions $\{f_n\}_{n\geq 1}$, $f_n \in \mathcal{O}(D_n)$, there exists $f \in \mathcal{O}(X)$ such that for every $n \geq 1$ we have $\|f - f_n\|_{K_n} < \varepsilon_n$.

Definition 2. A sequence $\{K_n\}_{n\geq 1}$ of disjoint compact sets in a Stein space X is called holomorphically separated if there exist $f \in \mathcal{O}(X)$ and an increasing sequence of real numbers $\{\alpha_n\}_{n\geq 0}$ such that $\lim \alpha_n = \infty$ and for each $n \geq 1$ one has $\alpha_{n-1} < \mathrm{Re(f_{|K_n})} < \mathrm{\alpha_n}$.

Remark 3. If $\{K_n\}_{n\geq 1}$is a sequence of disjoint holomorphically convex compact sets in a Stein space X and if there exist an increasing sequence of real numbers $\{\alpha_n\}_{n\geq 0}$ and a holomorphic function $f \in \mathcal{O}(X)$ such that $\alpha_{n-1} < \mathrm{Re(f_{|K_n})} < \mathrm{\alpha_n}$ for each $n \geq 1$, then $\bigcup K_n$ has a fundamental system of Runge neighborhoods. This follows from Lemma 1 in [13]. However, if $\lim \alpha_n \neq \infty$, $\{K_n\}_{n\geq 1}$ might not be holomorphically separated. This was noticed in [10].

Propositions 1 and 2 were proved in [10].

Proposition 1. *Suppose that $\{D_n\}_{n\geq 1}$ is a sequence of disjoint Runge domains in a Stein space X. The following are equivalent:*

i) $\{D_n\}_{n\geq 1}$ is uniformly Runge

ii) Every sequence of compact sets $\{K_n\}_{n\geq 1}$, $K_n \subset D_n$, is holomorphically separated.

Proposition 2. *A sequence $\{K_n\}_{n\geq 1}$ of disjoint holomorphically convex compact sets in a Stein space X is holomorphically separated if and only if there exists an exhaustion $\{P_n\}_{n\geq 1}$ of X with holomorphically convex compact subsets such that the following conditions hold:*

a) $\overset{\circ}{P}_n$ is Runge in X for all n,

b) $K_n \subset \overset{\circ}{P}_n$ for all n,

c) For every j and n, $j > n$, we have that $P_n \cap K_j = \emptyset$,

d) For every finite set $S \subset \{n+1, n+2\dots\}$, we have that $P_n \cup \bigcup_{j\in S} K_j$ is holomorphically convex.

Remark 4. In [10] Propositions 1 and 2 were stated for $\mathbb{C}^n$. However exactly the same proofs work for any Stein space. Also Proposition 2 is stated with $K_n \subset P_n$ instead of $K_n \subset \overset{\circ}{P}_n$ and without asking that $\overset{\circ}{P}_n$ is Runge. However, it is easy to see that one can slightly modify the proof in order to satisfy these two conditions.

Theorem 6 was proved in [12].

Theorem 6. *If X is a Stein space and $\phi : X \to \mathbb{R}$ is a continuous plurisubharmonic exhaustion function then, for every $\alpha \in \mathbb{R}$, the open set $\{x \in X : \phi(x) < \alpha\}$ is Runge in X and the compact set $\{x \in X : \phi(x) \leq \alpha\}$ is holomorphically convex.*

Assuming that in the main result of [2] the complex space X is Stein one obtains the following theorem.

Theorem 7. *Let X be a Stein space. Then there exists $\phi : X \to \mathbb{R}$ a smooth strictly plurisubharmonic exhaustion function such that the set of local minima of ϕ is discrete in X.*

Remark 5. Suppose that X is a Stein space and $\phi : X \to \mathbb{R}$ is a continuous plurisubharmonic function. If x_0 is an interior point of $\{x \in X : \phi(x) = \alpha\}$ such that $\phi(x_0) = \alpha$ it follows that x_0 is a local maximum point of ϕ and therefore ϕ is constant on a neighborhood of x_0. In particular every point in this neighborhood is a local minimum. It follows that if the set of local minima of ϕ is discrete then interior of $\{x \in X : \phi(x) \leq \alpha\}$ is precisely $\{x \in X : \phi(x) < \alpha\}$. Hence we have:

Corollary 1. *Every Stein space X has an exhaustion $\{K_n\}$ with holomorphically convex compact sets such that each $\overset{\circ}{K}_n$ is Runge in X.*

We recall the following well-known due to J. H. C. Whitehead, [15].

Theorem 8. *A CW-complex X is contractible if and only if $\pi_n(X) = 0$ for every n.*

Recall that a topological space X has covering dimension (topological dimension) $\leq n$ if any finite covering $\mathcal{U}$ has a refinement $\mathcal{V}$ such that for any $V_1, \ldots, V_{n+2} \in \mathcal{V}$, $V_i \neq V_j$ for $i \neq j$, we have $\cap_{j=1}^{n+2} V_j = \emptyset$. The covering dimension of X is n if it is $\leq n$ but not $\leq n-1$. Also recall that a complex space X (countable at infinity) of bounded dimension $\dim X = n < \infty$ has covering dimension $2n$.

The following theorem is due to P. A. Ostrand. For a proof see, for example, [5] page 228.

Theorem 9. *A normal topological space X has covering dimension at most n if and only if any locally finite covering $\mathcal{U} = \{U_s\}_{s\in S}$ can be written as the union of $n+1$ families $\mathcal{V}_1, \ldots, \mathcal{V}_{n+1}$ such that $\mathcal{V}_j = \{V_{j,s}\}_{s\in S}$, $V_{j,s} \subset U_s$ and $V_{i,s} \cap V_{j,s} = \emptyset$ for $i \neq j$.*

Remark 6. If X is a complex space and $\{x_k\}_{k\in\mathbb{N}}$ is a discrete set, it is not possible in general to find $\gamma_k : [0,1] \to X$ continuous paths with $\gamma_k(0) = x_k$ and $\gamma_k(1) = x_{k+1}$ such that the sequence $\{\gamma_k([0,1])\}_{k\in\mathbb{N}}$ is locally finite. For example if $X = \mathbb{C}^*$ and $x_{2n} = 2n$, $x_{2n+1} = \frac{1}{2n+1}$ then for any continuous path with γ with $\gamma(0) = x_k$ and $\gamma(1) = x_{k+1}$ we have that $\gamma([0,1]) \cap S^1 \neq \emptyset$. The same type of of construction works in every manifold with at least two ends.

Instead we can connect the points $\{x_k\}_{k\in\mathbb{N}}$ using a locally finite connected graph and this is the content of the following lemma.

Lemma 1. *Suppose that X is a connected complex space and $A \subset X$ is a discrete set. Then there exists $\mathcal{E}$, a countable collection of continuous 1-1 paths $\gamma : [0,1] \to X$, such that:*
- for each $\gamma \in \mathcal{E}$ we have that $\gamma(0), \gamma(1) \in A$ and for each $a \in A$ there exist $\gamma \in \mathcal{E}$ with $a \in \{\gamma(0), \gamma(1)\}$
- $G := \bigcup_{\gamma\in\mathcal{E}} \gamma([0,1])$ is connected,
- $\{\gamma([0,1]) : \gamma \in E\}$ is locally finite.

Proof. For a compact set K we denote by $co(K)$ the union of K with all relatively compact connected components of $X \setminus K$. It follows that $co(K)$ is also compact. Note that if K is a compact set such that $K = co(K)$ then $X \setminus K$ has finitely many connected components. Indeed if F is another compact subset of X such that $K \subset \overset{\circ}{F}$ then the union of all connected component of $X \setminus K$ covers ∂F and each one of these components intersects ∂F.

We consider $\{K_n\}$ an exhaustion of X with compact subsets such that $K_n = co(K_n)$. We will construct $\mathcal{E}$ inductively.

Without loss of generality we can assume that $K_1 \cap A \neq \emptyset$ and let $a \in K_1 \cap A$. For each point $x \in K_1 \cap A$ we choose a continuous 1-1 path joining x and a and we let $\mathcal{E}_0$ be the (finite) set of all these paths.

Let $X_1^1, \ldots, X_{p_1}^1$ be the connected components of $X \setminus K_1$. Note that $X_j^1 \cap (K_2 \setminus K_1) \neq \emptyset$. Without loss of generality we can assume that if $X_j^1 \cap A \neq \emptyset$ for some j then $X_j^1 \cap (K_2 \setminus K_1) \cap A \neq \emptyset$. For each such j we choose $a_j^1 \in X_j^1 \cap (K_2 \setminus K_1) \cap A$. We consider continuous 1-1 paths joining a_j^1 and a and for each $x \in X_j^1 \cap A \cap (K_2 \setminus K_1)$ we choose a 1-1 path in X_j^1 (hence not intersecting K_1) joining x and a_j^1. We denote by $\mathcal{E}_1$ the (finite) set of all this paths.

Suppose that we have constructed $\mathcal{E}_m$ and the points $a_j^m \in (K_{m+1} \setminus K_m) \cap A$, $j = 1, \ldots, p_m$ and we will construct $\mathcal{E}_{m+1}$. Let $X_1^{m+1}, \ldots, X_{p_{m+1}}^{m+1}$ be the connected components of $X \setminus K_{m+1}$. Without loss of generality we can assume that if $X_j^{m+1} \cap A \neq \emptyset$ then $X_j^{m+1} \cap (K_{m+2} \setminus K_{m+1}) \cap A \neq \emptyset$. For each such j we choose $a_j^{m+1} \in X_j^{m+1} \cap (K_{m+2} \setminus K_{m+1}) \cap A$ and let $l(j) \in \{1, \ldots, p_m\}$ be such that a_j^{m+1} and $a_{l(j)}^m$ are in the same connected component of $X \setminus K_m$. We choose a 1-1 path joining a_j^{m+1} and $a_{l(j)}^m$ that does not intersect K_m. For each $x \in X_j^{m+1} \cap A \cap (K_{m+2} \setminus K_{m+1})$ we choose a 1-1 path in X_j^{m+1} joining x and a_j^{m+1}. We let $\mathcal{E}_{m+1}$ be the set of all these paths.

We set $\mathcal{E} = \bigcup_{m \geq 0} \mathcal{E}_m$. It is clear that $\mathcal{E}$ has the required properties. □

3 Connected Stein coverings for irreducible complex spaces

In this section we prove that every irreducible complex space X can be written as a finite union of contractible Stein open subsets. Let us say first a few words about the idea of the proof. As X is irreducible, $\mathrm{Reg}(X)$ is connected and for it we can apply Fornaess and Stout result (Theorem 2). This means that we have to cover $\mathrm{Sing}(X)$ with finitely many contractible Stein open subsets of X. For simplicity let's consider the case in which $\mathrm{Sing}(X)$ is smooth. It may have however infinitely many connected components. Let $\mathrm{Sing}(X) = \bigcup_{j \in \mathbb{N}} Y_j$ be the decomposition of $\mathrm{Sing}(X)$ into connected components. Hence each Y_j is a connected complex manifold of dimension $< \dim(X)$. By Theorem 2 each Y_j can be written as $Y_j = \bigcup_{k=1}^{\lambda} Q_j^k$ where $\lambda = \lambda(\dim(X) - 1)$ is the bound given by Theorem 2 and Q_j^k are biholomorphic to polydisks. For each fixed $k \in \{1 \ldots, \lambda\}$ we want to show that there exists a contractible open Stein subset U_k of X such that $U_k \supset \bigcup_{j \in \mathbb{N}} Q_j^k$. To do this we construct an countable set $\{D_p^k\}$ of holomorphic disks is X (see bellow what we mean by that) such that $Z_k := \bigcup_{j \in \mathbb{N}} Q_j^k \cup \bigcup_{p \in \mathbb{N}} D_p^k$ is a contractible, *locally closed* complex subspace of X. This means that Z_k is a closed complex subspace of an open subset V_k of X. Once we constructed Z_k, we can apply Theorem 3 to find U_k, a contractible open Stein subset of X such that $Z_k \subset U_k \subset V_k$.

By a holomorphic disk D in a complex space X we will mean a 1-dimensional locally closed and locally irreducible analytic subset of X whose normalization is the unit disk Δ in $\mathbb{C}$. In particular D is contractible.

Lemma 2. *Let $A \subset \mathbb{C}^N$ be an analytic subset of pure dimension n and $x_0 \in A$. We denote by $\pi : \mathbb{C}^N \to \mathbb{C}^n$ the projection on the first n coordinates and let $P \subset \mathbb{C}^N$ be an open polydisk with $x_0 \in P$. We assume that:*
a) $\pi_{|P \cap A} : P \cap A \to \mathbb{C}^n$ is a branched covering;
b) $\pi^{-1}(\pi(x_0)) \cap P \cap A = \{x_0\}$.
Then $P \cap A$ is connected.

Proof. We assume that $P \cap A$ is not connected and therefore $P \cap A = A_1 \cup A_2$ where A_1 and A_2 are disjoint, open and closed subsets of $P \cap A$. Then A_1 and A_2 and closed analytic subsets of $P \cap A$. It follows that $\pi(A_1)$ and $\pi(A_2)$ are closed analytic subsets of $\pi(P)$ which is a polydisk in $\mathbb{C}^n$, and hence connected. This implies that $\pi(A_1) = \pi(A_2) = \pi(P)$. In particular $\pi(x_0) \in \pi(A_1)$ and $\pi(x_0) \in \pi(A_2)$. As $\pi^{-1}(\pi(x_0)) \cap P \cap A = \{x_0\}$, we deduce that $x_0 \in A_1 \cap A_2$ which contradicts the fact that A_1 and A_2 are disjoint. □

Remark 7. In the previous Lemma we did not assume A to be irreducible. It follows from the proof that if a) and b) are satisfied then every irreducible component of $P \cap A$ passes thorough x_0.

Lemma 3. *Suppose that X is a complex space and $x_0 \in X$ is a point such that* $\dim_{x_0}(X) \geq 1$. *Then x_0 has an open neighborhood U such that for every point $x \in U$ there exists a holomorphic disk $D \subset U$ with $x_0, x \in D$.*

Proof. The statement is local so we can assume that X is a closed analytic subset of an open subset W in $\mathbb{C}^N$. Also, as it suffices to prove the statement for each irreducible component of X, we may assume that X is pure dimensional and let $n = \dim(X)$

First we prove that there exist a system of coordinates on $\mathbb{C}^N$, a polydisk P centered at x_0, and $X_1, \ldots X_{n-1}$ closed analytic subsets of X such that if we denote by $\pi^{(j)} : \mathbb{C}^N \to \mathbb{C}^j$ the projection on the first j coordinates and set $X_n := X$ then we have that

1. X_j is of pure dimension j or empty, and if $X_j = \emptyset$ then $X_k = \emptyset$ for $k < j$,

2. If $X_j \neq \emptyset$ then $x_0 \in X_j$ and $\pi^{(j)}_{|P\cap X_j} : P \cap X_j \to \mathbb{C}^j$ is a branched covering,

3. X_{j-1} is the branch locus of $\pi^{(j)}_{|P\cap X_j} : P \cap X_j \to \mathbb{C}^j$

4. If $X_j \neq \emptyset$, then the fiber of $\pi^{(j)}_{|P\cap X_j}$ above $\pi^{(j)}_{|P\cap X_j}(x_0)$ is $\{x_0\}$.

We choose first a system of coordinates on $\mathbb{C}^N$ and a polydisk P centered at x_0 such that the properties 2 and 4 are satisfied for $X_n = X$. Let $P = P' \times P''$ where P' is a polydisk in $\mathbb{C}^n$ and P'' is a polydisk in $\mathbb{C}^{N-n}$. Let $X_{n-1} \subset P\cap X_n$ be the branch locus of $\pi^{(n)}_{|P\cap X_n} : P\cap X_n \to \mathbb{C}^n$. Then, by The Purity of Branch Locus Theorem, X_{n-1} has pure dimension $n-1$ and let $Y_{n-1} = \pi^{(n)}(X_{n-1})$ which is a closed analytic subset of $P'$of pure dimension $n-1$. We consider a linear change of coordinates $\chi : \mathbb{C}^n \to \mathbb{C}^n$ and a polydisk P_1 centered at $\pi^{(n)}(x_0)$, $P_1 \subset P'$ such that the projection on the first $(n-1)$ coordinates in this new system induces a branched covering $P_1 \cap Y_{n-1} \to \mathbb{C}^{n-1}$ and the fiber above the projection of $\pi^{(n)}(x_0)$ is just $\pi^{(n)}(x_0)$.

Note now that χ induces a linear change of coordinates on $\mathbb{C}^N$. Namely $(\chi \circ \pi^{(n)}, \pi'') : \mathbb{C}^N \to \mathbb{C}^N$ where π'' is the projection on the last $N-n$ coordinates. In this new coordinate system the projection on the first n coordinates has the same branch locus as $\pi^{(n)}$, i.e. X_{n-1}. At the same time $P_1 \times P''$ is a polydisk in the new system of coordinates and X_n and X_{n-1} satisfy properties 1, 2, 3 and 4. We continue this construction and we obtain $X_{n-2}, \dots, X_1$. Of course, if at some step π^j is an unbranched covering then $X_k = \emptyset$ for $k < j$.

Note that by Lemma 2 we have that $P \cap X_j$ is connected for every j and every irreducible component of $P \cap X_j$ that passes thorough x_0.

Let $j_0 = \min\{j : X_j \neq \emptyset\}$. We will prove by induction on $j \geq j_0$ that for every point $x \in P\cap X_j$ there exists a holomorphic disk $D \subset P \cap X_j$ with $x_0, x \in D$.

If $j_0 = 1$ then all we have to do is to work with every irreducible component of $P \cap X_1$ and to consider its normalization. If $j_0 \geq 2$ then π^j is an unbranched covering and we consider a lifting of a disk in $\pi^j(P)$ (which is also a polydisk).

Suppose that we have proved the statement for X_{j-1} and we would like to prove it for X_j. Let $x \in X_j$. If $x \in X_{j-1}$ then the existence of D follows from the induction hypothesis. Suppose now that $x \in X_j \setminus X_{j-1}$.

Let L be the complex line in $\mathbb{C}^j$ passing through $\pi^{(j)}(x_0)$ and $\pi^{(j)}(x)$. Let D_0 be a connected and simply connected open subset of $L \cap \pi^{(j)}(P)$ with $\pi^{(j)}(x_0), \pi^{(j)}(x) \in D_0$ and $D_0 \cap \pi^{(j)}(X_{j-1}) = \{\pi^{(j)}(x_0)\}$. It follows that $\pi^{(j)}$ induces a unramified covering with finitely many sheets $(\pi^{(j)})^{-1}(D_0 \setminus \{\pi^{(j)}(x_0)\}) \to D_0 \setminus \{\pi^{(j)}(x_0)\}$. Since a connected finite unramified covering of a pointed disk is also a pointed disk it follows that every connected component of $(\pi^{(j)})^{-1}(D_0 \setminus \{\pi^{(j)}(x_0)\})$ is biholomorphic to $\Delta \setminus \{0\}$. Let D^* be the connected component that contains x and let $f : \Delta \setminus \{0\} \to D^*$ be a biholomorphism. As $D^* \subset P$ which is relatively compact in $\mathbb{C}^N$ it follows that f extends to a holomorphic function $f : \Delta \to X$. We have that $\pi^{(j)}(f(0)) = \pi^{(j)}(x_0)$ and as $(\pi^{(j)})^{-1}(\pi^{(j)}(x_0)) = \{x_0\}$ we deduce that $f(0) = x_0$. At the same time $x_0 \notin D^*$ and therefore $f : \Delta \to X$ is injective. It follows that $f(\Delta)$ is a holomorphic disk containing x_0 and x.

□

Proposition 3. *Every irreducible complex space X of dimension 1 can be written as the union of at most 6 contractible Stein open subsets.*

Proof. Because X is irreducible, $\mathrm{Reg}(X)$ is a connected smooth complex curve. Hence it can be written as the union of $\lambda(1) = 3$ disks, $\mathrm{Reg}(X) = \Delta_1 \cup \Delta_2 \cup \Delta_3$ and therefore we have to deal only with the singular locus.

As $\dim(X) = 1$, we have that $\mathrm{Sing}(X)$ is just a discrete set. We write $\mathrm{Sing}(X) = \{x_j : j \in J\}$, the index set J being at most countable. Let $\pi : \widetilde{X} \to X$ be the normalization of X. For each $j \in J$ we choose $a_j \in \pi^{-1}(x_j)$ and we denote by $\{b_j^k : k \in I_j\} := \pi^{-1}(x_j) \setminus \{a_j\}$. Note that I_j is finite and could be empty. For each b_j^k we choose $D_j^k \Subset \Omega_j^k \Subset \widetilde{X}$ open disks such that $\overline{\Omega}_j^k$,

$j \in J, k \in I_j$, are pairwise disjoint and the family $\{\overline{\Omega}_j^k\}_{j\in J,k\in I_j}$ is locally finite. In particular $\bigcup_{j\in J,k\in I_j} \overline{\Omega}_j^k$ is closed in $\widetilde{X}$. Note that $\pi_{|D_j^k}$ is a homeomorphism onto an open subset of a local irreducible component of X at x_j. Its image is not however open in X.

We have then that $\widetilde{X} \setminus \bigcup_{j\in J,k\in I_j} \overline{\Omega}_j^k$ is a connected smooth complex curve and therefore it can be written as the union of three disks $U_1 \cup U_2 \cup U_3$. Let $A_1 := \{j \in J : a_j \in U_1\}$ and similarly we define A_2 and A_3. Let $V_1 = U_1 \cup \left(\bigcup_{j\in A_1,k\in I_j} D_j^k\right)$ and $W_1 = \pi(V_1)$. It is easy to see that W_1 is open. Note that W_1 is obtained as follows: we start with $\pi(U_1)$ which is homeomorphic to U_1 and hence is a contractible complex space, in $\pi(U_1)$ we have a discrete set of points, namely A_1, and at each of these points, a_j, we attach a finitely many, pairwise disjoint contractible complex spaces, namely $\pi(D_j^k)$, $k \in I_j$. It clear that in this way we obtain a contractible Stein complex space.

Similarly we define W_2 and W_3 and we have that $\text{Sing}(X) \subset W_1 \cup W_2 \cup W_3$. Therefore $X = \Delta_1 \cup \Delta_2 \cup \Delta_3 \cup W_1 \cup W_2 \cup W_3$.

□

Theorem 10. *For every $n \in \mathbb{N}$ there exists a positive number $\mu(n) < \infty$ such that every irreducible complex space X with $\dim(X) = n < \infty$ can be written as the union of at most $\mu(n)$ contractible Stein open subsets.*

Proof. If $\dim(X) = 1$, the statement is the content of Proposition 3. We assume then that $\dim(X) \geq 2$.

By Theorem 2, $\text{Reg}(X)$ (which is a connected complex manifold) can be written as the union of $\lambda(n)$ open sets each one of them biholomorphic to a polydisk, and therefore Stein and contractible. We have therefore only to cover $\text{Sing}(X)$ with finitely many contractible Stein open subsets of X, their number depending only on $\dim(X)$.

We write $\text{Sing}(X)$ as $\text{Sing}(X) = A_1 \cup A_2 \cup \cdots \cup A_k$, $k \leq n$, where $A_1 = \text{Reg}(\text{Sing}(X))$, $A_2 = \text{Reg}(\text{Sing}(\text{Sing}(X)))$, and so forth. Note that $A_1, \ldots, A_k$ are pairwise disjoint complex manifolds and $A_p \cup A_{p+1} \cup \cdots \cup A_k$ is closed in X for each p.

It suffices to cover each A_p with contractible Stein open subsets of X. We fix $p \in \{1, 2, \ldots, k\}$. We consider $\{A_{p,j}\}_{j\in J}$ the connected components of A_p, the index set J being at most countable. We apply Theorem 2 for each $A_{p,j}$ which is a connected complex manifold of dimension less than n and therefore we write $A_{p,j}$ as $A_{p,j} = \bigcup_{l=1}^{\lambda(n)} Q_{j,l}$ where each $Q_{j,l}$ is biholomorphic to a polydisk. We fix l and we want to show that there exists Ω_l a Stein and contractible open subset of X such that $\Omega_l \supset \bigcup_{j\in J} Q_{j,l}$. This, of course, will end the proof. Note that

$$\widetilde{X} := X \setminus \left(\left(\bigcup_{j\in J} (A_{p,j} \setminus Q_{j,l}) \right) \cup A_{p+1} \cup A_{p+2} \cup \cdots \cup A_k \right)$$

is a connected open subset of X, $\widetilde{X} \supset \bigcup_{j\in J} Q_{j,l}$, $Q_{j,l}$ is closed in $\widetilde{X}$ and $\{Q_{j,l}\}_{j\in J}$ is locally finite in $\widetilde{X}$. For each $j \in J$ we choose $x_j \in Q_{j,l}$. It follows that $\{x_j\}_{j\in J}$ is a discrete subset of $\widetilde{X}$.

Note that $\text{Reg}(X) = \text{Reg}(\widetilde{X})$ and therefore $\widetilde{X}$ is irreducible. Let $\pi : \widehat{X} \to \widetilde{X}$ be a desingularization of $\widetilde{X}$. It follows that $\widehat{X}$ is connected. For each $j \in J$ we choose $y_j \in \widehat{X}$ such that $\pi(y_j) = x_j$. It follows that $\{y_j\}$ is discrete in $\widehat{X}$.

We apply Lemma 1 for this set and $\widehat{X}$ and we obtain a connected graph $\hat{G}$ whose vertices are $\{y_j\}_{j\in J}$ and whose set of edges we denote with $\hat{\mathcal{E}}$. It is easy to see that any two points in a real analytic connected manifold can be joint be a piecewise real analytic path and therefore we may assume that the edges of $\hat{G}$ are piecewise real analytic. Clearly we may assume no edge is included in $\pi^{-1}(\text{Sing}(\widetilde{X}))$. Let G be the graph in $\widetilde{X}$ whose vertices are $\{x_j\}$ and edges are $\{\gamma = \pi \circ \hat{\gamma} : \hat{\gamma} \in \hat{\mathcal{E}}\}$. It follows that G is connected, its edges are piecewise real analytic and locally finite and none of them is included in $\text{Sing}(\widetilde{X})$. We modify the graph G as follows:

- For an edge γ, if $\gamma(t_0) = \gamma(t_1)$, $0 \leq t_0 < t_1 \leq 1$, we replace γ by the concatenation of $\gamma_{|[0,t_0]}$ and $\gamma_{|[t_1,1]}$ and we may assume that the edges are one-to-one.
- Note that for each edge γ the set $\{t \in [0,1] : \gamma(t) \in \mathrm{Sing}(\widetilde{X})\}$ is finite. If $\gamma(t_0) \in \mathrm{Sing}(\widetilde{X})$ for some $t_0 \in (0,1)$, we add $\gamma(t_0)$ to the set of vertices of G and at the same time we replace γ with $\gamma', \gamma'' : [0,1] \to \widetilde{X}$, $\gamma'(s) = \gamma(st_0)$, $\gamma''(s) = \gamma((1-s)t_0 + s)$.
- If for some $j \in J$ we have that $\gamma(0), \gamma(1) \in Q_{j,l}$, we add $\gamma(1/2)$ to the set of vertices of G and at the same time we replace γ with $\gamma', \gamma'' : [0,1] \to \widetilde{X}$, $\gamma'(s) = \gamma(s/2)$, $\gamma''(s) = \gamma((1+s)/2)$
- For every two edges $\gamma_1, \gamma_2 : [0,1] \to \widetilde{X}$, by real analicity, the set $\{(t_1,t_2) \in [0,1] \times [0,1] : \gamma_1(t_1) = \gamma_2(t_2)\}$ is finite. If for some $t_1, t_2 \in [0,1]$, $(t_1,t_2) \notin \{(0,0),(0,1),(1,0),(1,1)\}$ we have that $\gamma_1(t_1) = \gamma_2(t_2)$ we add $\gamma_1(t_1)$ to the set of vertices of G and at the same we replace γ_1 with $\gamma_1', \gamma_1'' : [0,1] \to \widetilde{X}$ $\gamma_1'(s) = \gamma_1(st_0)$, $\gamma_1''(s) = \gamma_1((1-s)t_0 + s)$ and the same thing for γ_2.
- For every two edges $\gamma_1, \gamma_2 : [0,1] \to \widetilde{X}$, we have that $\#\gamma_1(\{0,1\}) \cap \gamma_2(\{0,1\}) \leq 1$. Otherwise we remove one of them.

After all these transformations we obtain a new connected graph, which we denote also with G that has the following properties:

- its set of vertices is discrete and contains $\{x_j : j \in J\}$,
- the set of edges $\mathcal{E}$ is locally finite,
- for every edge γ we have that $\gamma((0,1)) \cap \mathrm{Sing}(\widetilde{X}) = \emptyset$ and therefore $\gamma((0,1)) \cap Q_{j,l} = \emptyset$
- for every edge γ and every $j \in J$ we have that $\#\gamma(\{0,1\}) \cap Q_{j,l} \leq 1$,
- for every two edges γ_1 and γ_2 we have that $\gamma_1((0,1)) \cap \gamma_2((0,1)) = \emptyset$ and $\#\gamma_1(\{0,1\}) \cap \gamma_2(\{0,1\}) \leq 1$,
- $\left(\bigcup_{\gamma \in \mathcal{E}} \gamma([0,1])\right) \cup \left(\bigcup_{j \in J} Q_{j,l}\right)$ is connected.

We will construct a subset $\mathcal{E}'$ of $\mathcal{E}$ such that $\left(\bigcup_{\gamma \in \mathcal{E}'} \gamma([0,1])\right) \cup \left(\bigcup_{j \in J} Q_{j,l}\right)$ is contractible. We assume that $J = \mathbb{N}$, for J finite the proof being the same.

We will construct inductively $T_k = \left(\bigcup_{\gamma \in A_k} \gamma([0,1])\right) \cup \left(\bigcup_{j \in B_k} Q_{j,l}\right)$ where $A_k \subset \mathcal{E}$ and $B_k \subset \mathbb{N}$ are finite sets such that $A_k \subset A_{k+1}$, $B_k \subset B_{k+1}$ and hence $T_k \subset T_{k+1}$, T_k is contractible, and $\bigcup_{k \in \mathbb{N}} B_k = J = \mathbb{N}$. Once we have constructed $T_k, k \in \mathbb{N}$ with these properties we set $\mathcal{E}' = \bigcup A_k$ and we will have that $\bigcup_{k \in \mathbb{N}} T_k = \left(\bigcup_{\gamma \in \mathcal{E}'} \gamma([0,1])\right) \cup \left(\bigcup_{j \in J} Q_{j,l}\right)$ which clearly is connected. Also it is easy to see that all the homotopy groups of $\bigcup_{k \in \mathbb{N}} T_k$ are trivial and therefore by Theorem 8 it is contractible.

We set $T_0 = Q_{0,l}$ (i.e $A_0 = \emptyset$ and $B_0 = \{0\}$) and obviously T_0 satisfies the required conditions. We assume now that we have constructed T_k, A_k, B_k, and we want to construct T_{k+1}, A_{k+1}, B_{k+1}.

Let i to be smallest positive integer such that there exists a finite sequence of edges in $\mathcal{E}$, $\gamma_1, \gamma_2, \ldots, \gamma_q$ such that:
- $\gamma_1([0,1]) \cap Q_{i,l} \neq \emptyset$, hence $\#\gamma_1([0,1]) \cap Q_{i,l} = 1$, and $\gamma_q([0,1]) \cap T_k \neq \emptyset$,
- $\gamma_s([0,1]) \cap \gamma_{s+1}([0,1]) \neq \emptyset$ for $s = 2, \ldots, q-1$ and, therefore, $\#\gamma_s([0,1]) \cap \gamma_{s+1}([0,1]) = \#\gamma_s(\{0,1\}) \cap \gamma_{s+1}(\{0,1\}) = 1$
- for every $j \in \mathbb{N}$ such that $Q_{j,l} \neq Q_{i,l}$ and $Q_{j,l} \not\subset T_k$ we have that $(\gamma_1([0,1]) \cup \cdots \cup \gamma_q([0,1])) \cap Q_{j,l} = \emptyset$.

Let q_1 be the smallest number q with these property.

We set $A_{k+1} = A_k \cup \{\gamma_1, \ldots, \gamma_{q_1}\}$, $B_{k+1} = B_k \cup \{i\}$ and therefore $T_{k+1} = T_k \cup \gamma_1([0,1]) \cup \cdots \cup \gamma_{q_1}([0,1]) \cup Q_{i,l}$. Clearly T_{k+1} is connected. Note also the following:
- If $1 \leq s_1 < s_2 \leq q_1$ and $s_2 - s_1 \geq 2$ then $\gamma_{s_1}([0,1]) \cap \gamma_{s_2}([0,1]) = \emptyset$. Otherwise $\gamma_1, \ldots, \gamma_{s_1}, \gamma_{s_2}, \ldots, \gamma_{q_1}$ will be a shorter sequence of edges connecting T_k and $Q_{i,l}$, thus contradicting the minimality of q_1.

- If $s \geq 2$ then $T_k \cap \gamma_s([0,1]) = \emptyset$. Otherwise $\gamma_s, \dots, \gamma_{q_1}$ will be a shorter sequence of edges connecting T_k and $Q_{i,l}$.
- If $s \leq q_1 - 1$ then $Q_{i,l} \cap \gamma_s([0,1]) = \emptyset$. Otherwise $\gamma_1, \dots, \gamma_s$ will be a shorter sequence of edges connecting T_k and $Q_{i,l}$.
- $\#T_k \cap \gamma_1 = 1$. Indeed, if $q_1 \geq 2$, above, we have that $\#(T_k \cup \gamma_2([0,1])) \cap \gamma_1([0,1]) = \#(T_k \cup \gamma_2(\{0,1\})) \cap \gamma_1(\{0,1\}) \leq 2$. At the same time $\gamma_1([0,1]) \cap \gamma_2([0,1]) \neq \emptyset$ and $\gamma_2([0,1]) \cap T_k = \emptyset$. If $q_1 = 1$ we argue similarly.

All these facts imply that $Q_{i,l} \cap (T_k \cup (\bigcup_{s=1}^{q_1} \gamma_s([0,1])))$ is just one point and also that $\gamma_s([0,1]) \cap \left(T_k \cup \left(\bigcup_{r=1}^{s-1} \gamma_r([0,1])\right)\right)$ consists of one point for $1 \leq s \leq q_1$. As $Q_{i,l}$ and each edge are contractible it follows that T_{k+1} has a strong deformation retract onto $T_k \cup \gamma_1([0,1]) \cup \dots \cup \gamma_{q_1}([0,1])$, which has a deformation retract onto $T_k \cup \gamma_1([0,1]) \cup \dots \cup \gamma_{q_1-1}([0,1])$ and so forth. We conclude that T_{k+1} has a deformation retract onto T_k which is contractible by the induction hypothesis and therefore T_{k+1} is contractible.

It remains to show that $\left(\bigcup_{j\in\mathbb{N}} Q_{j,l}\right) \subset \left(\bigcup_{k\in\mathbb{N}} T_k\right)$. Note that if k and m are positive integers such that $Q_{m,l}$ is not a subset of T_k (and therefore not intersecting T_k) and if there exists a sequence of edges connecting $Q_{m,l}$ and T_k that do not intersect any other $Q_{j,l}$ which is not a subset of $T_k \cup Q_{m,l}$ then $Q_{m,l} \subset T_{k+m}$. Indeed, otherwise all $Q_{j,l}$ that have been added to T_k in order to obtain T_{k+m} have indices $j < m$ which is a contradiction. Let now j_0 be any positive integer. Let $Q_{j_1,l}, \dots, Q_{j_q,l}$ be such that there exists sequences of edges as above joining $Q_{j_0,l}$ and $Q_{j_1,l}$, $Q_{j_1,l}$ and $Q_{j_2,l}$, ..., $Q_{j_q,l}$ and T_0. This is possible because $\left(\bigcup_{\gamma\in\mathcal{E}} \gamma([0,1])\right) \cup \left(\bigcup_{j\in J} Q_{j,l}\right)$ is connected. Then $Q_{j_q,l} \subset T_{j_q}$, ..., $Q_{j_0,l} \subset T_{j_0+\dots+j_q}$. This concludes the construction of $\{T_k\}$.

For each $\gamma \in \mathcal{E}'$ we choose $\{D_\gamma^1, \dots D_\gamma^k\}$ (k depending on γ) a finite set of holomorphic disks in $\widetilde{X}$ such that $\gamma(0) \in D_\gamma^1$, $\gamma(1) \in D_\gamma^k$, and $D_\gamma^j \cap D_\gamma^{j+1}$ is non-empty and finite. To do this, if $\gamma([0,1]) \subset \mathrm{Reg}(\widetilde{X})$ we have just to cover it with finitely many open subsets of X biholomorphic to an open ball in $\mathbb{C}^n$ and to choose smooth holomorphic disks in these open sets with the required property. If $\gamma(0) \in \mathrm{Sing}(\widetilde{X})$ we apply Lemma 3 and we find a neighborhood of $\gamma(0)$ and a holomorphic disk in this neighborhood that contains $\gamma(0)$ and $\gamma(t_0)$ for $t_0 > 0$ close enough to 0. If $\gamma(1) \in \mathrm{Sing}(\widetilde{X})$ we do the same thing. Note that we are able to find these disks such that $D_\gamma^j \cap D_\gamma^{j+1}$ is finite because we have assumed that $\dim(X) \geq 2$.

By choosing in the construction above small enough open subsets of $\widetilde{X}$, and hence analytic disks D_γ^j included in a small enough neighborhood of $\gamma([0,1])$ we can assume that:
- $D_\gamma^j \cap \mathrm{Sing}(\widetilde{X}) = \emptyset$ for every $j \in \{2 \dots, k-1\}$ and if $\gamma(0) \notin \mathrm{Sing}(\widetilde{X})$ then $D_\gamma^1 \cap \mathrm{Sing}(\widetilde{X}) = \emptyset$ and similarly for $\gamma(1)$,
- if γ' is another edge in $\mathcal{E}'$ and $\{D_{\gamma'}^1, \dots D_{\gamma'}^{k'}\}$, are the corresponding holomorphic disks, then $D_\gamma^j \cap D_{\gamma'}^{j'} = \emptyset$ for $2 \leq j \leq k-1$ and $1 \leq j' \leq k'$. If $\gamma(0) \notin \{\gamma'(0), \gamma'(1)\}$ then $D_\gamma^1 \cap D_{\gamma'}^{j'} = \emptyset$ for $1 \leq j' \leq k'$ and similarly for $\gamma(1)$.

Let $z_j \in D_\gamma^j \cap D_\gamma^{j+1}$ and let $\pi_j : \Delta \to D_\gamma^j$ the normalization map of D_γ^j. Replacing D_γ^j with the $\pi_j(U)$ where U is a connected and simply connected open subset of Δ containing $\pi_j^{-1}(z_j)$ and $\pi_j^{-1}(z_{j-1})$ we can assume that $D_\gamma^j \cap D_\gamma^{j+1} = \{y_j\}$ and $D_\gamma^j \cap D_\gamma^i = \emptyset$ if $i \notin \{j-1, j+1\}$. (Recall that, to start with, $D_\gamma^j \cap D_\gamma^{j+1}$ is finite.) Using the same procedure we can assume that if $\gamma(0) \in \mathrm{Sing}(\widetilde{X})$ then $D_\gamma^1 \cap \mathrm{Sing}(\widetilde{X}) = \{\gamma(0)\}$ and similarly for $\gamma(1)$. Moreover we can assume that if γ' is another edge in $\mathcal{E}'$ and $\gamma(0) = \gamma'(0)$ then $D_\gamma^1 \cap D_{\gamma'}^1 = \{\gamma(0)\}$. Similarly if $\gamma(0) = \gamma'(1)$, $\gamma(1) = \gamma'(0)$, or $\gamma(1) = \gamma'(1)$.

Let $\{D_i : i \in I\}$ the set of all holomorphic disks constructed above, for all $\gamma \in \mathcal{E}'$, the index set I being at most countable. Let

$$Z = \left(\bigcup_{j\in J} Q_{j,l}\right) \cup \left(\bigcup_{i\in I} D_i\right).$$

By shrinking the disks D_i we may assume that if $i_1 \neq i_2 \in I$ then $\partial D_{i_1} \cap D_{i_2} = \emptyset$ where ∂D_{i_1}

denotes the boundary of D_{i_1} in $\widetilde{X}$. At the same time we can assume that, for $i \in I$ and $j \in J$, $\partial D_i \cap Q_{j,l} = \emptyset$. Recall that $Q_{j,l}$ is closed in $\widetilde{X}$.

Remark 8. In general the union of two locally closed subsets A and B in a topological space X is not locally closed. However it is easy to see that if $\partial A \cap B = \emptyset$ and $A \cap \partial B = \emptyset$ then $A \cup B$ is locally closed in X.

It follows that Z is locally closed in $\tilde{X}$ and therefore there exists an open set V of $\widetilde{X}$ such that Z is a closed analytic subspace of V. By construction Z is connected. Also Z is Stein because each irreducible component is Stein.

It follows from the way we constructed the holomorphic disks D_i, Theorem 8, and the fact that $\left(\bigcup_{\gamma \in \mathcal{E}'} \gamma([0,1])\right) \cup \left(\bigcup_{j \in J} Q_{j,l}\right)$ is contractible that Z is contractible as well.

We apply now Theorem 3 and we obtain Ω_l a Stein open subset of $\tilde{X}$ and hence of X such that $\Omega_l \supset Z$ (and, therefore, $\Omega_l \supset \bigcup_{j \in \mathbb{N}} Q_{j,l}$) and Ω_l has a strong deformation retract onto Z. By the invariance under homotopy of the homotopy groups (see, e.g., [9]) and Theorem 8, we have that Ω_l is contractible.

□

4 Connected Stein coverings for connected complex spaces

In this section we want to prove that every connected complex space X with $\dim(X) < \infty$ can be written as the union of finitely many connected Stein open subsets. The idea of the proof is the following. For an explicitly defined positive integer N (that depends only on the dimension of X), and for each $j = 1, \ldots, N$ we construct a countable family $\mathcal{K}_j = \{K_{j,p}\}_{p \in \mathbb{N}}$ of compact subsets of X such that: $\bigcup_{j=1}^{N} \bigcup_{p \in \mathbb{N}} K_{j,p} = X$, each family $\mathcal{K}_j$ is locally finite, for each $j = 1, \ldots, N$ we have that $\{K_{j,p}\}_{p \in \mathbb{N}}$ are pairwise disjoint, and each $K_{j,p}$ is a Stein compactum. For each j we construct Z_j, a pure 1-dimensional locally closed complex subspace of X, such that $Z_j \cup \bigcup_{p \in \mathbb{N}} K_{j,p}$ is connected. To finish the proof, we prove that, under some supplementary conditions on Z_j and $\mathcal{K}_j$, we have that $Z_j \cup \bigcup_{p \in \mathbb{N}} K_{j,p}$ has a Stein neighborhood. Of course, when we construct $\mathcal{K}_j$ and Z_j we have to make sure that they satisfy these supplementary conditions. For one compact set and a closed complex subspace the appropriate setting is given by Theorem 4. For an infinite sequence of compacts we give such conditions in Proposition 4.

Lemma 4. *Suppose that Y is a complex space and X_1 and X_2 are closed complex subspaces such that $X_1 \cap X_2 = \{a\}$. Let $X = X_1 \cup X_2$. Then there exists $p \in \mathbb{N}^*$ such that for every $f_1, g_1 \in \mathcal{O}(X_1)$ with $f_1(a) = 0$ and every $p_1 \geq p$ we have that the continuous function $f : X \to \mathbb{C}$ defined by $f_{|X_1} = f_1^{p_1} g_1$ and $f_{|X_2} = 0$ is holomorphic on X.*

Notation. We denote by $p(X_1, X_2, a)$ the smallest such integer.

Proof. The statement being local we can assume that $Y = \mathbb{C}^n$ and $a = 0$. Let $\{h_j^1 : j \in J_1\}$ be a finite set of generators for $\mathcal{I}(X_1, 0)$ and $\{h_j^2 : j \in J_2\}$ be a finite set of generators for $\mathcal{I}(X_2, 0)$. Let $I \subset \mathcal{O}_{\mathbb{C}^n,0}$ be the ideal generated by $\{h_j^1 : j \in J_1\} \cup \{h_j^2 : j \in J_2\}$. We have then $\mathcal{Z}(I) = \{0\}$ and therefore, by Hilbert Nullstellensatz, $\mathrm{rad}(I)$ is the maximal ideal of $\mathcal{O}_{\mathbb{C}^n,0}$. If we denote by $z_1, \ldots, z_n$ the coordinate functions in $\mathbb{C}^n$ we deduce that there exists $q \in \mathbb{N}^*$ such that $z_1^q, \ldots, z_n^q \in I$. Let $p = nq$ and $p_1 \geq p$. We let also F_1, G_1 be holomorphic extensions for f_1 and g_1 to a neighborhood of $0 \in \mathbb{C}^n$. We have that $F_1^{p_1} G_1 \in I$ and hence $F_1^{p_1} G_1 = \sum_{j \in J_1} a_j^1 h_j^1 + \sum_{j \in J_2} a_j^2 h_j^2$. We set

$$F = F_1^{p_1} G_1 - \sum_{j \in J_1} a_j^1 h_j^1 = \sum_{j \in J_2} a_j^2 h_j^2.$$

It follows that $F_{|X_1} = f_1^{p_1} g_1$, $F_{|X_2} = 0$ and therefore $F_{|X} = f$. It follows that f is holomorphic on X. □

Lemma 5. *Let X be a Stein space, $K \subset X$ a holomorphically convex compact set, $A \subset X$ a closed analytic set such that $A \cap K = \emptyset$, and $p \in \mathbb{N}^*$. Then for every holomorphic function f defined on a neighborhood of K and every $\varepsilon > 0$ there exists $g, h \in \mathcal{O}(X)$ such that $h_{|A} = 0$ and $\|f - gh^p\|_K < \varepsilon$.*

Proof. Since the compact K is holomorphically convex there exists $g \in \mathcal{O}(X)$ with $\|f-g\|_K < \varepsilon/2$. Note that it suffices to find $h \in \mathcal{O}(X)$ such that $h_{|A} = 0$ and $\|g\|_K \cdot \|h^p - 1\|_K < \varepsilon/2$ because then $\|f - gh^p\|_K \le \|f - g\|_K + \|g - gh^p\|_K \le \|f - g\|_K + \|g\|_K \cdot \|h^p - 1\|_K < \varepsilon$.

Let $u \in \mathcal{O}(X)$ such that $u_{|A} = 0$ and $u(x) \neq 0$ for every $x \in K$. Then $\frac{1}{u}$ is holomorphic on a neighborhood of K. Let $v \in \mathcal{O}(X)$ such that $\|v - \frac{1}{u}\|_K < \frac{1}{2^p} \cdot \frac{1}{\|u\|_K} \cdot \frac{\varepsilon}{2}$ and let $h = uv$. We have then that $h_{|A} = 0$. At the same time, we have that $\|h-1\|_K \le \|u\|_K \cdot \|v - \frac{1}{u}\|_K < \frac{1}{2^p} \cdot \frac{\varepsilon}{2}$. In particular, if ε is small enough, $\|h\|_K \le 2$. Therefore we get $\|h^p - 1\|_K \le \|h-1\|_K(\|h\|_K^{p-1} + \cdots + 1) < \frac{\varepsilon}{2}$. □

Lemma 6. *Let X be a Stein space of pure dimension 1. For each $n \in \mathbb{N}$ let X_n be an union of irreducible components of X such that $\dim(X_m \cap X_n) \le 0$ for $m \neq n$. Let X_n^c be the union of the irreducible components of X that are not irreducible components for X_n. We assume that $X_n \cap X_n^c$ is finite for every n. For each n, let Ω_n be an Runge open subset of X_n such that $\overline{\Omega}_n \cap X_n^c = \emptyset$ (in particular Ω_n is open in X). Then $\{\Omega_n\}_{n\in\mathbb{N}}$ is uniformly Runge in X.*

Proof. Let $K_n \subset \Omega_n$, $n \in \mathbb{N}$, be compact sets, ε_n be positive real numbers, and $f_n : \Omega_n \to \mathbb{C}$ be holomorphic functions. We want to find a holomorphic function $f : X \to \mathbb{C}$ such that $\|f - f_n\|_{K_n} < \varepsilon_n$. Clearly we can assume that K_n are holomorphically convex in Ω_n and hence in X_n. For each $n \in \mathbb{N}$, let $A_n = X_n \cap X_n^c$, which is a finite set. We have that $A_n \cap \Omega_n = \emptyset$.

For each $a \in A_n$, let $p(X_n, X_n^c, a)$ be the number given by Lemma 4. Let $p_n = \max p(X_n, X_n^c, a)$ where the maximum is taken over all $a \in A_n$.

By Lemma 5 we can find $g_n, h_n \in \mathcal{O}(X_n)$ such that $h_n(a) = 0$ for every $a \in A_n$ and $\|f_n - g_n h_n^{p_n}\|_{K_n} < \varepsilon_n$. Let $f : X \to \mathbb{C}$ defined by $f_{|X_n} = g_n h_n^{p_n}$. It follows from Lemma 4 that f is holomorphic on X as a locally finite sum of holomorphic functions.

□

Proposition 4. *Let X be a complex space, $Z \subset X$ a closed Stein subspace of X, and $\{K_n\}_{n\ge1}$ a locally finite sequence of disjoint compact subsets of X. We assume that the following conditions are satisfied:*

i) $\mathring{K}_n$ *is Stein for every n,*

ii) Each K_n has a Stein neighborhood U_n such that $\mathring{K}_n$ is Runge in U_n and K_n is holomorphically convex in U_n,

iii) $K_n \cap Z$ is holomorphically convex in Z and $\{K_n \cap Z\}$ is holomorphically separable.

Then $Z \cup \left(\bigcup_{n\ge1} K_n\right)$ has a Stein neighborhood Ω.

Proof. By condition iii), we can apply Proposition 2 and we find an exhaustion of Z with holomorphically convex compact subsets $\{P_n\}_{n\ge1}$ such that $\mathring{P}_n$ is Runge in Z, $K_n \cap Z \subset \mathring{P}_n$, for $j > n$, we have that $P_n \cap K_j = \emptyset$, and $P_n \cup \bigcup_{j\in S}(K_j \cap Z)$ is holomorphically convex for every finite set $S \subset \{n+1, n+2\dots\}$. We choose Q_n compact subsets of Z such that $Q_n \subset \mathring{P}_n$ and $\bigcup_{n\ge1} Q_n = Z$.

We will construct inductively a sequence $\{F_n\}_{n\ge1}$ of compact subsets of X with the following properties:

- $\mathring{F}_n$ is Stein and F_n has a Stein neighborhood V_n such that F_n is holomorphically convex in V_n and $\mathring{F}_n$ is Runge in V_n,

- $F_n \cap Z$ is holomorphically convex in Z and $F_n \cap Z \subset \overset{\circ}{P}_n$,
- $K_n \cup Q_n \subset \overset{\circ}{F}_n$
- $F_n \cap \left(\bigcup_{j \geq n+1} K_j\right) = \emptyset$,
- $F_n \subset \overset{\circ}{F}_{n+1}$, F_n is holomorphically convex in $\overset{\circ}{F}_{n+1}$ and $\overset{\circ}{F}_n$ is Runge in $\overset{\circ}{F}_{n+1}$.

To construct F_1 we apply first Theorem 4 and we find Ω_1 a Stein open subset of X such that $\Omega_1 \supset K_1 \cup Z$ and K_1 is holomorphically convex in Ω_1. It follows from Theorem 5 that we can find D_1, a Runge open subset of Ω_1 such that $D_1 \cap Z = \overset{\circ}{P}_1$, $D_1 \supset K_1$, and $D_1 \cap (\cup_{n \geq 2} K_n) = \emptyset$. By Corollary 1 we can find a holomorphically convex compact subset F_1 of D_1 such that $\overset{\circ}{F}_1$ is Stein and Runge in D_1, and $\overset{\circ}{F}_1 \supset K_1 \cup Q_1$. It follows that $F_1 \cap Z$ is holomorphically convex in Z.

Suppose now that we have constructed F_n and we will construct F_{n+1}. By the hypothesis of the proposition and by the induction hypothesis, there exist U_{n+1} and V_n Stein open subsets of X such that $K_{n+1} \subset U_{n+1}$, $F_n \subset V_n$, K_{n+1} is holomorphically convex in U_{n+1}, $\overset{\circ}{K}_{n+1}$ is Runge in U_{n+1}, F_n is holomorphically convex in V_n and $\overset{\circ}{F}_n$ is Runge in V_n (and F_n satisfies the other induction conditions). As $F_n \cap K_{n+1} = \emptyset$, by shrinking V_n and U_{n+1}, we may assume that $V_n \cap U_{n+1} = \emptyset$. Then $F_n \cup K_{n+1}$ is holomorphically convex in $V_n \cup U_{n+1}$ and its interior is Runge in $V_n \cup U_{n+1}$. At the same time, because $F_n \cap Z \subset P_n$, $F_n \cap Z$ is holomorphically convex in Z, and $P_n \cup (K_{n+1} \cap Z)$ is holomorphically convex in Z it follows that $(F_n \cup K_{n+1}) \cap Z$ is holomorphically convex in Z.

By Theorem 4 there exists Ω_{n+1} a Stein open subset of X such that $\Omega_{n+1} \supset F_n \cup K_{n+1} \cup Z$, $F_n \cup K_{n+1}$ is holomorphically convex in Ω_{n+1} and its interior is Runge in Ω_{n+1}. By Theorem 5 that we can find D_{n+1}, a Runge open subset of Ω_{n+1} such that $D_{n+1} \cap Z = \overset{\circ}{P}_{n+1}$, $D_{n+1} \supset F_n \cup K_{n+1}$, and $D_{n+1} \cap (\cup_{j \geq n+2} K_j) = \emptyset$. As before, we apply Corollary 1 and we can find a holomorphically convex compact subset F_{n+1} of D_{n+1} such that $\overset{\circ}{F}_{n+1}$ is Stein and Runge in D_{n+1}, and $\overset{\circ}{F}_{n+1} \supset F_n \cup K_{n+1} \cup Q_{n+1}$. It is easy to see that F_{n+1} satisfies all the required conditions.

We set $\Omega := \bigcup_{n \geq 1} \overset{\circ}{F}_n$. Because $\overset{\circ}{F}_n$ is Runge in $\overset{\circ}{F}_{n+1}$ for every n we deduce that Ω is Stein. At the same time as $\overset{\circ}{F}_n \supset Q_n$ and $\bigcup_{n \geq 1} Q_n = Z$ we obtain that $Z \subset \Omega$. Clearly $\bigcup_{n \geq 1} K_n \subset \Omega$ and hence $Z \cup \left(\bigcup_{n \geq 1} K_n\right) \subset \Omega$.

□

Remark 9. Since a compact set in a Stein space is holomorphically convex if and only if it has a fundamental system of Runge neighborhoods and having in mind Forstnerič's Theorem, a natural question is if in the statement of the previous proposition one can assume that $\bigcup K_n \cap Z$ has fundamental system of Runge neighborhoods instead of $\{K_n \cap Z\}$ being holomorphically separable. It turns out this is not the case. An example can be constructed using Example 1 in [10] as follows.

Let $\Sigma_n = \{(z_1, z_2) \in \mathbb{C}^2 : z_1(z_2 + \frac{1}{n}) = 1\}$, $n \geq 1$. Note that Σ_n is biholomorphic, via the projection on the first coordinate, to $\mathbb{C}^*$. Let $\{r_n\}$ be an increasing sequence of positive real numbers such that, $r_n > r_{n-1} + 1$, $r_n \to \infty$ and $\{z_2 \in \mathbb{C} : |z_2 + \frac{1}{n}| \leq \frac{1}{r_n}\}$ are disjoint disks in $\mathbb{C}$. Let $\mu_n = \{(z_1, z_2) \in \mathbb{C}^2 : z_1(z_2 + \frac{1}{n}) = 1, |z_1| = 1\}$, $\gamma_n = \{(z_1, z_2) \in \mathbb{C}^2 : z_1(z_2 + \frac{1}{n}) = 1, |z_1| = r_n\}$ and $\sigma_n = \{(z_1, z_2) \in \mathbb{C}^2 : z_1(z_2 + \frac{1}{n}) = 1, 1 \leq |z_1| \leq r_n\}$. It follows that $\mu_n \subset F := \{(z_1, z_2) \in \mathbb{C}^2 : |z_1| \leq 1, |z_2| \leq 2\}$, all γ_n are holomorphically convex and pairwise disjoint, and σ_n is the holomorphically convex hull of $\mu_n \cup \gamma_n$. Moreover $\cup \gamma_n$ has a fundamental system of Runge neighborhoods, see Remark 3. Let $p_n = (r_n - 1, \frac{1}{r_n - 1} - \frac{1}{n}) \in \Sigma_n$. It follows that $p_n \notin F \cup (\bigcup \gamma_n)$ but $p_n \in \sigma_n$. For $r > 0$, we denote by D_r the disk $\{z \in \mathbb{C} : |z| < r\}$. Let $K_n \subset \mathbb{C}^3$, $K_n = \gamma_n \times \overline{D}_1$. Let X be an open neighborhood of $\mathbb{C}^2 \cup (\bigcup K_n)$ such that $\{p_n\} \times \overline{D}_{\frac{1}{n}} \not\subset X$ for any n and let $Z = \mathbb{C}^2$. We have that $\{K_n\}$ satisfies the first two conditions of Proposition 4 and $\bigcup (K_n \cap Z)$ has a fundamental system of Runge neighborhoods. If $V \subset \mathbb{C}^3$ is a Stein neighborhood of $Z \cup (\bigcup K_n)$

then there exists $\epsilon > 0$ such that $F \times \overline{D}_\epsilon \subset V$ and therefore $(\mu_n \cup \gamma_n) \times D_\epsilon \bigcup \sigma_n \times \{0\} \subset V$ for each n. However $(\mu_n \cup \gamma_n) \times \overline{D}_\epsilon \bigcup \sigma_n \times \{0\}$ is a Hartogs figure and, as V is Stein, we deduce that $\sigma_n \times \overline{D}_\epsilon \subset V$ for all n. In particular $\{p_n\} \times \overline{D}_\epsilon \subset V$ for all n. Because we assumed that $\{p_n\} \times \overline{D}_{\frac{1}{n}} \not\subset X$ for any n, it follows that $V \not\subset X$. We conclude that we cannot find a Stein open subset of X that contains $Z \cup (\bigcup K_n)$

Theorem 11. *For every $n \in \mathbb{N}$ there exists a positive number $\mu(n) < \infty$ such that every connected complex space X with $\dim(X) = n < \infty$ can be written as the union of at most $\mu(n)$ connected open Stein subsets.*

Proof. We prove first the following claim.

Claim: If the dimension of each irreducible component of X is at least 2 and A is a discrete subset of X then X can be written as the union of $d(n)(2n+1)$ connected Stein open subsets, each one of them containing A.

Proof of the Claim. Since X has topological dimension $2n$, we apply Theorem 9 and we find a countable covering $\mathcal{U}$ of X with relatively compact open subsets such that $\mathcal{U} = \mathcal{U}_1 \cup \cdots \cup \mathcal{U}_{2n+1}$ and $\mathcal{U}_j = \{U_{j,k}\}_{k \in \mathbb{N}}$ satisfies $U_{j,k} \cap U_{j,l} = \emptyset$ for $k \neq l$. By Theorem 1, each $U_{j,k}$ can be written as the union of (at most) $d(n)$ Stein open subsets, $U_{j,k} = \bigcup_{s=1}^{d(n)} V_{j,k,s}$. Let $\mathcal{V}_{j,s} = \{V_{j,k,s}\}_{k \in \mathbb{N}}$ and $\mathcal{V} = \bigcup_{j=1}^{2n+1} \bigcup_{s=1}^{d(n)} \mathcal{V}_{j,s}$. Note that $V_{j,k,s} \cap V_{j,l,s} = \emptyset$ for $k \neq l$. Replacing each $V_{j,k,s}$ in $\mathcal{V}_{j,s}$ by its connected components we can assume that each $V_{j,k,s}$ is connected, Stein and relatively compact. Since X is paracompact out of $\mathcal{V}$ we can extract a locally finite subcovering.

Summarizing and, for simplicity, changing the indices we have obtained a locally finite open covering $\mathcal{V}$ such that $\mathcal{V} = \bigcup_{j=1}^{N} \mathcal{V}_j$, where $N = (2n+1)d(n)$, $\mathcal{V}_j = \{V_{j,p}\}_{p \in \mathbb{N}}$, each $V_{j,p}$ is connected, Stein and relatively compact, and $V_{j,p} \cap V_{j,q} = \emptyset$ for $p \neq q$.

For each j and p we choose a compact set $K_{j,p} \subset V_{j,p}$ such that:

- $\overset{\circ}{K}_{j,p}$ is Stein, connected, and non-empty,
- $K_{j,p}$ is holomorphically convex in $V_{j,p}$ and $\overset{\circ}{K}_{j,p}$ is Runge in $V_{j,p}$,
- $\bigcup_{j=1}^{N} \bigcup_{p \in \mathbb{N}} \overset{\circ}{K}_{j,p} = X$.

We fix now $j \in \{1, \ldots, N\}$ and we let $A_j = A \setminus \bigcup_{p \in \mathbb{N}} K_{j,p}$. Note that, since $\{K_{j,p}\}_p$ is locally finite we have that $\bigcup_{p \in \mathbb{N}} K_{j,p}$ is closed. Because A is discrete we can replace $V_{j,p}$ with smaller neighborhoods of $K_{j,p}$, denoted also with $V_{j,p}$ and satisfying the same properties, such that $A_j \bigcap \left(\bigcup_{p \in \mathbb{N}} \overline{V}_{j,p} \right) = \emptyset$.

Because $\bigcup_{p \in \mathbb{N}} \overline{V}_{j,p}$ is closed in X, for each $a \in A_j$ we can find a Stein open neighborhood $V_{j,a}$ of a such that $V_{j,a}$ is relatively compact in X, $V_{j,a} \cap V_{j,b} = \emptyset$ for $a \neq b \in A_j$, and $\overline{V}_{j,a} \bigcap \left(\bigcup_{p \in \mathbb{N}} \overline{V}_{j,p} \right) = \emptyset$. Choosing $K_{j,a} \subset V_{j,a}$ and replacing $\mathcal{V}_j$ with $\mathcal{V}_j \cup \{V_{j,a} : a \in A_j\}$ we can assume from the beginning that $A \subset \bigcup_{p \in \mathbb{N}} K_{j,p}$ for each $j = 1, 2, \ldots, N$.

Note now that it suffices to prove that for each $j = 1, \ldots, N$ there exists Ω_j, a Stein connected open subset of X such that $\Omega_j \supset \bigcup_{p \in \mathbb{N}} K_{j,p}$. We fix $j \in \{1, 2, \ldots, N\}$ and for simplicity we write V_p for $V_{j,p}$ and K_p for $K_{j,p}$. We choose D_p a connected Stein open subset of X such that $K_p \subset D_p \Subset V_p$ and D_p is Runge in V_p. For each p we pick a point $x_p \in \overset{\circ}{K}_p$ and we apply Lemma 1 for the discrete set $\{x_p : p \in \mathbb{N}\}$ and the connected complex space X. We obtain the connected graph G with a locally finite set of edges $\mathcal{E}$.

For each edge $\gamma : [0,1] \to X$, $\gamma \in \mathcal{E}$ we do the following construction. Because $\gamma([0,1])$ is compact and the covering $\mathcal{V}$ is locally finite it follows that $\gamma([0,1]) \cap \overline{D}_p \neq \emptyset$ for finitely many indices p. We consider a partition of $[0,1]$, $0 = a_0 < a_1 < a_2 < a_3, \cdots, a_{2k-1} < a_{2k} < a_{2k+1} = 1$ such that:

- for each $i \in \{0, 1, \ldots, k\}$ there exists $p(i) \in \mathbb{N}$ such that $a_{2i}, a_{2i+1} \in V_{p(i)}$

- $\gamma([a_{2i-1}, a_{2i}]) \bigcap (\bigcup_{p\in\mathbb{N}} \overline{D}_p) = \emptyset$ for each $i \in \{1, 2, \ldots, k\}$.

For each $i \in \{1, 2, \ldots, k\}$ we cover $\gamma([a_{2i-1}, a_{2i}])$ with a finite number of connected, relatively compact, open Stein subsets of X, $W_i^1, \cdots, W_i^q$, $q = q(i)$ such that

- $W_i^s \bigcap (\bigcup_{p\in\mathbb{N}} \overline{D}_p) = \emptyset$ for every $s = 1, \ldots, q$,
- $W_i^1 \cap V_{p(i-1)} \neq \emptyset$ and $W_i^q \cap V_{p(i)} \neq \emptyset$,
- $W_i^s \cap W_i^{s+1} \neq \emptyset$ for every $s = 1, \ldots, q-1$.

A particular case of the main result proved by A. Baran in [1] is the following proposition.

Proposition 5. *Suppose that X is a connected complex space and x_1, x_2 are two points in X. Then there exists a connected closed complex subspace Y of X, of pure dimension 1 such that $x_1, x_2 \in Y$.*

We apply this proposition and we choose in each W_i^s a connected closed subspace C_i^s, and in each $V_{p(i)}$ a connected closed subspace C_i, all of them of pure dimension 1, such that

- $C_i^s \bigcap (\bigcup_{p\in\mathbb{N}} \overline{D}_p) = \emptyset$ for every s and i,
- $C_{i_1}^{s_1} \cap C_{i_2}^{s_2}$ is finite (or empty) for $(i_1, s_1) \neq (i_2, s_2)$ and similarly for $C_{i_1} \cap C_{i_2}$ and $C_{i_1}^s \cap C_{i_2}$,
- $C_i^s \cap C_i^{s+1} \neq \emptyset$ for every s and i,
- $C_i^1 \cap C_{i-1} \neq \emptyset$ and $C_i^q \cap C_i \neq \emptyset$,
- $\gamma(0) \in C_0$ and $\gamma(1) \in C_k$.

Let Z be the union of all these 1-dimensional complex spaces C_i^s and C_i, for all edges in $\mathcal{E}$. Because the edges in $\mathcal{E}$ have locally finite images we can assume that $\{W_i^s\}_{i,s,\gamma}$ is locally finite. By shrinking the open sets W_i^s and V_p (and therefore the 1-dimensional complex spaces that form Z) we obtain that for any such two complex spaces C and C' we may assume that $\partial C \cap C' = \emptyset$ where ∂C denotes the boundary of C in X. We deduce that Z is locally closed and therefore there exists an open set Y of X such that $Y \supset Z$ and Z is a closed analytic subspace of Y. It follows from our construction that we can choose Y such that $\overline{D}_p \subset Y$ for every p. Clearly Z is connected.

For each $p \in \mathbb{N}$ let Z_p be the union of all irreducible components of Z that intersect $\overline{D}_p$. Let also Z_p^c be the union of the other irreducible components of Z. Note that Z_p satisfies the following properties:

- $D_p \cap Z_p$ is Runge in Z_p,
- $D_p \cap Z_p \Subset Z_p$,
- $\overline{D}_p \cap Z_p^c = \emptyset$,
- $Z_p \cap Z_p^c$ is finite.

From Lemma 6 we deduce that $\{D_p \cap Z_p\}_{p\in\mathbb{N}}$ is uniformly Runge in Z. It follows from Proposition 1 that $\{K_p \cap Z_p\}_{p\in\mathbb{N}}$ is holomorphically separated in Z. At this moment we are in the settings of Proposition 4 and therefore we deduce that there exists a Stein open Ω set that contains $Z \cup \left(\bigcup_{p\in\mathbb{N}} K_p\right)$. Replacing Ω by its connected component that contains $Z \cup \left(\bigcup_{p\in\mathbb{N}} K_p\right)$, we can assume that Ω is connected and our Claim is proved.

To finish the proof of the theorem, we let $\tilde{X}$ to be union of all irreducible components of X of dimension at least 2. Let $\{\widetilde{X}_j\}_{j\in J}$ be the connected components of $\widetilde{X}$ and let $\{X_l\}_{l\in\Lambda}$ be the irreducible components of X of dimension 1. For $j \in J$ we set $A_j := \widetilde{X}_j \cap \left(\bigcup_{l\in\Lambda} X_l\right)$. It follows that A_j is a discrete subset of $\widetilde{X}_j$. We use the Claim proved above and we write each $\widetilde{X}_j$ as $\widetilde{X}_j = \bigcup_{p=1}^N \Omega_{j,p}$ where $\Omega_{j,p}$ is open, Stein, connected, and $\Omega_{j,p} \supset A_j$ for each p.

For each $l \in \Lambda$ we choose $a_l, b_l \in X_l$ such that $a_l \neq b_l$ and a_l and b_l are not in any other irreducible component of X. We let $U_l = X_l \setminus \{a_l\}$ and $V_l = X_l \setminus \{b_l\}$. By removing these points, we do not have any compact irreducible component. It follows that U_l and V_l are Stein and connected and $U_l \cap \Omega_{j,p} = X_l \cap \widetilde{X}_j$, $V_l \cap \Omega_{j,p} = X_l \cap \widetilde{X}_j$.

We deduce that

$$\mathcal{U}_p = \left(\bigcup_{l \in \Lambda} U_l \right) \cup \left(\bigcup_{j \in J} \Omega_{j,p} \right)$$

and

$$\mathcal{V}_p = \left(\bigcup_{l \in \Lambda} V_l \right) \cup \left(\bigcup_{j \in J} \Omega_{j,p} \right)$$

are open, Stein, connected subsets of X and $X = \bigcup_{p=1}^{N} (\mathcal{U}_p \cup \mathcal{V}_p)$. Hence we have covered X with $2d(n)(2n+1)$ connected Stein open subsets.

□

The method we used in the proof of the previous theorem, allows us to give a short proof to the following proposition which is a version for connected complex spaces (instead of irreducible and locally irreducible complex spaces) of Theorem III.1 in [6].

Theorem 12. *For each $n \geq 1$ there exists $N(n) \in \mathbb{N}$ such that for every X, a connected complex space of dimension n, then there exists a connected Stein complex space $\tilde{X}$ and a surjective holomorphic, locally biholomorphic map $\pi : \tilde{X} \to X$ whose fibers have no more than N points.*

Proof. We assume that each irreducible component of X has dimension ≥ 2 because the general case follows easily from this one, as in the proof of Theorem 11.

Let $N = (2n+1)d(n) \geq 3$ and let $\mathcal{V}$ be the locally finite open covering of X constructed in the proof of Theorem 11. We have that $\mathcal{V} = \bigcup_{j=1}^{N} \mathcal{V}_j$, $\mathcal{V}_j = \{V_{j,p}\}_{p \in \mathbb{N}}$. For $j = 1, \ldots, N$ let $\{K_{j,p}\}_{p \in \mathbb{N}}$ and $Z(j)$ be the sets from the proof of Theorem 11.

Clearly we can assume that $(\bigcup_{p \in \mathbb{N}} V_{j,p}) \cup (\bigcup_{p \in \mathbb{N}} V_{j+1,p}) \neq X$ for $j = 1, \ldots, N-1$. We choose $a_j \in X \setminus (\bigcup_{p \in \mathbb{N}} V_{j,p}) \cup (\bigcup_{p \in \mathbb{N}} V_{j+1,p})$ such that $a_i \neq a_j$ for $i \neq j$. It is not difficult to see that we can arrange that $\dim(Z(j) \cap Z(j+1)) = 0$ and $a_j \in Z(j) \cap Z(j+1)$ (simply, in the construction of $Z = Z(j)$, we apply Lemma 1 for the discrete set $\{x_p : p \in \mathbb{N}\} \cup \{a_{j-1}, a_j\}$ instead of $\{x_p : p \in \mathbb{N}\}$). Let $F_j = Z(j) \cup (\bigcup_{p \in \mathbb{N}} K_{j,p})$ which is connected.

We choose B_j an open neighborhood of a_j such that $B_j \cap F_j \cap F_{j+1} = \{a_j\}$, $B_j \cap ((\bigcup_{p \in \mathbb{N}} K_{j,p}) \cup (\bigcup_{p \in \mathbb{N}} K_{j+1,p})) = \emptyset$ and $\overline{B}_i \cap \overline{B}_j = \emptyset$ for $i \neq j$. For each j, we choose M_j an open and connected neighborhood of F_j such that $\partial B_j \cap M_j \cap M_{j+1} = \emptyset$. Let $X_1 := \bigsqcup_{j=1}^{N} M_j / \sim$, where $\sim$ identifies $B_j \cap M_j \cap M_{j+1}$ viewed as a subset of M_j with $B_j \cap M_j \cap M_{j+1}$ viewed as a subset of M_{j+1}. By our assumptions, clearly X_1 is Hausdorff. Let also $\pi_1 : X_1 \to X$ be the map whose restriction to each $M_j \subset X_1$ is the inclusion. Then X_1 is a connected complex space and π_1 is a holomorphic, locally biholomorphic map whose fibers have no more than N points. Let $\tilde{F}_j$, $\tilde{K}_{j,p}$, and $\tilde{Z}(j)$ be the images in $\tilde{X}_1$ of F_j, $K_{j,p}$, and $Z(j)$ (viewed as subsets of M_j), respectively. Note that $\tilde{F} = \bigcup_{j=1}^{N} \tilde{F}_j$ is connected.

Note also that $\bigcup_{j=1}^{N} \tilde{Z}(j)$ and $\bigcup_{j=1}^{N} \bigcup_{p \in \mathbb{N}} \tilde{K}_{j,p}$ satisfy the requirements of Proposition 4 and therefore we can find $\tilde{X} \subset X_1$ an open, Stein, connected neighborhood of $\tilde{F}$. We let $\pi : \tilde{X} \to X$ be the restriction of π_1. What is left to notice is that π is surjective and this holds because $\bigcup_{j=1}^{N} \bigcup_{p \in \mathbb{N}} K_{j,p} = X$.

□

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