# The disk property of coverings of 1-convex surfaces *

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**Abstract**

Let $X$ be an 1-convex surface and $p : \tilde{X} \to X$ an (unbranched) covering map. We prove that if $\tilde{X}$ does not contain an infinite Nori string of rational curves then $\tilde{X}$ satisfies the discrete disk property.

## 1 Introduction

Let $X$ be a 1-convex surface, i.e. a two dimensional complex manifold which is strongly pseudoconvex. Then $X$ is a proper modification of a 2-dimensional normal Stein space at a finite set of points. Let $p : \tilde{X} \to X$ be an (unbranched) covering map. We are interested in this note to study the convexity properties of $\tilde{X}$.

It was remarked in [1] that in general $\tilde{X}$ is not weakly 1-complete (i.e. it might be possible that there is no plurisubharmonic exhaustion function on $\tilde{X}$). This is due to the fact that $\tilde{X}$ might contain an infinite Nori string (necklace), that is a 1-dimensional connected complex subspace which is non-compact and has infinitely many compact irreducible components. However the main result in [1] shows that $\tilde{X}$ can be exhausted by a sequence of relatively compact strongly pseudoconvex domains with smooth boundary ($\tilde{X}$ is $p_3$-convex in the terminology of [4]). In particular $\tilde{X}$ satisfies the continuous Kontinuitätssatz (the continuous disk property).

In this paper we investigate the discrete disk property for $\tilde{X}$. The main result can be stated as follows (see Theorem 2): *If $\tilde{X}$ does not contain an*

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*infinite Nori string of rational curves then $\tilde{X}$ satisfies the discrete disk property.*

It should be noted that the discrete disk property is a much stronger condition than the continuous one.

# 2 Preliminaries

All complex spaces are assumed of bounded dimension and countable at infinity.

Let $X$ be a complex manifold. We recall that $X$ is said to be 1-convex if there exists a $\mathcal{C}^\infty$ function $\phi : X \to \mathbb{R}$ such that:
a) $\phi$ is an exhaustion function (i.e. $\{\phi < c\} \subset\subset X$ for every $c \in \mathbb{R}$),
b) $\phi$ is strongly plurisubharmonic (its Levi form is positive definite) outside a compact subset of $X$.
It is known (see [9]) that this is equivalent to the following condition: there exists a proper surjective holomorphic map $\pi : X \to Y$ where $Y$ is a normal Stein space with finitely many singular points and there exists a finite subset of $Y$, $B$, such that $\pi$ induces a biholomorphism from $X \setminus \pi^{-1}(B)$ to $Y \setminus B$. $\pi^{-1}(B)$ is called the exceptional set of $X$.

A complex space $X$ is called holomorphically convex if for every discrete sequence $\{x_n\}$ in $X$ there exists a holomorphic function $f : X \to \mathbb{C}$ such that $\lim_{n\to\infty} |f(x)| = \infty$.

We denote by $\Delta$ the unit disk in $\mathbb{C}$, $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ and for $\epsilon > 0$ by $\Delta_{1+\epsilon}$ the disk $\Delta_{1+\epsilon} := \{z \in \mathbb{C} : |z| < 1 + \epsilon\}$.

**Definition 1.** *Suppose that $X$ is a complex space. We say that $X$ satisfies the discrete disk property if whenever $f_n : U \to X$ is a sequence of holomorphic functions defined on an open neighborhood $U$ of $\overline{\Delta}$ for which there exists an $\epsilon > 0$ and a continuous function $\gamma : S^1 = \{z \in \mathbb{C} : |z| = 1\} \to X$ such that $\Delta_{1+\epsilon} \subset U$, $\bigcup_{n\geq 1} f_n(\Delta_{1+\epsilon} \setminus \Delta)$ is relatively compact in $X$ and $f_{n|S^1}$ converges uniformly to $\gamma$ we have that $\bigcup_{n\geq 1} f_n(\overline{\Delta})$ is relatively compact in $X$.*

This definition of the disk property using coronae instead of the boundary $\partial\Delta$ of the unit disk is natural as one sees looking at the example $f_n : \mathbb{C} \to \mathbb{C}$, $f_n(z) = z^n$ which is a sequence that converges to 0 if $|z| < 1$ and diverges otherwise.

For $\epsilon > 0$ we define $H_\epsilon \subset \mathbb{C} \times \mathbb{R}$ as

$$H_\epsilon = \Delta_{1+\epsilon} \times [0, 1) \bigcup \{z \in \mathbb{C} : 1 - \epsilon < |z| < 1 + \epsilon\} \times \{1\}.$$

Having in mind the definition of the continuity principle (see for example [7], page 47) we introduce the following:

**Definition 2.** *A complex space $X$ is said to satisfy the continuous disk property if whenever $\epsilon$ is a positive number and $F : H_\epsilon \to X$ is a continuous function such that, for every $t \in [0,1)$, $F_t : \Delta_{1+\epsilon} \to X$, $F_t(Z) = F(z,t)$, is holomorphic we have that $F(H_{\epsilon_1})$ is relatively compact in $X$ for any $0 < \epsilon_1 < \epsilon$.*

Clearly the discrete disk property implies the continuous one but as it is shown by the example of Fornaess (see [6]) of an increasing union of Stein open subsets that does not satisfy the discrete disk property, the discrete disk property is a much stronger condition. (It is not difficult to see that an increasing union of open subsets of a complex space $X$, each one of them satisfying the continuous disk property, satisfies the continuous disk property. The proof is pretty much the same as the proof of Theorem 3.1, page 60 in [7].)

The following theorem was proved in [2] and [8].

**Theorem 1.** *Let $\pi : X \to T$ be a proper holomorphic surjective map of complex spaces, let $t_0 \in T$ be any point, and denote by $X_{t_0} := \pi^{-1}(t_0)$ the fiber of $\pi$ at $t_0$. Assume that $\dim X_{t_0} = 1$. Let $\sigma : \tilde{X} \to X$ be a covering space and let $\tilde{X}_{t_0} = \sigma^{-1}(X_{t_0})$. If $\tilde{X}_{t_0}$ is holomorphically convex, then there exists an open neighborhood $\Omega$ of $t_0$ such that $(\pi \circ \sigma)^{-1}(\Omega)$ is holomorphically convex.*

The next result was proved in [1].

**Proposition 1.** *Let $X$ be an 1-convex manifold with exceptional set set $S$ and $p : \tilde{X} \to X$ any covering. Then there exists a strongly plurisubharmonic function $\tilde{\phi} : \tilde{X} \to [-\infty, \infty)$ such that $p^{-1}(S) = \{\tilde{\phi} = -\infty\}$ and for any open neighborhood $U$ of $S$, the restriction $\tilde{\phi}_{|\tilde{X} \setminus p^{-1}(U)}$ is an exhaustion function on $\tilde{X} \setminus p^{-1}(U)$.*

We recall that a topological space $X$ is called an ENR (Euclidean Neighborhood Retract) if it is homeomorphic to a closed subset $X_1$ of $\mathbb{R}^n$ for some $n$ such that there is an open neighborhood $V$ of $X_1$ and a continuous retract $\rho : V \to X_1$. For basic results regarding ENR's we are referring to [5]. It is proved there that a locally compact and locally contractible subset of $\mathbb{R}^n$ is and ENR and that if a Hausdorff topological space $X$ is covered by countable family of locally compact open subsets each one of them homeomorphic with a subset of fixed Euclidean space then $X$ is an ENR. In particular every complex space of bounded dimension is an ENR.

Lemma 1 was proved in [3].

**Lemma 1.** *If $X$ is an ENR and $\{\gamma_n\}_{n\geq 1}$, $\gamma_n : S^1 :\to X$, is a sequence of null-homotopic loops converging uniformly to $\gamma : S^1 \to X$ then $\gamma$ is null-homotopic. Moreover given a covering $p : \tilde{X} \to X$ there exists liftings $\tilde{\gamma}_n$ for $\gamma_n$ and $\tilde{\gamma}$ for $\gamma$ such that $\tilde{\gamma}_n$ converges uniformly to $\tilde{\gamma}$*

This lemma can be applied in particular if $X$ is a complex space (by the above discussion).

**Definition 3.** *Let $L$ be a connected 1-dimensional complex space and $\cup L_i$ be its decomposition into irreducible components. $L$ is called an infinite Nori string if all $L_i$ are compact and $L$ is not compact*

It is clear that a 1-dimensional complex space is holomorphically convex if and only if it does not contain an infinite Nori string.

We recall that a compact complex curve is called rational if its normalization is $\mathbb{P}^1$.

# 3 The Results

**Lemma 2.** *If $L$ is 1-dimensional complex space that does not contain an infinite Nori string of rational curves then $L$ has a holomorphically convex covering space.*

*Proof.* **Step 1.** We assume that $L$ is connected and all its irreducible components are compact not rational. Let $L = \bigcup_{i\in I} L_i$ be the decomposition of $L$ into irreducible components and $A = \{x \in L : \exists i_1, i_2 \in I,\ i_1 \neq i_2, \text{ such that } x \in L_{i_1} \cap L_{i_2}\}$. Let $p_i : \tilde{L}_i \to L_i$ be the universal cover of $L_i$. Since $L_i$ is not rational for all $i \in I$ we have that $\tilde{L}_i$ is Stein. For each $x \in A \cap L_i$, we choose a bijection $\tau(x,i) : \mathbb{Z} \to p_i^{-1}(x)$ and we write $\tau(x,i)(k) = (k, x, i)$.

On the disjoint union $S := \bigsqcup_{i\in I} \tilde{L}_i$ we define the projection $\tilde{p} : S \to L$ induced by $p_i$, $i \in I$. We also define an equivalence relation on $S$ by defining the equivalence classes as follows: if $s \in S$ is such that $\tilde{p}(s) \notin A$ then $\hat{s} = \{s\}$. Note that on $\tilde{p}^{-1}(A)$ we have a map $\zeta : \tilde{p}^{-1}(A) \to \mathbb{Z}$ which is nothing else then the projection on the first component. Then if $\tilde{p}(s) \in A$ and $\zeta(s) = k$

we set $\hat{s} = \{(k, x, i) : x \in L_i, i \in I, \tilde{p}(s) = x\}$. We let $\tilde{L}$ to be the quotient space of $S$ and $p : \tilde{L} \to L$ the application induced by $\tilde{p}$. It is not difficult to see that $p$ is a covering and $\tilde{L}$ is Stein.

**Step 2.** All irreducible components of $L$ are compact and not rational. In this case we apply the above procedure to each connected component.

**Step 3.** The general case. Let $L'$ be the union of all irreducible components of $L$ that are compact and not rational and $\{C_j\}_{j \in J}$ the connected components of $L'$. For each $j \in J$ let $p_j : \tilde{C}_j \to C_j$ the Stein coverings obtained at the first step. Let also $L''$ be the union of the other irreducible components of $L$. We set $\Gamma_j = L'' \cap C_j$. (Note that $\Gamma_j$ might be empty.) If $\Gamma_j \neq \emptyset$ we consider in $\tilde{C}_j$ infinitely many disjoint copies $\{\Gamma_j^\alpha : \alpha \in \mathbb{Z}\}$ of $\Gamma_j$. (Each $\Gamma_j^\alpha$ is in bijection via $p_j$ with $\Gamma_j$.) We glue now infinitely many copies $\{L''_\alpha : \alpha \in \mathbb{Z}\}$ of $L''$ to $\tilde{C}_j$ on $\{\Gamma_j^\alpha : \alpha \in \mathbb{Z}\}$ and we obtain in this way a holomorphically convex covering of $L$.

□

**Lemma 3.** *Suppose that $\tilde{X}$ and $X$ are Hausdorff topological spaces and $p : \tilde{X} \to X$ is a covering. Let $\tilde{\gamma}_n : S^1 \to \tilde{X}$, $\gamma_n : S^1 \to X$, $n \geq 1$, $\tilde{\gamma} : S^1 \to \tilde{X}$, $\gamma : S^1 \to X$ be continuous functions such that $\gamma_n = p \circ \tilde{\gamma}_n$, $\gamma = p \circ \tilde{\gamma}$ and $a \in S^1$ be a fixed point. If $\gamma_n$ converges uniformly to $\gamma$ and $\tilde{\gamma}_n(a)$ converges to $\tilde{\gamma}(a)$ then $\tilde{\gamma}_n$ converges uniformly to $\tilde{\gamma}$.*

*Proof.* Let $\Omega = \{t \in S^1 : \lim_{n\to\infty} \tilde{\gamma}_n(t) = \tilde{\gamma}(t)\}$. We will show first that $\Omega = S^1$. As $a \in \Omega$ we have that $\Omega \neq \emptyset$. We prove that $\Omega$ is open in $S^1$. Let $t_0 \in \Omega$ and let $V_0$ a neighborhood of $\gamma(t_0)$ which is evenly covered by $\bigcup \tilde{V}_k$. Let $\Omega_0$ a connected open neighborhood of $t_0$ such that for $n$ large enough $\gamma_n(\Omega_0) \subset V_0$ and $\gamma(\Omega_0) \subset V_0$. It follows that $\tilde{\gamma}_n(\Omega_0) \subset \bigcup \tilde{V}_k$. As $\tilde{\gamma}_n(\Omega_0)$ and $\tilde{\gamma}(\Omega_0)$ are connected we have that for each $n$ there exists $k_n$ such that $\tilde{\gamma}_n(\Omega_0) \subset \tilde{V}_{k_n}$ and there exists $k_0$ such that $\tilde{\gamma}(\Omega_0) \subset \tilde{V}_{k_0}$. However we assumed that $t_0 \in \Omega$ and therefore for $n$ large enough $\tilde{\gamma}_n(t_0) \in \tilde{V}_{k_0}$, hence $V_{k_n} = V_{k_0}$. Since $p : V_{k_0} \to V_0$ is a homeomorphism we conclude that $\tilde{\gamma}_n$ converges uniformly to $\tilde{\gamma}$ on $\Omega_0$. This shows that $\Omega_0 \subset \Omega$ and hence $\Omega$ is open.

Next we show in a similar fashion that $\Omega$ is closed. Let $t_0 \in S^1 \setminus \Omega$. As before we choose $V_1$ a neighborhood of $\gamma(t_0)$ which is evenly covered by $\bigcup \tilde{V}_k$, $\Omega_0$ a connected open neighborhood of $t_0$ such that for $n$ large enough $\gamma_n(\Omega_0) \subset V_0$, $k_n$ and $k_0$ such that $\tilde{\gamma}_n(\Omega_0) \subset \tilde{V}_{k_n}$ and $\tilde{\gamma}(\Omega_0) \subset \tilde{V}_{k_0}$. If $V_{k_n} = V_{k_0}$ for every $n$ we would have that $\tilde{\gamma}_n$ converges uniformly to $\tilde{\gamma}$ on $\Omega_0$ hence at

$t_0$ and this would contradict $t_0 \notin \Omega$. There exists then a subsequence $\tilde{\gamma}_{n_p}$ such that $\tilde{\gamma}_{n_p}(\Omega_0) \cap \tilde{V}_{k_0} = \emptyset$ and from here we get that $\Omega_0 \subset S^1 \setminus \Omega$. Hence $\Omega$ is closed and therefore $\Omega = S^1$.

Note that when we proved that $\Omega$ is open we actually proved that each $t_0 \in \Omega = S^1$ has an open neighborhood $\Omega_0$ such that on this neighborhood $\tilde{\gamma}_n$ converges uniformly to $\tilde{\gamma}$. The compactness of $S^1$ implies then that $\tilde{\gamma}_n$ converges uniformly to $\tilde{\gamma}$ on $S^1$. □

**Lemma 4.** *Suppose that $S$ is a complex space that has a holomorphically convex covering $p : \tilde{S} \to S$. If $\Omega \subset \mathbb{C}$ is an open neighborhood of $\overline{\Delta}$ and $f_n : \Omega \to S$ is a sequence of holomorphic functions which converges uniformly on $\{z \in \mathbb{C} : |z| = 1\}$ then $\bigcup_n f_n(\overline{\Delta})$ is relatively compact in $S$. (In particular $S$ satisfies the discrete disk property.)*

*Proof.* Let $\gamma : \{z \in \mathbb{C} : |z| = 1\} \to S$ be the limit of $\{f_n\}$ on $\{z \in \mathbb{C} : |z| = 1\}$. It follows from Lemma 1 that $\gamma$ is null-homotopic. We choose a point $a \in \{z \in \mathbb{C} : |z| = 1\}$. Let $\epsilon$ be a positive number such that $\Delta_{1+\epsilon} \subset \Omega$, $\tilde{f}_n : \Delta_{1+\epsilon} \to \tilde{S}$ and $\tilde{\gamma} : \{z \in \mathbb{C} : |z| = 1\} \to \tilde{S}$ be liftings of $f_n$ and $\gamma$ respectively. We choose $\tilde{f}_n$ and $\tilde{\gamma}$ such that $\tilde{f}_n$ converges uniformly to $\tilde{\gamma}$ on $\{z \in \mathbb{C} : |z| = 1\}$. This is possible by Lemma 1. As $\tilde{S}$ is holomorphically convex this implies that $\bigcup_n \tilde{f}_n(\overline{\Delta})$ is relatively compact in $\tilde{S}$ and therefore $\bigcup_n f_n(\overline{\Delta})$ is relatively compact in $S$. □

**Theorem 2.** *Let $X$ be a 1-convex surface and $p : \tilde{X} \to X$ be a covering map. If $\tilde{X}$ does not contain an infinite Nori string of rational curves then $\tilde{X}$ satisfies the discrete disk property.*

*Proof.* Let $L$ be the exceptional set of $X$. Without loss of generality we may assume that $L$ is connected. Let $Y$ be a normal Stein complex space of dimension 2 and $\pi : X \to Y$ a proper surjective holomorphic map such that $\pi(L) = \{y_0\}$ and $\pi : X \setminus L \to Y \setminus \{y_0\}$ is a biholomorphism. We may assume that $Y$ is an analytic closed subset of $\mathbb{C}^N$.

We put $\tilde{L} := p^{-1}(L)$. Since we assumed that $\tilde{L}$ does not contain an infinite Nori string of rational curves it follows from Lemma 2 that $\tilde{L}$ has a holomorphically convex covering $\hat{p} : \hat{L} \to \tilde{L}$. We choose $U_1$ an open neighborhood of $L$ in $X$ that has a continuous deformation retract on $L$. We let $\tilde{U}_1 = p^{-1}(U_1)$ and then $\tilde{U}_1$ has a continuous deformation retract on $\tilde{L}$, $\rho : \tilde{U}_1 \to \tilde{L}$. By considering the fiber product of $\rho$ and $\hat{p}$ we obtain a covering $\hat{U}_1$ of $\tilde{U}$ such that the pull-back of $\tilde{L}$ is $\hat{L}$. We apply then Theorem 1 and we deduce that there exists $U \subset Y$ an open neighborhood of $y_0$ such that $(\pi \circ p)^{-1}(U)$ has a holomorphically convex covering.

Let $f_n : \Delta_{1+\epsilon} \to \tilde{X}$, $n \geq 1$ be a sequence of holomorphic functions such that $\bigcup_{n\geq 1} f_n(\Delta_{1+\epsilon} \setminus \Delta)$ is relatively compact in $\tilde{X}$ and $f_{n|S^1}$ is uniformly convergent. We argue by contradiction and we assume that $\cup_{n\geq 1} f_n(\overline{\Delta})$ is not relatively compact in $\tilde{X}$ and hence $\cup_{n\geq 1} f_n(\Delta_{1+\epsilon})$ is not relatively compact. By passing to a subsequence we may assume that $\cup_{k\geq 1} f_{n_k}(\Delta_{1+\epsilon})$ is not relatively compact in $\tilde{X}$ for every subsequence $\{f_{n_k}\}_k$ of $\{f_n\}_n$.

Since $\bigcup(\pi \circ p \circ f_n)(\Delta_{1+\epsilon} \setminus \Delta)$ is relatively compact in $Y$, hence in $\mathbb{C}^N$, by the maximum modulus principle we have that $\bigcup(\pi \circ p \circ f_n)(\Delta_{1+\epsilon})$ is relatively compact in $Y$ and then it follows that there exists a subsequence of $\{f_n\}_n$, $\{f_{n_k}\}_k$ such that $\pi \circ p \circ f_{n_k}$ converges uniformly on compacts to a holomorphic function $g : D_\epsilon \to Y$. Without loss of generality we can assume that $\pi \circ p \circ f_n$ converges uniformly on compacts to $g$. We distinguish three cases.

**Case 1.** $g \equiv y_0$. Then $(\pi \circ p \circ f_n)(\overline{\Delta}_{1+\epsilon_1}) \subset U$ for $\epsilon_1 \in (0, \epsilon)$ and $n$ large enough. Therefore we can apply Lemma 4.

**Case 2.** $g(z) \neq y_0$ for every $z \in \Delta_{1+\epsilon}$. Then there exists a neighborhood $V$ of $y_0$ and $\epsilon_1 \in (0, \epsilon)$ such that $g(\overline{\Delta}_{1+\epsilon_1}) \cap \overline{V} = \emptyset$. Then for $n$ large enough we get that $\pi \circ p \circ f_n(\overline{\Delta}_{1+\epsilon_1}) \cap \overline{V} = \emptyset$ and hence $f_n(\overline{\Delta}_{1+\epsilon_1}) \cap (\pi \circ p)^{-1}(V) = \emptyset$. We consider now a plurisubharmonic function $\tilde{\phi}$ on $\tilde{X}$ with the properties given in Proposition 1. In particular its restriction to $\tilde{X} \setminus (\pi \circ p)^{-1}(V)$ is an exhaustion. Applying the maximum principle to $\tilde{\phi} \circ f_n$ we obtain immediately that $\bigcup f_n(\overline{\Delta})$ is relatively compact.

**Case 3.** $g \not\equiv y_0$ and $y_0 \in g(\Delta_{1+\epsilon})$. Then $g^{-1}(y_0)$ is a (non-empty) discrete subset of $\Delta_{1+\epsilon}$ and therefore there exists $\epsilon_1 \in (0, \epsilon)$ such that $g^{-1}(y_0) \cap \{z \in \mathbb{C} : |z| = 1+\epsilon_1\} = \emptyset$. We set $g^{-1}(y_0) \cap \{z \in \mathbb{C} : |z| < 1+\epsilon_1\} = \{\lambda_1, \lambda_2, \ldots, \lambda_s\}$. Let $r_1, \ldots, r_s$ be positive numbers such that the discs $\overline{\Delta}_j = \{z \in \mathbb{C} : |z - \lambda_j| \leq r_j\}$, $j = 1, 2, \ldots, s$ are pairwise disjoint, $\overline{\Delta}_j \subset \Delta_{1+\epsilon_1}$ and $g(\overline{\Delta}_j) \subset U$. Let $V \subset Y$ be an open neighborhood of $y_0$ such that $\overline{V} \subset U$ and $g(\{z \in \overline{\Delta}_{1+\epsilon_1} : |z - \lambda_j| \geq r_j \ \forall j = 1, .., s\}) \cap \overline{V} = \emptyset$. Then for $n$ large enough we have that $(\pi \circ p \circ f_n)(\{z \in \overline{\Delta}_{1+\epsilon_1} : |z - \lambda_j| \geq r_j \ \forall j = 1, .., s\}) \cap \overline{V} = \emptyset$. We apply again Proposition 1 and we obtain a plurisubharmonic function $\tilde{\phi} : \tilde{X} \to [-\infty, \infty)$ such that $\tilde{L} = \{\tilde{\phi} = -\infty\}$ $\tilde{\phi}_{|\tilde{X} \setminus (\pi \circ p)^{-1}(V)}$ is an exhaustion function on $\tilde{X} \setminus (\pi \circ p)^{-1}(V)$. Since $\bigcup f_n(\Delta_{1+\epsilon} \setminus \Delta)$ is relatively compact there exists a positive constant $M$ such that $\tilde{\phi} \circ f_n(z) \leq M$ for every $n$ and every $z \in \Delta_{1+\epsilon} \setminus \Delta$. From the plurisubharmonicity of $\tilde{\phi}$ we have that $\tilde{\phi} \circ f_n(z) \leq M$ for every $z \in \Delta_{1+\epsilon}$, hence in particular for $z \in \overline{\Delta}_{1+\epsilon_1} \setminus \bigcup_{j=1}^{s} \Delta_j$.

As $f_n(\overline{\Delta}_{1+\epsilon_1} \setminus \bigcup_{j=1}^s \Delta_j) \subset \tilde{X} \setminus (\pi \circ p)^{-1}(V)$ and $\tilde{\phi}_{|\tilde{X}\setminus(\pi\circ p)^{-1}(V)}$ is an exhaustion we deduce that $\bigcup f_n(\overline{\Delta}_{1+\epsilon_1} \setminus \bigcup_{j=1}^s \Delta_j)$ is relatively compact in $\tilde{X}$.

For $j = 1, \ldots, s$ we set $S_j = \{z \in \Delta_{1+\epsilon_1} : |z - \lambda_j| = r_j\}$ and we pick a point $a_j \in S_j$. As $\bigcup_{n\geq 1} f_n(S_j)$ is relatively compact, by passing to a subsequence we can assume that each sequence $\{f_n(a_j)\}_n$ is convergent and we denote by $x_j$ its limit. We have that $(\pi \circ p \circ f_n)_{|S_j}$ converges uniformly to $g_{|S_j}$ and $(\pi \circ p \circ f_n)(S_j) \subset Y \setminus \{y_0\}$, $g(S_j) \subset Y \setminus \{y_0\}$. Because $\pi : X \setminus L \to Y \setminus \{y_0\}$ is a biholomorphism we deduce that $(p \circ f_n)_{|S_j}$ converges uniformly to $(\pi^{-1} \circ g)_{|S_j}$. Note now that $(p \circ f_n)_{|S_j}$ is in fact a null-homotopic loop. It follows from Lemma 1 that $(\pi^{-1} \circ g)_{|S_j}$ is null-homotopic as well and therefore there exists a loop $\gamma_j : S_j \to \tilde{X}$ such that $p \circ \gamma_j = (\pi^{-1} \circ g)$ on $S_j$. From Lemma 3 we conclude that $f_n$ converges uniformly to $\gamma_j$ on $S_j$. Finally Lemma 4 implies that $\bigcup f_n(\overline{\Delta}_j)$ is relatively compact for every $j = 1, \ldots, s$. □

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