Convexity properties of intersections of decreasing sequences of q-complete domains in complex spaces *

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Abstract

We construct a decreasing sequence of 3-complete open subsets in $\mathbb{C}^5$ such that the interior of their intersection is not 3-complete. We also prove that, for every $q \geq 2$ there exists a normal Stein space X with only one isolated singularity and a decreasing sequence of open sets that are 2-complete, but the interior of their intersection is not q-complete with corners. In the concave case we show that, for every integer $n > 1$, there exists a connected complex manifold M of dimension n such that M is an increasing union of 1-concave open subsets and M is not weakly $(n-1)$-concave.

1 Introduction

Suppose that $\{D_\nu\}$ is a sequence of open subsets of $\mathbb{C}^n$ and let $D := \mathrm{Int}\,(\bigcap D_\nu)$. If each D_ν is a domain of holomorphy, then D is also a domain of holomorphy. More generally, if each D_ν is Hartogs q-convex (see Definition 6) then D has the same property. However Hartogs q-convexity is not a very useful notion since one does not get vanishing results for the cohomology groups of a Hartogs q-convex domain with values in a coherent sheaf. Andreotti and Grauert [1] introduced the notion of q-complete complex spaces and proved that they are cohomologically q-complete. In their setting, 1-complete spaces are precisely the Stein spaces. In general the intersection of finitely many q-complete domains is not q-complete. Therefore, for $q > 1$, we consider *decreasing* sequences of q-complete open subsets of a Stein space and we want to study the convexity properties of the interior of their intersection.

We prove, by means of a counterexample, that for a decreasing sequence $\{D_\nu\}$ of q-complete domains in $\mathbb{C}^n$, $\mathrm{Int}\,(\bigcap D_\nu)$ is not necessarily q-complete (Theorem 5).

On the other hand, because for domains in $\mathbb{C}^n$, or more generally in Stein manifolds, Hartogs q-convexity is equivalent to q-completness with corners (see [15]), it follows that, in the above setting, $\mathrm{Int}\,(\bigcap D_\nu)$ is q-complete with corners.

*Mathematics Subject Classification (2010): 32F10, 32E10

Key words: Stein space, q-complete space, q-concave space

We show that a similar statement does not hold for singular complex spaces. Namely, for each $q \geq 2$, we give an example of a normal Stein X space with only one singular point and a decreasing sequence $\{D_\nu\}$ of 2-complete domains in X, such that $\mathrm{Int}\,(\bigcap D_\nu)$ is not q-complete with corners (Theorem 6).

As a dual statement, in the concave case, we show that for every integer $n > 1$ there exists a connected complex manifold M of dimension n such that M is an increasing union of 1-concave open subsets and is not weakly $(n-1)$-concave (Theorem 8).

2 Decreasing sequences of q-complete domains

Definition 1. Suppose that D is an open subset of $\mathbb{C}^n$. A smooth function $\varphi : D \to \mathbb{R}$ is called weakly q-convex if its Levi form $\sum_{j,k=1}^{n} \frac{\partial^2 \varphi}{\partial z_j \partial \bar{z}_k}(p)\xi_j \overline{\xi}_k$ has at least $n-q+1$ nonnegative eigenvalues at every point $p \in D$. The function φ is called strictly q-convex if its Levi form has at least $n-q+1$ positive eigenvalues at every point $p \in D$.

Using local embeddings these notions can be extended to complex spaces.

Definition 2. Suppose that X is a complex space and q a positive integer:
(a) The space X is called q-convex if there exists a continuous exhaustion function $\varphi : X \to \mathbb{R}$ (i.e., $\{x \in X : \varphi(x) < c\} \Subset X$ for every $c \in \mathbb{R}$) and a compact set $K \subset X$ such that φ is strictly q-convex on $X \setminus K$.
(b) If we can choose $K = \emptyset$ in the above definition, X is called q-complete.

Definition 3. If $H^p(X, \mathcal{F}) = 0$ for every coherent sheaf $\mathcal{F}$ on a complex space X and every $p \geq q$, then X is called cohomologically q-complete.

By the results of Andreotti and Grauert [1] we have the following theorem.

Theorem 1. *Every q-complete complex space is cohomologically q-complete.*

Definition 4.
(a) A continuous function $\varphi : X \to \mathbb{R}$ defined on a complex space is called q-convex with corners if, for every $x \in X$, there exists a neighborhood U of x and finitely many strictly q-convex $\mathcal{C}^\infty$ functions $\varphi_1, \ldots, \varphi_l$, defined on U, such that $\varphi_{|U} = \max\{\varphi_1, \ldots, \varphi_l\}$.
(b) A complex space X is called q-complete with corners if there exists a q-convex with corners exhaustion function $\varphi : X \to \mathbb{R}$.

The next result is a particular case of a theorem due to Diederich and Fornaess [8]. It was generalized to the singular case in [9].

Theorem 2. *If M is an n-dimensional q-complete with corners complex manifold then M is $\tilde{q} = (n - \left[\frac{n}{q}\right] + 1)$-complete.*

For $q \geq 1$ and $r > 0$, we denote by $P^q(r) \subset \mathbb{C}^q$ the polydisk centered at the origin with multiradius $(r, \cdots, r)$. For the following definition, see, e.g., [18].

Definition 5.
(a) For $1 \leq q < n$ and $0 < r, r_1 < 1$, we let $H^q \subset \mathbb{C}^n$ be defined by $H^q := P^q(1) \times P^{n-q}(r) \bigcup \left[P^q(1) \setminus \overline{P^q(r_1)} \right] \times P^{n-q}(1)$. The pair $(H^q, P^n(1))$ is called a standard Hartogs q-figure.
(b) If M is an n-dimensional complex manifold and $V \subset U \subset M$ are open subsets, the pair (V, U) is called a Hartogs q-figure if there exists a standard Hartogs q-figure $(H^q, P^n(1))$ and a biholomorphism $F : P^n(1) \to U$ such that $F(H^q) = V$.

Definition 6. Let $\Omega \subset \mathbb{C}^n$ be an open set. If for every Hartogs q-figure (V, U) we have that $V \subset \Omega$ implies $U \subset \Omega$, then Ω is called Hartogs q-convex.

As we mentioned in the introduction, it was proved in [15] that a domain in $\mathbb{C}^n$ is Hartogs q-convex if and only if it is q-complete with corners.

The following result is Satz 2.3 in [16].

Proposition 3. *If X is a complex space and U and V are open subsets of X such that U is p-complete and V is q-complete, then $U \cup V$ is $(p+q)$-complete.*

Proposition 4 was proved in [19] in the smooth case and in [11] and [14] in the singular case.

Proposition 4. *Suppose that X is a complex space of dimension n. If X is cohomologically q-complete then $H_{n+i}(X, \mathbb{C}) = 0$ for every $i \geq q$.*

Our first result is the following theorem.

Theorem 5. *There exists a sequence $\{D_\nu\}$ of 3-complete open subsets of $\mathbb{C}^5$ such that $D_{\nu+1} \subset D_\nu$ for every ν and $\operatorname{Int}\left(\bigcap D_\nu\right)$ is not cohomologically 3-complete.*

Proof. We consider the following two planes in $\mathbb{C}^5$:

$$L_1 = \{z = (z_1, z_2, z_3, z_4, z_5) \in \mathbb{C}^5 : z_1 = z_2 = z_3 = 0\},$$

$$L_2 = \{z = (z_1, z_2, z_3, z_4, z_5) \in \mathbb{C}^5 : z_1 = z_4 = z_5 = 0\}.$$

Let $U_1 = \mathbb{C}^5 \setminus L_1$, $U_2 = \mathbb{C}^5 \setminus L_2$. It follows that U_1 and U_2 are 3-complete. At the same time, since $L_1 \cap L_2 = \{0\}$, we have $U_1 \cup U_2 = \mathbb{C}^5 \setminus \{0\}$. Then we have: $H_8(U_1, \mathbb{C}) = H_8(U_2, \mathbb{C}) = H_9(U_1, \mathbb{C}) = H_9(U_2, \mathbb{C}) = 0$ and $H_9(U_1 \cup U_2, \mathbb{C}) = \mathbb{C}$ since $H_9(S^9, \mathbb{C}) = \mathbb{C}$. From the Mayer-Vietoris exact sequence

$$H_9(U_1, \mathbb{C}) \oplus H_9(U_2, \mathbb{C}) \to H_9(U_1 \cup U_2, \mathbb{C}) \to H_8(U_1 \cap U_2, \mathbb{C}) \to H_8(U_1, \mathbb{C}) \oplus H_8(U_2, \mathbb{C})$$

it follows that $H_8(U_1 \cap U_2, \mathbb{C}) = \mathbb{C}$.

Let W be a relatively compact open subset of $U_1 \cap U_2$ such that the inclusion $W \hookrightarrow U_1 \cap U_2 = \mathbb{C}^5 \setminus (L_1 \cup L_2)$ induces an isomorphism $H_8(W, \mathbb{C}) \to H_8(U_1 \cap U_2, \mathbb{C})$. In fact we have an exhaustion $\{W_k\}$ of $U_1 \cap U_2$ such that the inclusion $W_k \hookrightarrow U_1 \cap U_2$ induces an isomorphism at all homology and homotopy groups.

For $\nu \geq 1$ we define:

$$L_{1,\nu} = \{z \in \mathbb{C}^5 : z_1 = \frac{1}{\nu}, z_2 = z_3 = 0\},$$

$$L_{2,\nu} = \{z \in \mathbb{C}^5 : z_1 = \frac{\sqrt{2}}{\nu}, z_4 = z_5 = 0\}.$$

It follows that $L_{i,\mu} \cap L_{j,\nu} = \emptyset$ if $(i, \mu) \neq (j, \nu)$. Because W is relatively compact in $\mathbb{C}^5 \setminus (L_1 \cup L_2)$, it follows that there exists $\nu_0 \geq 1$ such that, for $\nu \geq \nu_0$, $L_{1,\nu} \cap \overline{W} = \emptyset$ and $L_{2,\nu} \cap \overline{W} = \emptyset$.

For $\nu \geq \nu_0$, let $D_\nu = \mathbb{C}^5 \setminus \bigcup_{j=\nu_0}^{\nu} (L_{1,j} \cup L_{2,j})$. Since $L_{i,\mu} \cap L_{j,\nu} = \emptyset$, it follows that D_ν are 3-complete. Let $D = \operatorname{Int}\left(\bigcap_{\nu \geq \nu_0} D_\nu\right)$. It follows that $W \subset D \subset \mathbb{C}^5 \setminus (L_1 \cup L_2)$. Hence we have

$$H_8(W, \mathbb{C}) \to H_8(D, \mathbb{C}) \to H_8(\mathbb{C}^5 \setminus (L_1 \cup L_2), \mathbb{C}),$$

where the morphisms are induced by inclusions. Since $H_8(W, \mathbb{C}) \to H_8(\mathbb{C}^5 \setminus (L_1 \cup L_2), \mathbb{C})$ is surjective, it follows that $H_8(D, \mathbb{C}) \to H_8(\mathbb{C}^5 \setminus (L_1 \cup L_2), \mathbb{C})$, is surjective as well. In particular, we have $H_8(D, \mathbb{C}) \neq 0$. Proposition 4 implies that D is not cohomologically 3-complete.

□

As we mentioned in the introduction, if $\{D_\nu\}$ is a decreasing sequence of q-complete open subsets of $\mathbb{C}^n$, it follows that $\operatorname{Int}(\bigcap D_\nu)$ is q-complete with corners. This is not the case for singular complex spaces, as the following result shows.

Theorem 6. *For every integer $q \geq 2$, there exists a normal Stein complex space X with only one isolated singularity, and $\{D_\nu\}$ a decreasing sequence of open subsets of X such that each D_ν is 2-complete and $\operatorname{Int}(\bigcap D_\nu)$ is not q-complete with corners.*

Proof. Let q be an integer, $q \geq 2$. Let $\pi : F \to \mathbb{P}^1$ be a negative vector bundle of rank $r \geq 3q - 1$ and let S be the zero section of F (hence S is biholomorphic to $\mathbb{P}^1$). Let X be the blow-down of $S \subset F$ and $\tau : F \to X$ be the contraction map. We let $x_0 = \tau(S)$. We fix a point $a \in S$ and we set $U = S \setminus \{a\}$ (hence U is biholomorphic to $\mathbb{C}$) and $W = \pi^{-1}(U)$. We have that $\pi : W \to U$ is a trivial holomorphic vector bundle and therefore W is biholomorphic to $U \times \mathbb{C}^r$ (in particular W is Stein). We consider $W_\nu \subset F$ a fundamental system of Stein open neighborhoods of a and we define

$$D_\nu = \tau(W \cup W_\nu).$$

Note that D_ν are open neighborhoods of x_0 in X and, since $\bigcap W_\nu = \{a\}$, we have that $\operatorname{Int}(\bigcap D_\nu) = \tau(W \setminus S)$. Hence $\operatorname{Int}(\bigcap D_\nu)$ is biholomorphic to $W \setminus S$ and therefore to $U \times (\mathbb{C}^r \setminus \{0\})$.

Note the following points:

• $n = \dim X = r + 1 \geq 3q$.

• As W and W_ν are Stein, by Theorem 3 we have that $W \cup W_\nu$ is 2-complete. Therefore D_ν is 2-convex and since X is Stein, we deduce that D_ν is 2-complete.

• We have that $U \times (\mathbb{C}^r \setminus \{0\})$ is not cohomologically $(n-2)$ complete since $\mathbb{C}^r \setminus \{0\}$ is not cohomologically $(n-2)$-complete.

Because $n \geq 3q$, we have that $\tilde{q} = n - \left[\frac{n}{q}\right] + 1 \leq n - 2$. Using Theorems 1 and 2 we deduce that $U \times (\mathbb{C}^r \setminus \{0\})$ is not q-complete with corners.

Hence although each D_ν is 2-complete, the interior of their intersection is not q-complete with corners.

□

Next we would like to say a few things about the intersection of Stein open subsets of a normal Stein space. Let X be a normal Stein complex space, and $\{D_\nu\}$ be a sequence of Stein open subsets of X. It is a completely open problem whether the interior of their intersection is Stein or not, even if X has dimension 2; see [2]. Of course, the problem is due to singularities. However, we have the following proposition.

Proposition 7. *Let X be a normal Stein complex space and $\{D_\nu\}$ a sequence of Stein open subsets of X. If $D = \mathrm{Int}\,(\bigcap D_\nu)$, then we have*
(a) $\mathrm{Reg}(X) \cap \partial D$ is dense in ∂D.
(b) D is a domain of holomorphy in X.

Proof. (a) Suppose that this is not the case and let $x_0 \in \partial D$ and W a Stein neighborhood of x_0 such that $\partial D \cap W \subset \mathrm{Sing}(X)$. As X is normal and therefore locally irreducible, we have that $W \setminus \partial D$ is connected. Since $W \setminus \partial D = (W \cap D) \cup (W \setminus \overline{D})$, we deduce that $W \setminus \partial D = W \cap D$. Therefore $W \setminus \mathrm{Sing}(X) \subset D$. Using again the normality of X, the Riemann Second Extension Theorem, and the fact that each D_ν is Stein, we deduce that the inclusion $W \setminus \mathrm{Sing}(X) \hookrightarrow D_\nu$ extends to W (with values in D_ν) and therefore $W \subset D_\nu$ for every ν. Hence $W \subset D$. In particular $x_0 \in D$, which contradicts our choice of x_0.

(b) Obviuously, D is locally Stein at every point $x \in \partial D \cap \mathrm{Reg}(X)$. Then for every sequence $\{x_k\}$, $x_k \in D$, such that $x_k \to x \in \partial D \cap \mathrm{Reg}(X)$ there exists $f \in \mathcal{O}(D)$ which is unbounded on $\{x_k\}$. This was proved for relatively compact domains $D \Subset X$ in [12] and extended to arbitrary domains in [17]. From this fact and part (a), we deduce that D is a domain of holomorphy in X. □

Remark 1. Using the method in [7] it can be proved that, in the same setting, if $\dim(X) = 2$ then D satisfies the disk property. This means that if $\overline{\Delta} = \{z \in \mathbb{C} : |z| \leq 1\}$ is the closed unit disk and $f_n : \overline{\Delta} \to X$ is a a sequence of holomorphic functions converging uniformly to a holomorphic function $f : \overline{\Delta} \to X$ and if $f_n(\overline{\Delta}) \subset D$ and $f(\partial\Delta) \subset D$, then $f(\overline{\Delta}) \subset D$.

3 Increasing sequences of q-concave domains

We want to discuss a dual question, namely concavity properties of a union of q-concave open subsets of a complex manifold.

For the next definition, see [1].

Definition 7. A complex space X is called q-concave if there exists a continuous function $\varphi : X \to (0, \infty)$ and a compact set $K \subset X$ such that φ is strictly q-convex on $X \setminus K$ and $\{x \in X : \varphi(x) > c\} \Subset X$ for every $c > 0$.

By analogy with the notion of weakly q-convex space, we introduce the following definition:

Definition 8. A complex space X is called weakly q-concave if there exists a continuous function $\varphi : X \to (0, \infty)$ and a compact set $K \subset X$ such that φ is weakly q-convex on $X \setminus K$ and $\{x \in X : \varphi(x) > c\} \Subset X$ for every $c > 0$.

Remark. A proper modification of a q-concave manifold is weakly q-concave.

Example: The following example appears in [3]. Let $a \in \mathbb{P}^2$ and $\{x_n\}_{n\geq 1}$ be a sequence in $\mathbb{P}^2 \setminus \{a\}$ converging to a, and M be the blow-up of $\mathbb{P}^2 \setminus \{a\}$ at this sequence. Then M is weakly 1-concave but it is not 1-concave. Moreover, M is an increasing sequence of 1-concave open subsets. Indeed, we let M_k, $k \geq 1$, be the blow-up of $\mathbb{P}^2 \setminus (\{a\} \cup \{x_n : n \geq k+1\})$ at $x_1, \ldots, x_k$. Then M_k is an open subset of M, $M_k \subset M_{k+1}$ and $\bigcup_{k\geq 1} M_k = M$. It was noticed in [10] that if $\{A_n\}$ is a countable set of closed, completely pluripolar subsets of a complex manifold Ω such that $A := \bigcup A_n$ is closed in Ω and Ω' is an open subset of Ω such that $\Omega' \Subset \Omega$ then $A \cap \Omega'$ is completely pluripolar in Ω'. It follows then that $\mathbb{P}^2 \setminus (\{a\} \cup \{x_n : n \geq k+1\})$ is 1-concave and hence M_k is 1-concave.

Theorem 8. *For every integer $n > 1$ there exists a connected complex manifold M of dimension n such that M is an increasing union of 1-concave open subsets and M is not weakly $(n-1)$-concave.*

Proof. The following construction was used in [5] and [6]. We start with $\Omega_0 := \mathbb{P}^n$ (or any compact complex manifold of dimension n) and we choose $a_0 \in \Omega_0$ to be any point. We set $M_0 := \Omega_0 \setminus \{a_0\}$ and let $p_0 : \Omega_1 \to \Omega_0$ be the blow-up of Ω_0 at a_0. Let a_1 be a point on the exceptional divisor of p_0, $M_1 = \Omega_1 \setminus \{a_1\}$ and $p_1 : \Omega_2 \to \Omega_1$ be the blow-up of Ω_1 at a_1. Suppose now that we have defined inductively:

- Ω_j for $j = 0, \ldots, k$;
- $a_j \in \Omega_j$, $p_j : \Omega_{j+1} \to \Omega_j$ and $M_j = \Omega_j \setminus \{a_j\}$ for $j = 0, \ldots, k-1$.

We choose a point a_k on the exceptional divisor of $p_{k-1} : \Omega_k \to \Omega_{k-1}$ such that a_k is not on the proper transform of the exceptional divisor of $p_{k-2} : \Omega_{k-1} \to \Omega_{k-2}$. We let $M_k = \Omega_k \setminus \{a_k\}$ and $p_{k+1} : \Omega_{k+1} \to \Omega_k$ be the blow-up of Ω_k at a_k.

Note that M_k is an open subset of M_{k+1} for every $k \geq 0$. We set $\tilde{M} := \bigcup_{k\geq 0} M_k$. Each M_k is 1-concave since it is the complement of a point in a compact complex manifold. At the same time, $\tilde{M}$ contains a noncompact connected $(n-1)$-dimensional complex subspace X such that all irreducible components of X are compact. This subspace is the union of the (proper transforms of the) exceptional divisors of all the blow-ups defined above.

If $\varphi : M \to (0, \infty)$ is weakly $(n-1)$-convex outside a compact subset K of M, since each irreducible component of X has dimension $n-1$, we have, by the maximum principle, that φ must be constant on each irreducible component of X that does not intersect K. Therefore it is constant on at least one noncompact connected component of $X \setminus K$. Hence M cannot be weakly $(n-1)$-concave. $\square$

Remark 2. It was noticed in [3] that if a complex manifold is an increasing union of 1-concave open subsets then its cohomology with values in any locally free coherent sheaf is separated. It is an open question raised by R. Hartshorne [13] whether a complex connected manifold such that its cohomology with values in any locally free coherent sheaf is finite-dimensional is necessarily a compact manifold.

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