

On Runge-curved domains in Stein spaces *

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Abstract

We prove the following result: if X is a Stein complex space and $D \subset X$ is an open subset then D is Runge-curved in X if and only if the canonical map $H_c^1(D, \mathcal{F}) \rightarrow H_c^1(X, \mathcal{F})$ is injective for every $\mathcal{F} \in \text{Coh}(X)$. We also show that a Runge-curved open subset of a Stein manifold is necessarily Stein.

1 Introduction

Let X be a (non-necessarily reduced) Stein space and $D \subset X$ an open subset. D is called a Runge domain if D is Stein and the restriction map $\Gamma(X, \mathcal{O}_X) \rightarrow \Gamma(D, \mathcal{O}_X)$ has dense image in the topology of uniform convergence of compact subsets of D . The domain D (non necessarily Stein) is called Runge-curved if for any complex curve $\Gamma \subset X$ (closed complex space of dimension 1) the pair $(\Gamma, \Gamma \cap D)$ is a Runge pair.

For Stein spaces of dimension 1, a complete characterization has been given by N. Mihalache [10]. If $n = \dim X > 1$ the notions of Runge domain and Runge-curved domain do not coincide. G. Stolzenberg [14] has given an example of a domain $D \subset \mathbb{C}^2$ such that D is Runge-curved and D is not Runge.

The main purpose of this paper is to continue the study of Runge-curved domains. As a first step we prove in Theorem 1 that a Runge-curved domain in a Stein manifold is Stein. It is an open question if the same result holds in the singular case. The main result of the paper is Theorem 3 which gives

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a characterization of Runge-curved domains in Stein spaces using cohomology with compact support. More precisely we prove that the following two conditions are equivalent:

- 1) D is Runge-curved,
- 2) The natural map between cohomology groups with compact support $H_c^1(D, \mathcal{F}) \rightarrow H_c^1(X, \mathcal{F})$ is injective for every $\mathcal{F} \in \text{Coh}(X)$.

The proof is done by induction on $\dim(X)$ and because of this one needs to use complex spaces with nilpotent elements and properties of the torsion of a coherent sheaf defined on them. If one uses only torsion free sheaves the condition 2) is replaced by the weaker condition that $X \setminus D$ has no holes, see [6]. Therefore the use of torsion sheaves on complex spaces with nilpotents is essential for our proof.

2 Steiness of Runge curved domains in Stein manifolds

For the following statement see [1].

Lemma 1. *Let X be a Stein manifold of dimension n . Then there exists a finite number of holomorphic maps $\Phi_j : X \rightarrow \mathbb{C}^n$ with discrete fibers, $j = 1, 2, \dots, m$, such that if A_j is the branch locus of Φ_j then $\bigcap_{j=1}^m A_j = \emptyset$.*

The following statement is part of Theorem 6.8 in [10].

Lemma 2. *Let X be a pure 1-dimensional Stein space and $D \subset X$ an open subset. Then the following are equivalent:*

- 1) (X, D) is a Runge pair,
- 2) For any open neighborhood U of $X \setminus D$ and any irreducible component C of $\text{Red}(U)$, $C \setminus D$ is not compact.
- 3) $H_c^1(D, \mathcal{O}) \rightarrow H_c^1(X, \mathcal{O})$ is injective.

The proof of the following Lemma is similar to the proof of Proposition 5.5, page 95, in [12].

Lemma 3. *Let D be an open subset of \mathbb{C}^n . We assume that for every closed complex curve $\Gamma \subset \mathbb{C}^n$, biholomorphic to \mathbb{C} we have that $(\Gamma, \Gamma \cap D)$ is a Runge pair. Then D is Stein.*

Proof. We may assume, of course, that $D \neq \mathbb{C}^n$ and we denote by $\delta : D \rightarrow (0, \infty)$ the distance to the boundary of D which is a continuous function. For every $u \in \mathbb{C}^n$, $u \neq 0$, we define $\delta_u : D \rightarrow (0, \infty]$,

$$\delta_u(z) := \sup\{\tau : z + \eta u \in D \text{ for every } \eta \in \mathbb{C}, |\eta| \leq \tau\}.$$

We have that $\delta = \inf\{\delta_u : u \in \mathbb{C}^n, \|u\| = 1\}$. By Oka's theorem we must prove that $-\log \delta = \sup\{-\log \delta_u : u \in \mathbb{C}^n, \|u\| = 1\}$ is plurisubharmonic. Therefore it suffices to prove that $-\log \delta_u$ is plurisubharmonic for every u . Let $u \in \mathbb{C}^n$, $\|u\| = 1$. To prove that $-\log \delta_u$ is plurisubharmonic we have to show that if $z \in D$, $w \in \mathbb{C}^n \setminus \{0\}$, $r > 0$, and $g : \mathbb{C} \rightarrow \mathbb{C}$ is a polynomial function of one complex variable such that

$$-\log \delta_u(z + \lambda w) \leq \operatorname{Re} g(\lambda) \quad \forall \lambda \text{ such that } |\lambda| = r$$

then we have

$$-\log \delta_u(z + \lambda w) \leq \operatorname{Re} g(\lambda) \quad \forall \lambda \text{ such that } |\lambda| \leq r.$$

This is equivalent to proving that if

$$z + \lambda w + \mu e^{-g(\lambda)} u \in D \quad \forall \lambda, \mu \in \mathbb{C} \text{ such that } |\lambda| = r, |\mu| < 1$$

then we have

$$z + \lambda w + \mu e^{-g(\lambda)} u \in D \quad \forall \lambda, \mu \in \mathbb{C} \text{ such that } |\lambda| \leq r, |\mu| < 1.$$

Note that if u and w are linearly dependent over \mathbb{C} then this is just a standard 1-dimensional statement. Let's assume that u and w are linearly independent and let's fix $\mu \in \mathbb{C}$ with $|\mu| < 1$. We consider $h : \mathbb{C} \rightarrow \mathbb{C}^n$, $h(\lambda) = z + \lambda w + \mu e^{-g(\lambda)} u$. Then h is an embedding and therefore $\Gamma := h(\mathbb{C})$ is a closed complex curve in \mathbb{C}^n , biholomorphic to \mathbb{C} . Therefore $(\Gamma, \Gamma \cap D)$ is a Runge pair. At the same time we have that $\{\lambda \in \mathbb{C} : |\lambda| = r\} \subset h^{-1}(D)$. We deduce that $\{\lambda \in \mathbb{C} : |\lambda| \leq r\} \subset h^{-1}(D)$ and hence $z + \lambda w + \mu e^{-g(\lambda)} u \in D$, $\forall \lambda$ with $|\lambda| \leq r$. □

Theorem 1. *Let X be a n -dimensional Stein manifold and D an open subset of X . We assume that $(C, C \cap D)$ is a Runge pair for every closed complex curve $C \subset X$. Then D is Stein.*

Proof. Because X is a Stein manifold, it suffices to show that D is locally Stein in X . Let $x_0 \in \partial D$. By Lemma 1 there exists a holomorphic map with discrete fibers, $\Phi : X \rightarrow \mathbb{C}^n$, such that Φ is a locally biholomorphism around x_0 . Let U be a Stein open neighborhood of x_0 and V be an open neighborhood of $y_0 := \Phi(x_0)$ such that $\Phi : U \rightarrow V$ is a biholomorphism. Moreover, we choose U to be Runge in X . Replacing V by a small ball $B(y_0, \epsilon)$ centered at y_0 and U by the connected component of $\Phi^{-1}(B(y_0, \epsilon)) \cap U$ that contains x_0 , we may assume that U is Runge in X and V is Runge in \mathbb{C}^n .

In this setting, we would like to show that $U \cap D$ is Stein which is equivalent to $\Phi(D \cap U)$ being Stein. If Γ is a closed complex curve in \mathbb{C}^n then we let $C := \Phi^{-1}(\Gamma)$ which is a closed complex curve in X . Therefore $C \cap D$ is Runge in C . It follows that $C \cap D \cap U$ is Runge in $C \cap U$ and therefore $\Gamma \cap \Phi(D \cap U)$ is Runge in $\Gamma \cap V$. As $\Gamma \cap V$ is Runge in Γ we deduce that $\Gamma \cap \Phi(D \cap U)$ is Runge in Γ . In order to finish the proof, we apply Lemma 3. \square

Remark 1. We do not know if the above theorem is true if X is a Stein complex space.

Remark 2. If D is a Runge open subset of Stein manifold X , it was proved by A. Cassa in [4] that for every closed irreducible complex curve C in D one can find a sequence of closed irreducible curves $\{C_k\}_{k \geq 1}$ in X such that $C = \lim_{k \rightarrow \infty} (C_k \cap D)$. The convergence is the one induced by the currents. See also [5].

It was proved in [7] that if X is an $(n - 1)$ -convex irreducible complex space of $\dim X > 1$, $\rho : X \rightarrow \mathbb{R}$ is a smooth exhaustion function which is $(n - 1)$ -convex on $X_c = \{x \in X : \rho(x) > c\}$ for some $c \in \mathbb{R}$, D is a bordered Riemann surface, and $f : \bar{D} \rightarrow X$ is a \mathcal{C}^2 function which is holomorphic in D and satisfies $f(D) \not\subset X_{sing}$ and $f(\partial D) \subset X_c$, then there is a sequence of proper holomorphic maps $g_\nu : D \rightarrow X$ converging to f uniformly on compact subsets of D .

Example 1. Suppose that B_1 and B_2 are two balls in \mathbb{C}^n such that $B_1 \not\subset B_2$, $B_2 \not\subset B_1$, and $B_1 \cap B_2 \neq \emptyset$. Then $D := B_1 \cup B_2$ is not Stein. At the same time for every complex affine line, L in \mathbb{C}^n , $L \cap D$ is simply connected because $L \cap B_1$, $L \cap B_2$, $L \cap B_1 \cap B_2$ are connected and simply connected. Therefore $L \cap D$ is Runge in L . According to our theorem there must exist a curve C in \mathbb{C}^n such that $C \cap D$ is not Runge in C . We are going to exhibit such a curve. For simplicity we assume that $n = 2$, B_1 is the open ball of radius

3 centered at $(-2, 0) \in \mathbb{C}^2$ and B_2 is the open ball of radius 3 centered at $(2, 0)$. Let $C = \{(z, w) \in \mathbb{C}^2 : w = z^2 + \sqrt{5}\}$. We choose $r > 0$ small enough such that

$$r^3 + [2\sqrt{5}(2\lambda^2 - 1) + 1]r - 4\lambda < 0 \quad \text{for all } \lambda \text{ such that } \lambda \in [0, 1]. \quad (*)$$

Note that this is equivalent to

$$r^3 + [2\sqrt{5}(2\lambda^2 - 1) + 1]r + 4\lambda < 0 \quad \text{for all } \lambda \text{ such that } \lambda \in [-1, 0].$$

Then clearly $(0, \sqrt{5}) \in \partial D$ (hence $\notin D$) where $D := B_1 \cup B_2$. At the same time, we have $\{(z, w) \in C : z = re^{i\theta}, \theta \in [\frac{\pi}{2}, \frac{3\pi}{2}]\} \subset B_1$. Indeed, we have to check that for $z = re^{i\theta}$, $\theta \in [\frac{\pi}{2}, \frac{3\pi}{2}]$, we have that $|z + 2|^2 + |z^2 + \sqrt{5}|^2 < 9$.

We set $\lambda = \cos \theta$ and hence $\lambda \in [-1, 0]$ and $\cos 2\theta = 2\lambda^2 - 1$. We get that $|z + 2|^2 = r^2 + 4r\lambda + 4$ and $|z^2 + \sqrt{5}|^2 = r^4 + 2\sqrt{5}(2\lambda^2 - 1)r^2 + 5$. We need to check that $r^2 + 4r\lambda + 4 + r^4 + 2\sqrt{5}(2\lambda^2 - 1)r^2 + 5 < 9$. Which is the same as $r(r^3 + [2\sqrt{5}(2\lambda^2 - 1) + 1]r + 4\lambda) < 0$ and this exactly the above inequality.

Completely the same argument shows that $\{(z, w) \in C : z = re^{i\theta}, \theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]\} \subset B_2$. It follows then that $\{(z, w) \in C : z = re^{i\theta}\} \subset D$ and we deduce that $C \cap D$ is not Runge in C .

It remains to show that if $r > 0$ is small enough, then $(*)$ is satisfied. Indeed, let $\delta > 0$ be such that $2\sqrt{5}(2\delta^2 - 1) + 1 < 0$ (e.g $\delta = \frac{1}{2}$). If $\lambda \in [0, \delta]$ then $r^3 + [2\sqrt{5}(2\lambda^2 - 1) + 1]r - 4\lambda < 0$ for every $r > 0$. On the other hand, for r small enough $r^3 + [2\sqrt{5}(2\lambda^2 - 1) + 1]r \leq 2\delta$ for every $\lambda \in [\delta, 1]$ and therefore $r^3 + [2\sqrt{5}(2\lambda^2 - 1) + 1]r - 4\lambda < 0$ for every $\lambda \in [\delta, 1]$.

3 A characterization of Runge-curved domains in Stein spaces

Definition 1. Let X be a complex space, $\text{red } X$ its reduction and $\nu : \tilde{X} \rightarrow \text{red } X$ be the normalization of $\text{red } X$. If $A \subset X$ is a closed set we say that A has no irreducible compact component if $\tilde{A} := \nu^{-1}(A)$ has no compact connected component.

Lemma 4. *Let X be a Stein space and $D \subset X$ an open subset. We assume that $(\Gamma, \Gamma \cap D)$ is a Runge pair for every closed complex curve $\Gamma \subset X$. Then $A := X \setminus D$ has no compact irreducible component.*

Proof. Let $\nu : \tilde{X} \rightarrow \text{red } X$ be the normalization of $\text{red } X$ and let $\tilde{A} := \nu^{-1}(A)$, $\tilde{D} := \nu^{-1}(D)$. We assume, by reductio ad absurdum, that \tilde{A} has a compact connected component K_1 . Let $K := \nu(K_1)$.

Let $\Gamma \subset X$ be a closed complex curve such that $\Gamma \cap K \neq \emptyset$ and let $\tilde{\Gamma} := \nu^{-1}(\Gamma)$. Let $U_2 \Subset \tilde{X}$ be an open, relatively compact neighborhood of K_1 such that $\partial U_2 \cap \tilde{A} = \emptyset$. For the existence of U_2 see, for example, Proposition 2, page 112, in [11]. Let U_1 be an open neighborhood of K_1 such that $U_1 \Subset U_2$ and $(U_2 \setminus U_1) \cap \tilde{A} = \emptyset$.

If we put $U := \tilde{\Gamma} \cap U_1$ and $U' = \tilde{\Gamma} \cap (\tilde{X} \setminus \overline{U_2})$ we have that $U \cup U'$ is an open neighborhood of $\tilde{\Gamma} \setminus \tilde{D}$ in $\tilde{\Gamma}$, $U \cap U' = \emptyset$, and $U \setminus (\tilde{\Gamma} \cap \tilde{D})$ is compact. Therefore, for any irreducible component C of U , we have that C is an irreducible component of $U \cup U'$ and $C \setminus (\tilde{\Gamma} \cap \tilde{D})$ is compact. By Lemma 2 we deduce that the $(\tilde{\Gamma}, \tilde{\Gamma} \cap \tilde{D})$ is not a Runge pair and therefore $(\Gamma, \Gamma \cap D)$ is not a Runge pair, which contradicts our assumption. \square

For basic facts regarding the torsion of sheaves on complex spaces we refer to [13].

Suppose that X is a complex space and \mathcal{F} is a coherent analytic sheaf on X . We denote by $t\mathcal{F}$ the torsion sheaf of \mathcal{F} . More precisely, for $x \in X$, $t\mathcal{F}_x$ is the set of all germs $s_x \in \mathcal{F}_x$ for which there exists $g_x \in \mathcal{O}_{X,x}$ such that $\text{red } g_x$ is a non zero divisor (g_x is called an active germ) and $g_x s_x = 0$. We have that $t\mathcal{F}$ is also a coherent sheaf and $\text{supp}(t\mathcal{F})$ is a thin analytic subset of X . If $t\mathcal{F} = 0$ the sheaf \mathcal{F} is called torsion free.

For the proof of the following see for example [9], page 67.

Theorem 2. (Rückert Nullstellensatz) *Suppose that X is a complex space and \mathcal{S} is a coherent analytic sheaf on X . Let $f \in \Gamma(X, \mathcal{O}_X)$ which vanishes on $\text{supp}(\mathcal{S})$. Then for each point $x \in X$ there exists an open neighborhood U of x and a positive integer k such that $f^k \mathcal{S}_U = 0$.*

Lemma 5. *Suppose that X is a complex space, \mathcal{F} is a coherent sheaf on X and $s \in \Gamma(X, \mathcal{F})$ is a section. If $\text{supp}(s)$ is a thin analytic subset of X then $s \in \Gamma(X, t\mathcal{F})$.*

Proof. Let $S = \text{supp}(s)$ and $x \in S$. We choose $V \subset X$ an open Stein neighborhood of x and $f \in \mathcal{O}_X(V)$ such that $\text{red } f|_{S \cap V} = 0$ and $\text{red } f_x$ is not a zero divisor. Such an f exists because S is thin. Applying Theorem 2 to the subsheaf of \mathcal{F} generated by s we deduce that there exists a positive integer k and an open neighborhood $U \subset V$ of x such that $f^k s|_U = 0$. It follows that $s_x \in t\mathcal{F}_x$. \square

Corollary 1. *If \mathcal{F} is a torsion free coherent sheaf on X and $s \in \Gamma(X, \mathcal{F})$ is a section then $\text{supp}(s)$ is a union of irreducible components of $\text{red } X$.*

This means that the identity principle holds for torsion free coherent sheaves: if $U \subset X$ is an open subset and $s \in \Gamma(X, \mathcal{F})$ is a section such that $s \equiv 0$ on U then $s \equiv 0$ on every irreducible component of $\text{red } X$ that intersects U .

Proposition 1. *Suppose that X is a complex space and $A \subset X$ is a closed subset without compact irreducible components. Then, for every torsion free coherent analytic sheaf \mathcal{F} on X , we have $H_c^0(A, \mathcal{F}) = 0$.*

Proof. We assume that there exists $s \in H_c^0(A, \mathcal{F})$, $s \neq 0$. We choose a representative of s , $\tilde{s} \in \Gamma(U, \mathcal{F})$ where U is an open neighborhood of A and $\text{supp}(\tilde{s}) \cap A$ is compact and non empty. Because \mathcal{F} is torsion free it follows that $\text{supp}(\tilde{s})$ is an union of irreducible components of $\text{red } U$.

Let $\nu : \tilde{X} \rightarrow \text{red } X$ be the normalization map and let $\tilde{U} := \nu^{-1}(U)$, $\tilde{A} := \nu^{-1}(A)$. We consider B to be the union of the connected components of \tilde{U} whose images are the irreducible components of $\text{supp}(\tilde{s})$. The intersection $B \cap \tilde{A}$ is compact

It follows that \tilde{A} has compact connected components and therefore A has compact irreducible components, which contradicts our assumption. \square

The following theorem gives a characterization of Runge-curved domains using cohomology with compact supports. For basic definitions and results for cohomology with compact supports with values in a sheaf on a paracompact topological space, see [8] or [3].

Theorem 3. *Let X be a Stein complex space and $D \subset X$ an open subset. Then the following are equivalent:*

- 1) *for every closed complex curve $\Gamma \subset X$, we have that $(\Gamma, \Gamma \cap D)$ is a Runge pair,*
- 2) *the canonical map $H_c^1(D, \mathcal{F}) \rightarrow H_c^1(X, \mathcal{F})$ is injective for every $\mathcal{F} \in \text{Coh}(X)$.*

Proof. 2) \implies 1) is straightforward: By choosing \mathcal{F} to be $\mathcal{O}_X/\mathcal{I}_\Gamma$, where \mathcal{I}_Γ is the ideal sheaf determined by Γ we obtain that $H_c^1(\Gamma \cap D, \mathcal{O}_\Gamma) \rightarrow H_c^1(\Gamma, \mathcal{O}_\Gamma)$ is injective. It follows then from Lemma 2 that $(\Gamma, \Gamma \cap D)$ is a Runge pair.

1) \implies 2) We will prove this implication by induction on $\dim X$. If $\dim X = 1$, the statement follows from Lemma 2. We assume that the statement is true for all Stein complex spaces of dimension $\leq n - 1$ and we let X be a complex space with $\dim X = n$.

Let \mathcal{F} be a coherent sheaf on X and let $A := X \setminus D$. It follows from Lemma 4 that A has no compact irreducible component. We consider the following exact sequence:

$$0 \rightarrow t\mathcal{F} \rightarrow \mathcal{F} \rightarrow \mathcal{F}/t\mathcal{F} \rightarrow 0$$

which gives us the following commutative diagram with exact rows and columns, the rows being long exact sequences for the inclusion of an open set.

$$\begin{array}{ccccccccc}
& & & & & & & & H_c^0(X, \mathcal{F}/t\mathcal{F}) \\
& & & & & & & & \downarrow \\
0 & \longrightarrow & H_c^0(D, t\mathcal{F}) & \longrightarrow & H_c^0(X, t\mathcal{F}) & \longrightarrow & H_c^0(A, t\mathcal{F}) & \longrightarrow & H_c^1(D, t\mathcal{F}) & \xrightarrow{\beta} & H_c^1(X, t\mathcal{F}) \\
& & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \delta \\
0 & \longrightarrow & H_c^0(D, \mathcal{F}) & \longrightarrow & H_c^0(X, \mathcal{F}) & \longrightarrow & H_c^0(A, \mathcal{F}) & \longrightarrow & H_c^1(D, \mathcal{F}) & \xrightarrow{\alpha} & H_c^1(X, \mathcal{F}) \\
& & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & H_c^0(D, \mathcal{F}/t\mathcal{F}) & \longrightarrow & H_c^0(X, \mathcal{F}/t\mathcal{F}) & \longrightarrow & H_c^0(A, \mathcal{F}/t\mathcal{F}) & \longrightarrow & H_c^1(D, \mathcal{F}/t\mathcal{F}) & \xrightarrow{\gamma} & H_c^1(X, \mathcal{F}/t\mathcal{F})
\end{array}$$

Notice now that we have:

- $\text{supp}(t\mathcal{F})$ is a closed analytic subspace of X of dimension $< \dim X$ and $t\mathcal{F}|_{\text{supp}(t\mathcal{F})}$ is a coherent sheaf. On the other hand, we have that $H_c^1(D, t\mathcal{F}) = H_c^1(D \cap \text{supp}(t\mathcal{F}), t\mathcal{F})$ and $H_c^1(X, t\mathcal{F}) = H_c^1(\text{supp}(t\mathcal{F}), t\mathcal{F})$. Hence, according to our induction hypothesis, the map $\beta : H_c^1(D, t\mathcal{F}) \rightarrow H_c^1(X, t\mathcal{F})$ is injective,
- because A has no compact irreducible components and $\mathcal{F}/t\mathcal{F}$ has no torsion it follows that $H_c^0(A, \mathcal{F}/t\mathcal{F}) = 0$ and therefore $\gamma : H_c^1(D, \mathcal{F}/t\mathcal{F}) \rightarrow H_c^1(X, \mathcal{F}/t\mathcal{F})$ is injective,
- because $\mathcal{F}/t\mathcal{F}$ has no torsion it follows that $H_c^0(X, \mathcal{F}/t\mathcal{F})$ vanishes and therefore the map $\delta : H_c^1(X, t\mathcal{F}) \rightarrow H_c^1(X, \mathcal{F})$ is injective.

From the injectivity of β , γ , and δ , it follows that α is injective as well. \square

Remark 3. It is proved in [2] that X is a Stein space and $D \subset X$ is an open Runge subset then the map $H^i(D, \mathcal{F}) \rightarrow H^i(X, \mathcal{F})$ is injective for every $i \geq 0$ and every coherent sheaf \mathcal{F} .

Remark 4. Note that the ideal sheaf that defines $\text{supp}(t\mathcal{F})$ is $\text{Ann}(t\mathcal{F})$ and $\text{supp}(t\mathcal{F})$ might have nilpotent elements even if X is reduced. Hence, even if we wanted to prove Theorem 3 only for reduced complex spaces, in order to use an induction argument we had to consider non-reduced complex spaces as well.

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