# On the open immersion problem *

*Dedicated to the memory of Professor Hans Grauert*

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**Abstract**

We give two examples of complex spaces on which global holomorphic functions separate points and give local coordinates and they cannot be realized as open subsets of Stein spaces. At the same time we notice that these examples are open subsets of Stein schemes, a notion introduced by H. Grauert in [7]. In the context of complex schemes we notice that by contracting a Nori string one obtains a complex scheme and not a complex space. The covering spaces of 1-convex surfaces are divided in two categories: those that have an envelope of holomorphy and those that do not. More interesting are those in the second category and they correspond to covering spaces for singularities which in the desingularization with normal crossings contain cycles in the exceptional set.

## 1 Introduction

Let $Y$ be a Stein space and $X$ an open subset of $Y$. Then $X$ satisfies the following two properties:
a) the global holomorphic functions on $X$ separate the points of $X$ (i.e., for any $x, y \in X$, $x \neq y$, there exists $f \in \mathcal{O}(X)$ such that $f(x) \neq f(y)$);
b) the global holomorphic functions on $X$ give local coordinates on $X$ (i.e., for any $x \in X$ there exists $f : X \to \mathbb{C}^N$, $N = N(x)$, a holomorphic function, which is an immersion at $x$).

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In [2] it was raised the following question, called "the open immersion problem":

**Question.** Given a complex space $X$ that satisfies the two conditions a) and b) above, is it possible to realize $X$ as an open subset of a Stein space?

In this paper we give two counter-examples to this question. The first one is an irreducible complex space of dimension 2 with isolated singularities and the second example is a complex space of dimension 2 with hypersurface non-isolated singularities, having infinitely many irreducible components but which is a covering space of an open subset of a Stein space and, additionally, we prove that it can be realized as closed analytic subset of an open subset of $\mathbb{C}^4$.

We notice that the two examples are open sets of Stein schemes, a generalization of Stein complex spaces introduced by Grauert in [7].

We remark that the given examples have non-normal singularities and therefore the question remains open for normal complex spaces.

Related to the open immersion problem is the existence of envelope of holomorphy (see [4] for a discussion of this topic). A complex space is said to have an envelope of holomorphy (or to be precomplete) if there exists a Stein space $Y$ and a holomorphic map $f : X \to Y$ such that the natural map $\mathcal{O}(Y) \to \mathcal{O}(X)$ is an isomorphism. When $X$ satisfies the conditions a) and b), the map $f$ is an injective immersion. The first example of open subsets in Stein spaces of dimension 3 with normal isolated singularities, without envelope of holomorphy, was obtained by H. Grauert in [7]. Another example, satisfying additionally the hypersurface section condition was obtained by M. Colţoiu and K. Diederich in [3]. Related to the notion of envelope of holomorphy H. Grauert [7] generalized the notion of complex space to the notion of complex scheme. A complex scheme might be quite pathological: not locally compact, with non-noetherian local rings, etc. In Section 3 of our paper we consider a germ $(X, x_0)$ of a two-dimensional normal singularity. The desingularization $Y$ of $(X, x_0)$ is a 1-convex manifold which might contain cycles in its exceptional set $Z$. In this case, see [5], there exists coverings $\tilde{Y}$ of $Y$ which contain infinite Nori strings $\tilde{A} \subset \tilde{Z}$. (i.e., $\tilde{A}$ is a connected, non-compact one-dimensional complex space having only compact irreducible components). We show that $\tilde{Z}$ as above does not have an envelope of holomorphy and blowing-down the Nori string $\tilde{A}$ one gets a complex scheme instead of a complex space.

## 2 The counter-examples

**Example 1.** We consider two complex lines, $L_1$ and $L_2$, in $\mathbb{C}^2$ such that $0 \in L_1$, $L_1 \cap L_2 = \{p\}$ where $p \neq 0$. We consider also two sequences, $\{x_n\}_{n\geq 1}$ and $\{y_n\}_{n\geq 1}$, such that
- $x_n \in L_1 \setminus \{0\}$, $y_n \in L_2$ for every $n \geq 1$ and $x_1 = y_1 = p$,
- $x_n \neq x_m$ and $y_n \neq y_m$ for $n \neq m$,
- $\lim x_n = 0$ and $\lim |y_n| = \infty$.

We define $X = \mathbb{C}^2 \setminus \{0\}/ \backsim$ where $\backsim$ identifies $x_n$ and $y_n$ for every $n \geq 1$. Clearly $X$ is an irreducible complex space. Let $\pi : \mathbb{C}^2 \setminus \{0\} \to X$ be the canonical projection. We prove, using Cartan's Theorem B, that global holomorphic functions separate points and give local coordinates on $X$.

Let $a \neq b$ be two points in $\mathbb{C}^2 \setminus \{0\}$ such that $\pi(a) \neq \pi(b)$. If $a \notin L_1 \cup L_2$ we consider a holomorphic function $f : \mathbb{C}^2 \to \mathbb{C}$ such that $f(a) \neq f(b)$ and $f_{|L_1 \cup L_2} \equiv 0$. Then $f$ will induce a holomorphic function $\tilde{f}$ on $X$ and $\tilde{f}(\pi(a)) \neq \tilde{f}(\pi(b))$. If $a, b \in L_1$ we consider a holomorphic function $g_1 : L_1 \to \mathbb{C}$ such that $g_1(a) \neq g_1(b)$. Let $g_2 : L_2 \to \mathbb{C}$ be a holomorphic function such that $g_2(y_n) = g_1(x_n)$ for every $n$. This is possible because $\{y_n : n \geq 1\}$ is a discrete set in $L_2$. Then $g : L_1 \cup L_2 \to \mathbb{C}$, $g_{|L_1} = g_1$, $g_{|L_2} = g_2$ is a holomorphic function which can be extended to a holomorphic function on $\mathbb{C}^2$ which in turn induces a holomorphic function $\tilde{f}$ on $X$ and $\tilde{f}(\pi(a)) \neq \tilde{f}(\pi(b))$. The other remaining cases can be treated in a similar fashion.

To prove that the global holomorphic functions on $X$ give local coordinates we choose a point $x \in X$. We consider the case $x = \pi(x_{n_0}) = \pi(y_{n_0})$ for some $n_0$. We consider $(g_1, h_1) : L_1 \to \mathbb{C}^2$ a holomorphic function such that $(g_1, h_1)$ is a local embedding around $x_{n_0}$, $(g_2, h_2) : L_2 \to \mathbb{C}^2$ a holomorphic function such that $(g_2, h_2)$ is a local embedding around $y_{n_0}$ and $g_2(y_n) = g_1(x_n)$, $h_2(y_n) = h_1(x_n)$ for every $n$. Let $g : L_1 \cup L_2 \to \mathbb{C}$, $g_{|L_1} = g_1$, $g_{|L_2} = g_2$ and $h : L_1 \cup L_2 \to \mathbb{C}$, $h_{|L_1} = h_1$, $h_{|L_2} = h_2$. We choose now $f_1 : \mathbb{C}^2 \to \mathbb{C}$ and $f_2 : \mathbb{C}^2 \to \mathbb{C}$ extensions of $g$ and $h$, respectively, such that $(f_1, f_2) : \mathbb{C}^2 \to \mathbb{C}^2$ is a local biholomorphism around both $x_{n_0}$ and $y_{n_0}$ and let $\tilde{f}_1$ and $\tilde{f}_2$ be the induced functions on $X$. Then $(\tilde{f}_1, \tilde{f}_2, \tilde{f}_1, \tilde{f}_2) : X \to \mathbb{C}^4$ is a local embedding around $x$. Again the other cases are simpler and can be treated in a similar fashion.

To show that $X$ is not biholomorphic to an open subset of a Stein space we argue again by contradiction and we assume that there exists $Z$ a Stein space such that $X$ is an open subset of $Z$. Then the projection $\pi$ takes values in $Z$ and as $Z$ is Stein it extends to a holomorphic map $\pi : \mathbb{C}^2 \to Z$. Notice now that $\pi(x_n)$ are isolated singular points in $X$ and hence in $Z$. On the other hand because $\lim x_n = 0$ we have that $\lim \pi(x_n) = \pi(0)$. All these contradict the fact that the singular locus of $Z$ is a complex space.

**Example 2**. Let $L_1$ and $L_2$ be two complex lines in $\mathbb{C}^2$ both passing through the origin. For every $k \in \mathbb{Z}$ let $X_k$ be a copy of $\mathbb{C}^2$ and let $L_{1,k}$ and $L_{2,k}$ be the corresponding complex lines in $X_k$. We set $X = \bigsqcup_{k\in\mathbb{Z}} X_k \setminus \{0\}/\sim$ where $\sim$ identifies $L_{2,k} \setminus \{0\}$ with $L_{1,k+1} \setminus \{0\}$ via a linear isomorphism. We obtain in this way a complex space on which global holomorphic functions are easy to describe: they are given by $\{f_k\}_{k\in\mathbb{Z}}$ where $f_k : X_k \to \mathbb{C}$ are holomorphic functions such that $f_{k|L_{2,k}} = f_{k+1|L_{1,k+1}}$. Moreover, by Cartan's Theorem B, any holomorphic function $f : \bigsqcup_{m\leq k\leq n} X_k \setminus \{0\}/\sim\, \to \mathbb{C}$ can be extended to a holomorphic function on $X$. It is clear then that global holomorphic functions separate points and give local coordinates on $X$.

We assume that there exists a Stein space $Z$ such that $X$ is an open subset of $Z$. If $Z_2$ is the union of all two-dimensional irreducible components of $Z$ then $X$ is in fact an open subset of $Z_2$ and therefore we can assume from the beginning that $Z$ has pure dimension 2. Let $f_k : X_k \setminus \{0\} \to Z$ be the inclusion maps. As $Z$ is Stein it follows from the Hartogs theorem for functions with values in a Stein space that $f_k$ extends to a holomorphic function $f_k : X_k \to Z$. At the same time $f_{k|L_{2,k}} = f_{k+1|L_{1,k+1}}$ and this implies that $f_k(0) = f_{k+1}(0)$ for every $k \in \mathbb{Z}$ and therefore $f_k(0) = f_p(0)$ for every $k, p \in \mathbb{Z}$.

As obviously $f_k$ is one-to-one on $X_k \setminus \{0\}$ we have that 0 is isolated in $f_k^{-1}(f_k(0))$ and so there exists $U \subset X_k$ a neighborhood of 0 and $V \subset Z$ a neighborhood of $f_k(0)$ such that $f_k(U) \subset V$ and $f_k : U \to V$ is a finite map. Then $f_k$ maps the germ of $X_k$ at 0 onto an irreducible component of the germ of $Z$ at $f_k(0)$. As the images of $f_k$ and $f_s$ intersect only in one point for $|p - k| \geq 2$ we will have that the germ of $Z$ at $f_k(0)$ has infinitely many irreducible components which is of course a contradiction.

• Now we show that $X$ is a covering of an open subset of a Stein space. Let $M_1$ and $M_2$ be two copies of $\mathbb{C}^2$ and $L_{1,1}, L_{2,1} \subset M_1$, $L_{1,2}, L_{2,2} \subset M_2$ the lines that correspond to $L_1$ and $L_2$. We define $Y = M_1 \cup M_2/\sim$ where $\sim$ identifies $L_{1,1}$ with $L_{2,2}$ and $L_{2,1}$ with $L_{1,2}$ and we denote by $y_0$ the point that corresponds to the origin of $M_1$ (which is identified with the origin of $M_2$). Clearly $Y$ is a Stein space with two irreducible components. We consider, for every $k \in Z$, $f_{2k+1} : X_{2k+1} \to M_1$ and $f_{2k} : X_{2k} \to M_2$ linear isomorphisms such that $f_{2k+1}(L_{1,2k+1}) = L_{1,1}$, $f_{2k+1}(L_{2,2k+1}) = L_{2,1}$, $f_{2k}(L_{1,2k}) = L_{1,2}$, $f_{2k}(L_{2,2k}) = L_{2,2}$. Then $f : X \to Y \setminus \{y_0\}$, defined by $f_{|X_{2k+1} \setminus \{0\}} = f_{2k+1}$, $f_{|X_{2k} \setminus \{0\}} = f_{2k}$, is a covering map.

• We prove that $X$ is biholomorphic to a closed analytic subset of $\mathbb{C}^4 \setminus \mathbb{C}^2$. In $\mathbb{C}^4$ we set $e_1 = (1,0,0,0)$ and $e_2 = (0,1,0,0)$ which are orthogonal. We show that there exists a sequence $\{a_k\}_{k\in\mathbb{Z}}$, $a_k \in \mathbb{C}^4$, with the following properties:
1) $\|a_k\| = 1$
2) for every $k_1, k_2, k_3, k_4 \in \mathbb{Z}$, $k_1 < k_2 < k_3 < k_4$, we have that $a_{k_1}, a_{k_2}, a_{k_3}$ and $a_{k_4}$ are linearly independent (over $\mathbb{C}$).
3) for every $k_1, k_2 \in \mathbb{Z}$, $k_1 < k_2$, we have that $a_{k_1}, a_{k_2}, e_1$ and $e_2$ are linearly independent.
4) $\|a_{2k+1} - e_1\| \leq \frac{1}{|k|+1}$ and $\|a_{2k} - e_2\| \leq \frac{1}{|k|+1}$ for every $k \in \mathbb{Z}$

We will do the construction recursively. We set $a_0 = (0,0,1,0)$. We assume now that for $n \in \mathbb{Z}$, $n \geq 0$, we have constructed $a_{-n}, a_{-n+1} \ldots, a_n$ and we will construct $a_{-n-1}$ and $a_{n+1}$. We denote by $S^7$ the unit sphere in $\mathbb{C}^4$ and we let

$$\mathcal{A}_1 = \bigcup_{-n \leq k_1 < k_2 < k_3 \leq n} Span_{\mathbb{C}}\{a_{k_1}, a_{k_2}, a_{k_3}\}; \quad \mathcal{A}_2 = \bigcup_{-n \leq k \leq n} Span_{\mathbb{C}}\{e_1, e_2, a_k\}.$$

As both $\mathcal{A}_1$ and $\mathcal{A}_2$ are finite unions of complex linear spaces of complex dimension 3 they have 7-dimensional Hausdorff measure equal to 0 and hence $(\mathcal{A}_1 \cup \mathcal{A}_2) \cap S^7$ is nowhere dense in $S^7$. We choose $a_{n+1} \in S^7 \setminus \mathcal{A}_1 \cup \mathcal{A}_2$ such that it satisfies Property 4 above. In a completely similar manner we construct $a_{-n-1}$.

Let now $\tilde{X}_k = Span\{a_k, a_{k+1}\}$ for $k \in \mathbb{Z}$ and $Y = Span\{e_1, e_2\}$. We notice that
- $\tilde{X}_k \cap \tilde{X}_{k+1} = \mathbb{C} \cdot a_{k+1}$
- if $|p-k| \geq 2$ then $\tilde{X}_k \cap \tilde{X}_p = \{0\}$
- $\tilde{X}_k \cap Y = \{0\}$.

We verify now that $\tilde{X} = \bigcup_{k\in\mathbb{Z}} \tilde{X}_k \setminus \{0\}$ is a closed analytic subset of $\mathbb{C}^4 \setminus Y$. We will show that $\bigcup_{k\geq 0} \tilde{X}_k \setminus \{0\}$ is a closed analytic subset of $\mathbb{C}^4 \setminus Y$ because a completely similar proof will work for $\bigcup_{k\leq 0} \tilde{X}_k \setminus \{0\}$. It is enough to show that if $\{k_p\}$ is an increasing sequence of positive numbers, $x_{k_p} \in \tilde{X}_{k_p}$ and $\lim_{p\to\infty} x_{k_p} = y$ then $y \in Y$. Passing to a subsequence we can assume that every $k_p$ is odd (the same proof works if every $k_p$ is even). Then $\lim_{p\to\infty} a_{k_p} = e_1$ and $\lim_{p\to\infty} a_{k_p+1} = e_2$. And then, for $p$ large enough, $| < a_{k_p}, a_{k_p+1} > | < \frac{1}{2}$ since $e_1$ and $e_2$ are orthogonal. If we set $x_{k_p} = \alpha_p a_{k_p} + \beta_p a_{k_p+1}$ then

$$\|x_{k_p}\|^2 =< x_{k_p}, x_{k_p} >= |\alpha_p|^2 + |\beta_p|^2 + 2\Re(\alpha_p \overline{\beta_p} < a_{k_p}, a_{k_p+1} >)$$

and therefore $\|x_{k_p}\|^2 \geq |\alpha_p|^2 + |\beta_p|^2 - |\alpha_p \beta_p| \geq \frac{1}{2}(|\alpha_p|^2 + |\beta_p|^2)$. As $\{\|x_{k_p}\|^2\}_p$ is bounded it follows that $\{\alpha_p\}_p$ and $\{\beta_p\}_p$ are bounded. We choose now $\{k_{p_s}\}_s$ a subsequence of $\{k_p\}_p$ such that $\lim_{s\to\infty} \alpha_{p_s} = \alpha$ and $\lim_{s\to\infty} \beta_{p_s} = \beta$. We deduce that $y = \alpha e_1 + \beta e_2$ and therefore $y \in Y$.

Obviously $X$ and $\tilde{X}$ are biholomorphic.

Motivated by these two examples we raise the following two problems:

**Problem 1.** Suppose that $X$ is a complex space of bounded Zariski dimension such that global holomorphic functions separate points and give local coordinates on $X$. Does it follow that $X$ is a closed analytic subset of an open subset some Euclidean space $\mathbb{C}^N$?

**Problem 2.** Suppose that $Y$ is a normal Stein space and $p : X \to Y$ is an unbranched Riemann domain. We assume that global holomorphic functions separate points and give local coordinates on $X$. Does it follow that $X$ is an open subset of a Stein space?

**Remark:** It is well-known that global holomorphic functions on $X$ ($X$ as in Problem 2) might not separate points, see for example [10], Chapter 6. We provide here a very simple example. Let $B \subset \mathbb{C}^2$ be the open unit ball and $S^3 = \partial B$ its boundary. Choose two points $p, q \in \partial B$, $p \neq q$, and attach a handle to $\partial B$ inside $B$. That means that we choose a real analytic arc $\gamma$ inside $B$ such that $\gamma(0) = p$ and $\gamma(1) = q$. A small enough open neighborhood $Y$ of $\gamma \cup S^3$ in $\overline{B}$ can be deformed continuously to $\gamma \cup S^3$ and therefore $\pi_1(Y) = \mathbb{Z}$. In particular $\pi_1(Y)$ admits subgroups of any finite index and hence $Y$ admits finite coverings of order $\nu$ for every $\nu \in \mathbb{N}$. Let $p : X \to Y$ a covering with $\nu$ sheets, $\nu \geq 2$. Since holomorphic functions on $Y$ extend uniquely to holomorphic functions on $B$ (by Hartogs' theorem) and $B$ is simply connected, it follows from Proposition 1.3 in [6] that the holomorphic functions on $X$ are pull-backs of holomorphic functions on $Y$. In particular they do not separate the points of $X$.

We would like to show now that the two examples above are actually open subsets of Stein schemes. For the convenience of the reader we recall briefly the basic definitions regarding complex schemes according to Grauert [7] (see also [9], [11]).

Suppose that $X$ is a normal, reduced complex space and $I^* \subset \mathcal{O}(X)$ is a complex algebra of holomorphic functions with $1 \in I^*$. On $X$ we introduce the following equivalence relation: $x_1 \sim x_2$ if and only if $f(x_1) = f(x_2)$ for every $f \in I^*$. Let $Y$ be the quotient space $X/\sim$ endowed with the quotient topology and $\pi : X \to Y$ the quotient map. Every function $f \in I^*$ induces a continuous function on $Y$, denoted also by $f$. For any open subset $U$ of $Y$ let $\Gamma_U$ to be the complex algebra of all continuous functions that locally can be written as $\sum a_{\nu_1 \cdots \nu_k} f_1^{\nu_1} \cdots f_k^{\nu_k}$ with uniform convergence on $\pi^{-1}(U)$ where $f_1, \ldots, f_k \in I^*$. We let $\mathcal{O}_Y$ to be the sheaf generated by the pre-sheaf $\{\Gamma_U\}$.

**Definition 1.** *Then the ringed space $(Y, \mathcal{O}_Y)$, which is also denoted by $S(X, I^*)$ is called the spectrum of $(X, I^*)$.*

**Definition 2.** *A complex scheme is a ringed space $(Y, \mathcal{O})$ on a Hausdorff topological space $Y$, which locally is isomorphic to a spectrum.*

**Definition 3.** *A complex scheme $(Y, \mathcal{O})$ is called holomorphically convex if for every compact subset $K$ of $Y$ its holomorphically convex hull, defined by $\hat{K} := \{y \in Y : |f(y)| \leq \sup |f(K)|, \forall f \in \mathcal{O}(Y)\}$ is compact as well.*

**Definition 4.** *A complex scheme $(Y, \mathcal{O})$ is called Stein if it is holomorphically convex and for every $y_1, y_2 \in Y$, $y_1 \neq y_2$, there exists $f \in \mathcal{O}(Y)$ such that $f(y_1) \neq f(y_2)$.*

• For Example 1 we consider $X_1 = \mathbb{C}^2$ and the following algebra of holomorphic functions $I^* = \{f \in \mathcal{O}(\mathbb{C}^2) : f(x_n) = f(y_n)\ \forall n \in \mathbb{N}\}$. Let $(Y, \mathcal{O}_Y)$ be the spectrum of $(X_1, I^*)$ and $\pi : X_1 \to Y$ be the quotient map. Obviously the space $X$ constructed in this example is an open subset of $Y$, namely $X = Y \setminus \{\pi(0)\}$. It is clear that functions in $\mathcal{O}_Y(Y)$ separate the points of $Y$. To show that $Y$ is holomorphically convex, as $Y$ is second countable, it is enough to prove that for every discrete sequence $\{a_k\}_{k \in \mathbb{N}}$ in $Y$ there exists $f \in \mathcal{O}_Y(Y)$ such that $\{|f(a_k)|\}_{k \in \mathbb{N}}$ is unbounded. So let $\{a_k\}_{k \in \mathbb{N}}$ be a discrete sequence in $Y$ and $b_k \in \mathbb{C}^2$ be such that $\pi(b_k) = a_k$. As $\pi(x_n) \to \pi(0)$ we have that $\{k \in \mathbb{N} : \exists n \in \mathbb{N}\ a_k = \pi(x_n)\}$ is finite and hence we can assume that $\{b_k : k \in \mathbb{N}\} \bigcap (\{x_n : n \in \mathbb{N}\} \cup \{y_n : n \in \mathbb{N}\}) = \emptyset$. We choose a holomorphic function $g_1 : L_1 \to \mathbb{C}$ such that $g_1(b_k) = k$ whenever $b_k \in L_1$ and then a holomorphic function $g_2 : L_2 \to \mathbb{C}$ such that $g_2(y_n) = g_1(x_n)$ for every $n$ and $g_2(b_k) = k$ whenever $b_k \in L_2$. We use Cartan's Theorem B to get a holomorphic function $f : \mathbb{C}^2 \to \mathbb{C}$ such that on $L_1$ one has $f = g_1$, on $L_2$ one has $f = g_2$ and $f(b_k) = k$. Then $f$ induces a function in $\mathcal{O}_Y(Y)$ with the desired property.

We notice that $Y$ is not locally compact. Indeed if $V$ is any neighborhood of $\pi(0)$ we choose $z_n \in \pi^{-1}(V)$ such that $0 < \|z_n - y_n\| < 1$. We will have then that $\pi(z_n) \in V$ and the sequence $\{\pi(z_n)\}_{n\in\mathbb{N}}$ is discrete in $Y$.

We also notice that the local ring of $\mathcal{O}_{Y,\pi(0)}$ is not Noetherian. Indeed let $A_n$ be the set of positive integers of the form $2^n m$ where $m$ is a positive integer. We have that $A_n \supset A_{n+1}$ and, moreover, $A_n \setminus A_{n+1}$ is infinite for every $n \geq 0$. Let $J_n$ be the ideal of $\mathcal{O}_{Y,\pi(0)}$ that contains all those germs of functions $f \in \mathcal{O}_Y(V)$ such that $f \circ \pi$ vanishes identically on a neighborhood of $y_m$ for every $m \in A_n$ and on the intersection of $L_1$ with a neighborhood of 0. Then these ideals form an ascending chain. To show that the chain is non-stationary we consider $\mathcal{I}$ the coherent sheaf of ideals in $\mathcal{O}_{\mathbb{C}^2}$ consisting of holomorphic functions that vanish on $L_1 \cup \{y_m : m \geq 1\}$ ($\{y_m\}$ being discrete $L_1 \cup \{y_m : m \geq 1\}$ is an analytic subset of $\mathbb{C}^2$). Let $W_0$ be a connected Stein neighborhood of 0 and $W_m$ be a connected Stein neighborhood of $y_m$ for every $m \geq 1$ such that $W_m$, $m \geq 0$, are pairwise disjoint and the sequence $\{W_m\}_{m\geq 0}$ is uniformly Runge in $\mathbb{C}^2$.

We recall (see [8]) that a sequence, $\{D_n\}_{n\geq 1}$ of pairwise disjoint Runge domains in a Stein space $X$ is called uniformly Runge if for every sequence of positive real numbers $\{\epsilon_n\}_{n\geq 1}$, every sequence of compact sets $\{K_n\}_{n\geq 1}$, $K_n \subset D_n$, and every sequence of holomorphic functions $\{f_n\}_{n\geq 1}$, $f_n \in \mathcal{O}(D_n)$, there exist $f \in \mathcal{O}(X)$ such that for every $n \geq 1$ we have $\|f - f_n\|_{K_n} < \epsilon_n$. In our case we choose $h \in \mathcal{O}(\mathbb{C}^2)$ such that $h(0) = 0$ and $h(y_m) = m$ for $m \geq 1$. We choose then $W_m$ to be Runge in $\mathbb{C}^2$ and $m - \frac{1}{2} < \Re(h_{|\overline{W}_m}) < m + \frac{1}{2}$. It follows from Propositions 3 and 4 in [8] that $\{W_m\}_{m\geq 0}$ is uniformly Runge.

Let $U_m \Subset W_m$ be such that $0 \in U_0$ and $y_m \in U_m$ and let $U := \bigcup_{m\geq 0} U_m$ and $W := \bigcup_{m\geq 0} W_m$. We consider a function $f \in \mathcal{I}(W)$ such that $f_{|W_m} \equiv 0$ for $m \in A_{n+1}$ and $f_{|W_m} \not\equiv 0$ for $m \in A_n \setminus A_{n+1}$. As $\{W_m\}_{m\geq 0}$ is uniformly Runge in $\mathbb{C}^2$ we have that $f$ can be approximated uniformly on $U$ by functions in $\mathcal{I}(\mathbb{C}^2)$. Note now that $\mathcal{I}(\mathbb{C}^2) \subset I^*$ and that $f$ induces a continuous functions on an open subset of $Y$ containing $\pi(0)$. It follows that $f$ induces a function in $\mathcal{O}_Y(V)$ where $V$ is a neighborhood of $\pi(0)$. Clearly its germ $f_{\pi(0)} \in J_{n+1}$. As $A_n \setminus A_{n+1}$ is infinite we have that $f_{\pi(0)} \notin J_n$ as well. Therefore $\mathcal{O}_{Y,\pi(0)}$ is not Noetherian.

• For Example 2 we consider $\tilde{X} = \bigsqcup_{k\in\mathbb{Z}} X_k$ and $I^* = \{f \in \mathcal{O}(\tilde{X}) : f_{|L_2,k} = f_{L_1,k+1}\}$, $(Y, \mathcal{O}_Y)$ be the spectrum of $(\tilde{X}, I^*)$ and $\pi : \tilde{X} \to Y$ the quotient map. We have that $X = Y \setminus \{\pi(0)\}$ where 0 is the origin in any $X_k$. Clearly the functions in $\mathcal{O}_Y(Y)$ separate the points of $Y$. To show that $(Y, \mathcal{O}_Y)$ is holomorphically convex we consider again a discrete sequence $\{a_s\}_{s\in\mathbb{N}}$ in $Y$, we split the problem in two cases:
- there exists $k \in \mathbb{Z}$ such that $\{s \in \mathbb{N} : a_s \in \pi(X_k)\}$ is infinite
- there exists an infinite set $A \subset \mathbb{Z}$ such that for every $k \in A$ there exists $s \in \mathbb{N}$ such that $a_s \in \pi(X_k)$,
and we use Cartan's Theorem B.

As for Example 1 the scheme $Y$ is not locally compact (with the same type of argument) and the local ring $\mathcal{O}_{Y,\pi(0)}$ is not Noetherian. Here the non-stationary, ascending chain of ideals is $\{J_n\}_{n\geq 1}$ where $J_n$ consists of those germs at $\pi(0)$ of functions $f \in \mathcal{O}_Y(V)$ such that $f \circ \pi$ vanishes identically on $\bigcup_{|k|\geq n} \pi(X_k)$.

**Remark.** Note that in both examples the quotient map has 0-dimensional fibers and still the complex scheme is not a complex space. This shows that the condition on the connectivity of the fibers required by Grauert in [7], Satz 2, page 389, is essential.

# 3 Nori strings and envelopes of holomorphy

We have seen in the previous section that there exist complex spaces that satisfy the conditions a) and b) but are not open subsets of Stein spaces. In particular they do not have envelopes of holomorphy. For the first time Grauert [7] noted the connection between envelopes of holomorphy and complex schemes. We want now to relate the coverings of a desingularization of a germ of a two-dimensional normal singularity $(X, x_0)$ to a complex scheme. Let $\pi : Y \to X$ be a desingularization of $(X, x_0)$ with normal crossings and denote by $A$ the exceptional set of $\pi$, i.e., $A = \pi^{-1}(x_0)$. If $A$ contains no cycles then any covering $\tilde{Y}$ of $Y$ is holomorphically convex, therefore $\mathcal{O}(\tilde{Y})$ is a Stein algebra. We are interested in the more difficult case when $A$ contains cycles. For simplicity we assume that $A$ is a cycle (the proof is the same if $A$ contains a cycle). As in [5] we can construct a covering space $p : \tilde{Y} \to Y$ such that $\tilde{A} := p^{-1}(A)$ is a Nori string.

The following result is due to A. Brudnyi [1].

**Theorem 1.** *Let $X$ be a 1-convex manifold, $D \Subset X$ a strongly pseudoconvex domain with smooth boundary, containing the exceptional set $A$ of $X$. Let $p : \tilde{X} \to X$ be a covering map, $\tilde{D} = p^{-1}(D)$, and $\tilde{A} = p^{-1}(A)$. Then if $x_1, x_2 \in \tilde{D} \setminus \tilde{A}$, $x_1 \neq x_2$ and $p(x_1) = p(x_2)$ there exists $f \in \mathcal{O}(\tilde{D})$ such that $f(x_1) \neq f(x_2)$.*

**Remark.** In fact the author shows that the function $f$ can be chosen to be $L^2$ on the covering with respect to the pull-back of a Riemannian metric on $X$.

Let $I^* = \mathcal{O}(\tilde{Y})$ and let $(Z, \mathcal{O}_Z)$ be the spectrum of $(\tilde{Y}, I^*)$.

**Corollary 1.** *The topological space $Z$ is the (topological) contraction of $\tilde{A}$ to a point $P_0$ and $(\mathcal{O}_Z)_{|Z \setminus \{P_0\}} = (\mathcal{O}_{\tilde{Y}})_{|\tilde{Y} \setminus \tilde{A}}$.*

We want to prove the following:

**Theorem 2.** *$\tilde{Y}$ has no envelope of holomorphy in the category of complex spaces and $Z$ is not a complex space.*

*Proof.* Let us assume by reductio ad absurdum that $\tilde{Y}$ has an envelope of holomorphy $\tau : \tilde{Y} \to W$ in the category of complex spaces. It follows from Theorem 1 that on $\tilde{Y} \setminus \tilde{A}$ the map $\tau$ is an injective immersion. At the same time since $\tilde{A}$ is connected and all its irreducible components are compact it follows that $\tau(\tilde{A})$ is just a point.

Let $f$ be a holomorphic function on the germ $(X, x_0)$ such that $f(x_0) = 0$ and $\{f = 0\}$ is smooth outside $\{x_0\}$. On the desingularization $Y$ we let $A_1$ be the union of all non-compact irreducible components of $\{f \circ \pi = 0\}$, therefore the union of those components that do not belong to the decomposition of the exceptional set $A$ into irreducible components. Let $\tilde{A}_1 = p^{-1}(A_1)$. The map $f \circ \pi \circ p$ has a unique extension $f_1 \in \mathcal{O}(W)$. Since $\tau$ is injective on $\tilde{Y} \setminus \tilde{A}$ and $p$ is infinitely sheeted this would imply that $\{f_1 = 0\}$ had infinitely many local irreducible components at $\tau(\tilde{A})$, which is, of course, a contradiction.

Completely the same argument shows that $Z$ is not a complex space. □

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## References


[1] A. Brudnyi: On holomorphic $L^2$ functions on coverings of strongly pseudoconvex manifolds. *Publ. Res. Inst. Math. Sci.* **43** (2007), no. 4, 963–976.

[2] M. Colţoiu: Some open problems concerning Stein spaces. Analyse complexe (Bucharest, 1989). *Rev. Roumaine Math. Pures Appl.* **36** (1991), no. 5-6, 225–229.

[3] M. Colţoiu; K. Diederich: Open sets with Stein hypersurface sections in Stein spaces. *Ann. of Math. (2)* **145** (1997), no. 1, 175–182

[4] M. Colţoiu; K. Diederich: On Levi's problem on complex spaces and envelopes of holomorphy. *Math. Ann.* **316** (2000), no. 1, 185–199.

[5] M. Colţoiu; M. Tibăr: Steinness of the universal covering of the complement of a 2-dimensional complex singularity. *Math. Ann.* **326** (2003), no. 1, 95–104.

[6] E. Falbel: Nonembeddable CR-manifolds and surface singularities. *Invent. Math.* **108** (1992), no. 1, 49–65

[7] H. Grauert: Bemerkenswerte pseudokonvexe Mannigfaltigkeiten. *Math. Z.* **81** (1963), 377–391.

[8] C. Joiţa: On uniformly Runge domains. *J. Math. Kyoto Univ.* **47** (2007), no. 4, 875–880.

[9] I. Lieb: Über komplexe Räume and komplexe Spektren. *Invent. Math.* **1** (1966), 45–58.

[10] R. Narasimhan. Several complex variables. Chicago Lectures in Mathematics. The University of Chicago Press, Chicago, Ill.-London, 1971.

[11] O. Suzuki: A new class of domains of holomorphy. I. The concepts of boundary resolutions and $L$-manifolds. *Publ. Res. Inst. Math. Sci.* **13** (1977/78), no. 2, 497–521.

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