On a Problem of Bremermann Concerning Runge Domains *

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Abstract

In this paper we give an example of a bounded Stein domain in $\mathbb{C}^n$, with smooth boundary, which is not Runge and whose intersection with every complex line is simply connected.

1 Introduction

In [2] Bremermann asked the following question:

"Suppose that D is a Stein domain in $\mathbb{C}^n$ such that for every complex line l in $\mathbb{C}^n$, $l \setminus D$ is connected. Is it true that D is Runge in $\mathbb{C}^n$?"

The question remained open and was mentioned again in a recent book by T. Ohsawa ([5], page 81). In this paper we will give a negative answer to Bremermann's question. Namely, we will give an example of a bounded, strictly pseudoconvex domain in $\mathbb{C}^n$ with real analytic boundary which is not Runge in $\mathbb{C}^n$ but whose intersection with every complex line is simply connected.

Note that if D is bounded the hypothesis of the problem means simply that for every complex line l, $l \cap D$ is Runge in l. If, in addition, one requires that $l \cap D$ is connected as well then it does follow that D is Runge. See for example [1], page 79, Theorem 3.1.5 or [3], page 309, Theorem 4.7.8.

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For simplicity our construction will be done in $\mathbb{C}^2$ but it can be easily adapted to $\mathbb{C}^n$ for $n \geq 2$. To produce our example we will construct first a bounded, strictly pseudoconvex domain $W \subset \mathbb{C}^2$ with smooth, real analytic boundary which is Runge but its closure is not holomorphically convex. (Note that this is not possible in $\mathbb{C}$.) Next we show that, in fact, we can construct W as above and moreover it has the following geometric property: for every complex line l the set of points where l is tangent to ∂W is at most finite. If this is the case, then one can show that $l \cap \overline{W}$ is polynomially convex, again for every complex line l. Finally, we show that an appropriate neighborhood of $\overline{W}$ is a counterexample to Bremermann's problem.

2 The Example

The construction will be done in several steps.

First we prove that there exists a bounded domain in $\mathbb{C}^2$ with smooth, real analytic boundary which is strictly pseudoconvex, Runge in $\mathbb{C}^2$, and its closure is not polynomially convex. To our knowledge, such an example does not exist in the literature.

J. Wermer [6] proved that there exists a biholomorphic map F from a polydisc $P = \{(z_1, z_2) \in \mathbb{C}^2 : \ |z_1| < a, |z_2| < b\}$ in $\mathbb{C}^2$ onto an open set $F(P)$ of $\mathbb{C}^2$ such that $F(P)$ is not polynomially convex. (Wermer's original result was in $\mathbb{C}^3$ but it can be modified to hold in $\mathbb{C}^2$ as well; see [4] or [5].) We start with such a map and let $U_n := \{z \in \mathbb{C}^2 : \ |\frac{z_1}{a}|^n + |\frac{z_2}{b}|^n < 1\}$. Since $U_n \subset U_{n+1} \subset P$ and $\cup U_n = P$, it follows that there exists $m \in \mathbb{N}$ such that $F(U_m)$ is not polynomially convex. Set $U = U_m$ and $V = F(U)$. If we define $\varphi : U \to \mathbb{R}$ by $\varphi(z) = \dfrac{1}{1 - |z_1/a|^{2m} - |z_2/b|^{2m}} + |\dfrac{z_1}{a}|^2 + |\dfrac{z_2}{b}|^2 - 1$ then φ is a strictly plurisubharmonic real analytic function and has only one critical point. Since F is a biholomorphism, $\varphi \circ F^{-1} : V \to \mathbb{R}$ has the same properties and it is an exhaustion function for V. For $\alpha > 0$ let $V_\alpha = \{z \in V : \ \varphi \circ F^{-1}(z) < \alpha\}$. It follows that there exists $\alpha > 0$ such that V_α is not polynomially convex. On the other hand, if $z_0 = F(0)$ (this is the minimum point and the only critical point of $\varphi \circ F^{-1}$ and $\varphi \circ F^{-1}(z_0) = 0$) and we choose $B \subset V$ a ball centered at z_0, then there exists $\alpha > 0$ such that $V_\alpha \subset B$. It follows that V_α is Runge in B (because $\varphi \circ F^{-1}$ is defined on B) and therefore is polynomially convex. Put $r := \sup\{\alpha \in \mathbb{R} : \ V_\alpha \text{ is polynomially convex}\}$. From the above observations we deduce that $0 < r < \infty$.

We claim that V_r is the example that we are looking for. Indeed V_r is Runge in $\mathbb{C}^2$ as an increasing union of Runge domains and it has smooth, real analytic boundary because $\varphi \circ F^{-1}$ has no critical point on the ∂V_r. We only need to convince ourselves that $\overline{V}_r = \{z \in V : \ \varphi \circ F^{-1}(z) \leq r\}$ is not polynomially convex. If $\overline{V}_r$ were polynomially convex then it would have a Runge (in $\mathbb{C}^2$) neighborhood W with $W \subset V$. If this were the case then for $\epsilon > 0$ small enough, we would have $V_{r+\epsilon} \subset W$ and $V_{r+\epsilon}$ would be Runge in W and therefore in $\mathbb{C}^2$. This would contradict the choice of r.

Let us rephrase what we have done so far. We proved that if V is a domain in $\mathbb{C}^2$ and $\phi : V \to \mathbb{R}$ is a strictly plurisubharmonic function such that there exist $a_0 < a_1$ real numbers with the following properties:

$$\left.\begin{array}{l} \{x \in V : \phi(x) < a_1\} \subset\subset V, \\ \{x \in V : \phi(x) < a_0\} \text{ is connected and contains } C(\phi) := \text{the set of} \\ \text{critical points of } \phi, \\ \{x \in V : \phi(x) < a'\} \text{ is Runge in } \mathbb{C}^2, \text{ for some } a' > a_0, \\ \{x \in V : \phi(x) < a_1\} \text{ is not Runge in } \mathbb{C}^2 \end{array}\right\} (*)$$

then there exists a unique real number $r = r(\phi) \in [a_0, a_1)$ such that $V_{r(\phi)} := \{x \in V : \phi(x) < r(\phi)\}$ is Runge and $\overline{V}_{r(\phi)} = \{x \in V : \phi(x) \leq r(\phi)\}$ is not holomorphically convex. Note that $V_{r(\phi)}$ must be connected since each of its components contains a (minimum) critical point, $V_{r(\phi)}$ contains $\{x \in V : \phi(x) < a_0\}$ which is connected and $\{x \in V : \phi(x) < a_0\} \supset C(\phi)$. We also proved that there exists a real analytic function ϕ satisfying (*). We fix such a ϕ. Shrinking V we can assume that $\overline{V}$ is compact and that ϕ is defined on a neighborhood of $\overline{V}$.

Next we want to show that there exists ψ, a small perturbation of ϕ, which satisfies (*) and in addition it has the following geometric property: for every complex line l the set $\mathcal{T}(\psi, l) := \{x \in \partial V_{r(\psi)} \cap l : l \text{ is tangent to } \partial V_{r(\psi)} \text{ at } x\}$ is finite.

Indeed: let U be an open and connected set such that $C(\phi) \subset U \subset\subset V_{r(\phi)}$ and let W be an open and relatively compact neighborhood of $\partial V_{r(\phi)}$ and $0 < \delta < \delta' < a_1 - r(\phi)$ two real numbers such that $U \subset\subset \{x \in V : \phi(x) < r(\phi) - \delta\}$ and $\{x \in V : r(\phi) - \delta < \phi(x) < r(\phi) + \delta\} \subset\subset W \subset\subset \{x \in V : \phi(x) < r(\phi) + \delta'\}$.

If $\epsilon > 0$ is small enough then, for every $\psi : V \to \mathbb{R}$, a $\mathcal{C}^\infty$ function, if the sup norms on $\overline{V}$ of $\psi - \phi$, $\frac{\partial(\psi-\phi)}{\partial x_j}$, $\frac{\partial^2(\psi-\phi)}{\partial x_j \partial x_k}$, $j, k = 1, ..., n$ (here we denote $z_j = x_{2j-1} + ix_{2j}$) are less than ϵ, then ψ is strictly plurisubharmonic and satisfies (*). Moreover $C(\psi) \subset U$, $r(\psi) \in [r(\phi) - \delta, r(\phi) + \delta]$ and $\partial\{x \in V : \psi < s\} \subset W$ for every $s \in [r(\phi) - \delta, r(\phi) + \delta]$. We claim that there exists a real analytic ψ such that for every complex line l and for every $s \in [r(\phi) - \delta, r(\phi) + \delta]$ the following set $\mathcal{T}(\psi, l, s) := \{x \in \partial\{x \in V : \psi(x) < s\} \cap l : l \text{ is tangent to } \partial\{x \in V : \psi(x) < s\} \text{ at } x\}$ is finite. Indeed, suppose that there exists x_0 which is not isolated in some $\mathcal{T}(\psi, l, s)$ (note that s is then $\psi(x_0)$ and l is the complex tangent line at x_0 to $\{\psi = \psi(x_0)\}$). If we denote by $u := \psi_{|l}$ then x_0 is not isolated in $\{z \in l : u(z) = u(x_0),\ \nabla u(z) = 0\}$. On the other hand u is real analytic and strictly subharmonic. It follows that around x_0 at least one of the sets $\{z \in l \cap V : \frac{\partial u}{\partial x}(z) = 0\}$ or $\{z \in l \cap V : \frac{\partial u}{\partial y}(z) = 0\}$ is smooth and then the smooth one is contained in the other one. Hence there exists around x_0 a smooth real analytic curve C such that $u_{|C} = \text{constant} = u(x_0)$ and $\nabla u_{|C} = 0$. If $\{f = 0\}$ is a local equation for C and a is any point on C it follows that in a neighborhood of a, u can be written as $u = u(a) + f^2 g$. This shows that if

(*) There exists l and s such that $\mathcal{T}(\psi, l, s)$ is not finite, then

(**) There exists a complex line L, a constant k, and a germ of smooth real analytic curve C in L such that in a neighborhood of each point of C $\psi_{|L} = k + f^2 g$ where f is a local defining function for C.

Assume that (**) holds.

Let a be any point on C. It follows that L is the complex tangent line at a to $\{\psi = k\}$ and therefore a parametrization for L is

$$\lambda \to \Big(\frac{\partial\psi}{\partial z_2}(a)\lambda, -\frac{\partial\psi}{\partial z_1}(a)\lambda\Big) + a$$

We write $\lambda = s + it$ and we consider the one complex variable function ψ_1 given by $\psi_1(\lambda) = \psi((\frac{\partial\psi}{\partial z_2}(a)\lambda, -\frac{\partial\psi}{\partial z_1}(a)\lambda + a)$. Then ψ_1 is strictly subharmonic around 0 hence at least one of $\frac{\partial^2\psi_1}{\partial s^2}(0)$, $\frac{\partial^2\psi_1}{\partial t^2}(0)$ is positive. Without loss of generality we assume that $\frac{\partial^2\psi_1}{\partial s^2}(0)$ is positive. Then, around 0, we have: $\psi_1 = \psi_1(0) + \Big((2\frac{\partial^2\psi_1}{\partial s^2}(0))^{-1/2}\frac{\partial\psi_1}{\partial s}\Big)^2 g(\lambda)$, where g is a real analytic function with $g(0) = 1$. It follows that

(1) $\det Hess(\psi_1)(0,0) = 0$

and if we consider the linear change of coordinates $\chi(s,t) = (\sigma,\tau)$ with $\sigma = (2\frac{\partial^2\psi_1}{\partial s^2}(0))^{-1/2}(\frac{\partial^2\psi_1}{\partial s^2}(0)s + \frac{\partial^2\psi_1}{\partial s\partial t}(0)t)$, $\tau = t$, and we set $\psi_2 = \psi_1 \circ \chi^{-1}$ we also have, at 0,

(2) $\frac{\partial^3\psi_2}{\partial\tau^3}(0,0) = 0$

(3) $\frac{2}{3}\frac{\partial^4\psi_2}{\partial\tau^4}(0,0) = [\frac{\partial^3\psi_2}{\partial\sigma\partial\tau^2}(0,0)]^2$

(4) $\frac{4}{5!}\frac{\partial^5\psi_2}{\partial\tau^5}(0,0) = [\frac{1}{6}\frac{\partial^4\psi_2}{\partial\sigma\partial\tau^3}(0,0) - \frac{1}{8}\frac{\partial^3\psi_2}{\partial\sigma\partial\tau^2}(0,0)\frac{\partial^3\psi_2}{\partial\sigma^2\partial\tau}(0,0)]\frac{\partial^3\psi_2}{\partial\sigma\partial\tau^2}(0,0)$.

Note also that all these four relations can be expressed only in terms of ψ, without involving L, s, t, σ or τ.

$\Big($E.g. $\frac{\partial^2\psi_1}{\partial s^2}(0,0) = \frac{\partial^2\psi}{\partial z_1^2}(a)\Big(\frac{\partial\psi}{\partial z_2}(a)\Big)^2 + \frac{\partial^2\psi}{\partial \bar{z}_1^2}(a)\Big(\overline{\frac{\partial\psi}{\partial z_2}(a)}\Big)^2 + 2\frac{\partial^2\psi}{\partial z_1\partial\bar{z}_1}(a)\Big|\frac{\partial\psi}{\partial z_2}(a)\Big|^2 + \frac{\partial^2\psi}{\partial z_2^2}(a)\Big(\frac{\partial\psi}{\partial z_1}(a)\Big)^2 + \frac{\partial^2\psi}{\partial\bar{z}_2^2}(a)\Big(\overline{\frac{\partial\psi}{\partial z_1}(a)}\Big)^2 + 2\frac{\partial^2\psi}{\partial z_2\partial\bar{z}_2}(a)\Big|\frac{\partial\psi}{\partial z_1}(a)\Big|^2 - 2\frac{\partial^2\psi}{\partial z_1\partial z_2}(a)\frac{\partial\psi}{\partial z_2}(a)\frac{\partial\psi}{\partial z_1}(a) - 2\frac{\partial^2\psi}{\partial z_1\partial\bar{z}_2}(a)\frac{\partial\psi}{\partial z_2}(a)\overline{\frac{\partial\psi}{\partial z_1}(a)} - 2\frac{\partial^2\psi}{\partial\bar{z}_1\partial z_2}(a)\overline{\frac{\partial\psi}{\partial z_2}(a)}\frac{\partial\psi}{\partial z_1}(a) - 2\frac{\partial^2\psi}{\partial\bar{z}_1\partial\bar{z}_2}(a)\overline{\frac{\partial\psi}{\partial z_2}(a)\frac{\partial\psi}{\partial z_1}(a)}\Big)$

They are, in fact, four differential equations that must be satisfied by ψ at each point $a \in C$. For a generic real analytic function ψ the intersection of the four real anaytic sets defined by this four equations is discrete (ψ is defined on $\mathbb{C}^2 \equiv \mathbb{R}^4$) hence it cannot contain C (the germ of a real analytic curve). As a matter of fact, from $\psi_{|L} = k + f^2 g$ one can come up with as many differential equations as one wants, but their are more and more complicated.

We fix now a ψ which satisfies (*) and the geometrical property mentioned above.

Our next goal will be to show that, for every complex line l in $\mathbb{C}^2$, $l \cap \overline{V}_{r(\psi)}$ is polynomially convex (although $\overline{V}_{r(\psi)}$ is not). Note that $(l \cap \overline{V}_{r(\psi)}) \setminus \overline{l \cap V_{r(\psi)}}$ is a finite set (as a subset of $\mathcal{T}(\psi, l)$). Hence it suffices to show that $\overline{l \cap V_{r(\psi)}}$ is polynomially convex. Let's assume that it is not. Note that $l \cap V_{r(\psi)}$ is Runge in l (since $V_{r(\psi)}$ is Runge in $\mathbb{C}^2$) and that it has a smooth boundary except at a finite set of points (the set of points of non-smoothness is also a subset of $\mathcal{T}(\psi, l)$). As we assumed that $\overline{l \cap V_{r(\psi)}}$ is not polynomially convex it follows that there exists a rectifiable loop γ in l such that $\gamma \setminus (l \cap V_{r(\psi)})$ contains only points where the boundary of $l \cap V_{r(\psi)}$ in l is not smooth and therefore is finite and $\widehat{\gamma} \cap (l \setminus (\overline{l \cap V_{r(\psi)}})) \neq \emptyset$ (in fact it has a nonempty interior). Using again the finiteness of $\mathcal{T}(\psi, l)$ it follows that $\widehat{\gamma} \cap (\mathbb{C}^2 \setminus \overline{V}_{r(\psi)}) \neq \emptyset$. We claim that there exists a $\mathcal{C}^\infty$ family of biholomorphisms $\{f_\epsilon : \mathbb{C}^2 \to \mathbb{C}^2\}_{\epsilon \in \mathbb{R}}$ such that f_0 is the identity and for $\epsilon > 0$ small enough $f_\epsilon(\gamma) \subset V_{r(\psi)}$. Without loss of generality we can assume that $l = \{z = (z_1, z_2) \in \mathbb{C}^2 : z_2 = 0\}$. We write $\gamma \setminus (l \cap V_{r(\psi)}) =: \{(p_1, 0), \dots, (p_s, 0)\}$ and we denote by $(0, q_1), \dots (0, q_s)$ the unit inner normals to $\partial V_{r(\psi)}$. We choose $h : \mathbb{C} \to \mathbb{C}$ a holomorphic function such that $h(p_j) = q_j$ and we define $f_\epsilon(z) = (z_1, z_2 + \epsilon h(z_1))$. It is obvious that f_ϵ are biholomorphisms and since $\dfrac{\mathrm{d} f_\epsilon}{\mathrm{d}\epsilon}(p_j, 0) = (0, q_j)$ it follows that f_ϵ have the sought properties. Because $\widehat{\gamma} \cap (\mathbb{C}^2 \setminus \overline{V}_{r(\psi)}) \neq \emptyset$ and $\{f_\epsilon\}$ is a continuous family we deduce that for ϵ small enough $f_\epsilon(\widehat{\gamma}) \not\subset \overline{V}_{r(\psi)}$. On the other hand $f_\epsilon(\widehat{\gamma}) = \widehat{f_\epsilon(\gamma)}$ and $f_\epsilon(\gamma) \subset V_{r(\psi)} \cap f_\epsilon(l)$. It follows from here that $V_{r(\psi)} \cap f_\epsilon(l)$ is not Runge in $f_\epsilon(l)$ which is a contradiction since $V_{r(\psi)}$ is Runge in $\mathbb{C}^2$ and $f_\epsilon(l)$ is a closed analytic submanifold in $\mathbb{C}^2$.

We are now ready to produce our example. For $\epsilon > 0$ we set $W_\epsilon := \{x \in V : \psi(x) < r(\psi) + \epsilon\}$. It follows from the definition of $r(\psi)$ that W_ϵ is not Runge in $\mathbb{C}^2$. We wish to prove that there exists $\epsilon > 0$ such that for every complex line l, $W_\epsilon \cap l$ is Runge in l.

Suppose that this is not the case. Then for $n \in \mathbb{N}$ large enough there exists a complex line l_n such that $W_{\frac{1}{n}} \cap l_n$ is not Runge in l_n. Note that $\{l_n\}$ is a sequence of lines that intersect a given compact subset of $\mathbb{C}^2$. It contains then a convergent subsequence. By passing to this subsequence we can assume that $\{l_n\}$ converges to a line l.

We already proved that $l \cap \overline{V}_{r(\psi)}$ is holomorphically convex and this implies that there exists Ω a Runge open subset of $\mathbb{C}^2$ such that $l \cap \overline{V}_{r(\psi)} \subset \Omega \subset V$. As $\cap W_{\frac{1}{n}} = \overline{V}_{r(\psi)}$ and l_n converges to l we deduce that there exists $n_0 \in \mathbb{N}$ such that, for every $n \geq n_0$, $W_{\frac{1}{n}} \cap l_n \subset \Omega$. Hence $W_{\frac{1}{n}} \cap l_n = (W_{\frac{1}{n}} \cap \Omega) \cap l_n$. On the other hand, ψ is a plurisubharmonic function defined on the whole Ω which is Stein and therefore $W_{\frac{1}{n}} \cap \Omega$, which is a level set for $\psi_{|\Omega}$, is Runge in Ω. Since Ω is Runge in $\mathbb{C}^2$ it follows that $W_{\frac{1}{n}} \cap \Omega$ is also Runge in $\mathbb{C}^2$ and from here we obtain that $W_{\frac{1}{n}} \cap l_n$ is Runge in l_n. This contradicts our assumption.

In conclusion, we proved that for $\epsilon > 0$ small enough W_ϵ is bounded, strictly pseudoconvex, is not Runge in $\mathbb{C}^2$ and for every complex line l in $\mathbb{C}^2$, $W_\epsilon \cap l$ is Runge in l. In the same way as before W_ϵ must be connected since each of its components contains a critical point of ψ.

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