Analytic cohomology groups in top degrees of Zariski open sets in $\mathbb{P}^n$

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Dedicated to Professor Cabiria Andreian Cazacu on her 80^{th} birthday

1 Introduction

Let $A \subset \mathbb{P}^n$ be a closed analytic subset of pure codimension q. We denote by $\tilde{q} = n - \left[\frac{n}{q}\right] + 1$ and $\hat{q} = n - \left[\frac{n-1}{q}\right]$ (here $[x]$ stands for the integer part of the real number x). Therefore $\hat{q} = \tilde{q}$ if $q|n$ and $\hat{q} = \tilde{q} - 1$ if $q \not| n$. It follows from M. Peternell's results [16] that $\mathbb{P}^n \setminus A$ is q-complete with corners and from Diederich and Fornaess approximation theorem [7] that $\mathbb{P}^n \setminus A$ is $\tilde{q}$-complete in the sense of Andreotti-Grauert [1]. In particular for every coherent analytic sheaf $\mathcal{F} \in Coh(\mathbb{P}^n \setminus A)$ the cohomology groups $H^i(\mathbb{P}^n \setminus A, \mathcal{F})$ are trivial for $i \geq \tilde{q}$.

In this paper we will improve this result, in cohomological setting, for Zariski open subsets of $\mathbb{P}^n$ by replacing $\tilde{q}$ with $\hat{q}$. However we will do this only for coherent algebraic sheaves $\mathcal{F} \in Coh(\mathbb{P}^n)$. The integer $\hat{q}$ was introduced by K. Matsumoto [13] in the study of cohomologic convexity of some open sets in complex manifolds which are finite intersection of q-complete open sets. The integer $\tilde{q}$ was introduced by Diederich and Fornaess [7] in connection with the results of G. Faltings [9]. Moreover the vanishing of cohomology spaces will be replaced by their finite-dimensionality. Finite dimensionality conditions for cohomology spaces were considered in some cases in algebraic context by R. Hartshorne [10], A. Ogus [14] and have as analytic correspondent the notion of q-convex space [1].

More precisely, we prove:

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Theorem 1. *Let $A \subset \mathbb{P}^n$ be a closed analytic subset of pure codimension q. Then $\dim_{\mathbb{C}} H^i(\mathbb{P}^n \setminus A, \mathcal{F}) < \infty$ for every $\mathcal{F} \in Coh(\mathbb{P}^n)$ and every $i \geq \hat{q}$.*

This result is optimal, as Proposition 3 shows and represents an improvement on the previously known results when $q \not| n$ (hence $\hat{q} = \tilde{q} - 1$). It is not known if, under the assumptions of Theorem 1, $\mathbb{P}^n \setminus A$ is $\hat{q}$-convex in the sense of Andreotti-Grauert or if at least $\dim_{\mathbb{C}} H^i(\mathbb{P}^n \setminus A, \mathcal{F}) < \infty$ for every $\mathcal{F} \in Coh(\mathbb{P}^n \setminus A)$ if $i \geq \hat{q}$ (i.e. if $\mathbb{P}^n \setminus A$ is cohomologically $\hat{q}$-convex).

For small codimension, that is for $\dim A \geq \frac{n}{2}$, using Peternell's comparison theorem ([18]) between algebraic and analytic cohomology spaces H^{n-2} (the density of the algebraic cohomology in the the analytic one), and the results of C. Hunecke and G. Lyubeznik [11] regarding the algebraic cohomology we deduce the following theorem:

Theorem 2. *Let $A \subset \mathbb{P}^n$ be an irreducible closed analytic subset of codimension q where q satisfies $n - 1 \geq 2q$. Then for every $\mathcal{F} \in Coh(\mathbb{P}^n)$ we have, for analytic cohomology spaces, $H^{n-2}(\mathbb{P}^n \setminus A, \mathcal{F}) = 0$.*

Cohomology in top degree (i.e. $H^n(X, \mathcal{F})$ for a n-dimensional complex space X) is essentially described, in the algebraic case, in Hartshorne-Lichtenbaum Theorem [10] (see also G. Chiriacescu [5]) and in analytic context by the results of Y.-T. Siu [20] and T. Ohsawa [15].

2 Results

We recall first a few notions from the theory of analytic q-convexity, theory initiated by Andreotti and Grauert [1].

A complex space X is called q-convex if there exists $\varphi : X \to \mathbb{R}$ a smooth exhaustion function which is strictly q-convex outside a compact set K. If one could choose K to be the empty set then X is called q-complete. For example if $A \subset \mathbb{P}^n$ is a q-codimensional closed complex submanifold then $\mathbb{P}^n \setminus A$ is q-convex ([3]) and $2q - 1$-complete ([17]). In general $\mathbb{P}^n \setminus A$ is not q-complete. However if A is a complete intersection then $\mathbb{P}^n \setminus A$ is q-complete.

For q-convex spaces Andreotti and Grauert [1] have proved the following result which is fundamental for finite dimensionality of cohomology spaces: if X is a q-convex complex space then it is cohomologically q-convex, i.e.

$\dim_{\mathbb{C}} H^i(X, \mathcal{F}) < \infty$ for every coherent analytic sheaf $\mathcal{F} \in Coh(X)$ and every integer $i \geq q$. For q-complete complex spaces they proved a similar theorem (a vanishing theorem), replacing finite dimensionality with the vanishing of cohomology spaces.

A continuous function $\varphi : X \to \mathbb{R}$ is called q-convex with corners (see [16], [7]) if locally it is equal to the maximum of a finite set of smooth q-convex functions. Analogously to the notions of q-convex and q-complete spaces one can define q-convex with corners and q-complete with corners spaces. For example, it was proved in [16] that the complement of a closed analytic subset of $\mathbb{P}^n$ of pure codimension q is q-complete with corners. A fundamental result in the theory of q-convexity with corners is Diederich and Fornaess approximation theorem [7], [8]. According to this theorem q-convex functions with corners can be approximated by $\tilde{q}$-convex functions where $\tilde{q} = n - \left[\frac{n}{q}\right] + 1$ and n is the dimension of X, the complex space that the function is defined on. As a direct consequence of this theorem one gets that for every closed analytic subset A of $\mathbb{P}^n$ (possibly with singularities) of pure codimension q, its complement $\mathbb{P}^n \setminus A$ is $\tilde{q}$-complete. In particular it follows from [1] that $H^i(\mathbb{P}^n \setminus A, \mathcal{F}) = 0$ for every coherent analytic sheaf $\mathcal{F} \in Coh(\mathbb{P}^n \setminus A)$ and every $i \geq \tilde{q}$.

In [13] K. Matsumoto has studied the vanishing of cohomology groups for $i \geq \hat{q}$ for those open subsets U of a non-compact complex manifold X that are finite intersection of q-complete open sets. She proved that in this setting the cohomology groups $H^i(U, \mathcal{F})$ vanish for every $\mathcal{F} \in Coh(X)$ and every $i \geq \hat{q} = n - \left[\frac{n-1}{q}\right]$.

In what follows we would like to study the finite dimensionality of cohomology spaces in this range, $i \geq \hat{q}$, for Zariski open subsets $U \subset \mathbb{P}^n$, $U = \mathbb{P}^n \setminus A$ where A is a pure q-codimensional analytic subset.

To prove Theorem 1 we will need some preliminary results. Following [2] and [19] a linear map between two $\mathbb{C}$-vector spaces $f : E \to F$ is called Φ-injective if its kernel is finite dimensional, f is called Φ-surjective if its cokernel is finite dimensional and Φ-bijective if it is both Φ-injective and Φ-surjective. If M is a topological space, $\mathcal{F}$ is a sheaf of $\mathbb{C}$-vector spaces on M and $\mathcal{D} = (D_1, D_2, \ldots, D_t)$ is a finite (ordered) tuple of open subsets of M we denote by $\delta^j(\mathcal{D}, \mathcal{F}) : H^j(D_1 \cap \cdots \cap D_t, \mathcal{F}) \to H^{j+t-1}(D_1 \cup \cdots \cup D_t, \mathcal{F})$ the following composition of boundary maps in corresponding Mayer-Vietoris sequences:

$$H^j(D_1\cap\cdots\cap D_t,\mathcal{F})\to H^{j+1}\left(\bigcap_{s=1}^{t-1}(D_s\cup D_t),\mathcal{F}\right)\to$$

$$H^{j+2}\left(\bigcap_{s=1}^{t-2}(D_s\cup D_{t-1}\cup D_t),\mathcal{F}\right)\to\cdots\to H^{j+t-1}(D_1\cup\cdots\cup D_t,\mathcal{F})$$

The proof of the following lemma is identical to that of Proposition 1 in [13].

Lemma 1. *Let $p\in\mathbb{N}$ be fixed. In the above setting, we assume that for every k with $1\le k\le t-1$ for every $i_1,\dots,i_k\in\{1,\dots,t\}$ and every $j\ge p$ the cohomology groups $H^j(D_{i_1}\cap\cdots\cap D_{i_k},\mathcal{F})$ are finite dimensional. Then:*
1) $\delta^j(\mathcal{D},\mathcal{F}):H^j(D_1\cap\cdots\cap D_t,\mathcal{F})\to H^{j+t-1}(D_1\cup\cdots\cup D_t,\mathcal{F})$ is Φ-bijective for every $j\ge p$
2) $\delta^{p-1}(\mathcal{D},\mathcal{F}):H^{p-1}(D_1\cap\cdots\cap D_t,\mathcal{F})\to H^{p+t-2}(D_1\cup\cdots\cup D_t,\mathcal{F})$ is Φ-surjective.

Assume now that X is a compact irreducible complex space of dimension n, $\mathcal{F}\in Coh(X)$ and $D_1,\dots,D_t$ are open q-complete subsets of X.

Lemma 2. *In the above setting the cohomology spaces $H^j(D_1\cap\cdots\cap D_t,\mathcal{F})$ are finite dimensional for every $j\ge\hat{q}$.*

The proof of this lemma is identical to that of Lemma 2 of [13] using Lemma 1 above and the fact that for a compact complex space its cohomology groups with values in a coherent analytic sheaf are finite dimensional, see [4]. Note that in Lemma 2 if $D_1\cup\cdots\cup D_t\ne X$ (hence it is not compact) then $H^j(D_1\cap\cdots\cap D_t,\mathcal{F})=0$ for $j\ge\hat{q}$ (one uses here a theorem of Y.-T. Siu, [20], that states that $H^n(D_1\cup\cdots\cup D_t,\mathcal{F})=0$).

The following result will play a crucial role in the proof of Theorem 1. It is due to M. Peternell, [18].

Lemma 3. *Let $A\subset\mathbb{P}^n$ be a closed analytic subset of pure codimension q. Then there exist:*
1) An irreducible algebraic variety X, $\dim(X)=n$, together with a finite surjective holomorphic mapping $\phi:X\to\mathbb{P}^n$
2) Closed analytic subsets $A_1,\dots,A_r$ of X
3) Ample line bundles $L_1,\dots,L_r$ on X and, for each L_i, q holomorphic sections $s_{i_1},\dots,s_{i_q}\in\Gamma(X,L_i)$ such that:

a) $\phi^{-1}(A) = A_1 \cup \cdots \cup A_r$
b) $A_i = \{s_{i_1} = \cdots = s_{i_q} = 0\}$ *for* $i = 1, \ldots, r$.

Remarks: i) With the notations of Lemma 3, it follows that each $X \setminus A_i$ is q-complete as a union of q Stein open subsets
ii) $r = \deg(A)$ and if A is connected then $\phi^{-1}(A)$ is connected as well, however we will not need these facts.

Now we are ready to prove:

Theorem 1. *Let* $A \subset \mathbb{P}^n$ *be a closed analytic subset of pure codimension* q. *Then* $\dim_{\mathbb{C}} H^i(\mathbb{P}^n \setminus A, \mathcal{F}) < \infty$ *for every* $\mathcal{F} \in Coh(\mathbb{P}^n)$ *and every* $i \geq \hat{q}$.

Proof. We apply Lemma 3 and we find an irreducible projective variety X together with a finite surjective holomorphic map $\phi : X \to \mathbb{P}^n$ satisfying the conditions of that lemma. Because ϕ is finite and surjective it follows that in order to prove the theorem it is enough to show that for every coherent analytic sheaf $\mathcal{G} \in Coh(X)$ and every $i \geq \hat{q}$ we have that $\dim H^i(X \setminus \phi^{-1}(A), \mathcal{G}) < \infty$. However $\phi^{-1}(A) = \cup \phi^{-1}(A_i)$ and hence $X \setminus \phi^{-1}(A)$ is a finite intersection of q-complete domains. The finite dimensionality of $H^i(X \setminus \phi^{-1}(A), \mathcal{G})$ follows now from Lemma 2. □

Remarks: 1) Let $A \subset \mathbb{C}^n$ be a closed analytic subset of pure codimension q and let $x_0 \in A$ be an arbitrary point. Then there exists a small open neighborhood U of x_0 such that, by a convenient choice of the coordinate system, the restriction of the canonical projection $\pi : \mathbb{C}^n \to \mathbb{C}^{n-q}$ induces a finite map $U \cap A \to V$ where V is an open subset of $\mathbb{C}^{n-q}$. Using the same argument as in the proof of Lemma 3 (see [18]) one deduces that there exists an irreducible complex space X, $\dim(X) = n$, and a finite and surjective holomorphic map $\phi : X \to U$ such that $X \setminus \phi^{-1}(A)$ is a finite intersection of q-complete open subsets of X (each of them is the union of q open Stein subsets). We deduce that $H^i(U \setminus A, \mathcal{F}) = 0$ for every $\mathcal{F} \in Coh(U)$ and every $i \geq \hat{q}$. It would be interesting to know if one has this vanishing result for every $\mathcal{F} \in Coh(U \setminus A)$ or, more generally, if $U \setminus A$ is $\hat{q}$-complete.
2) If $A \subset \mathbb{C}^n$ is a closed algebraic subvariety of pure codimension q then we can choose the coordinate system in $\mathbb{C}^n$ such that the restriction to A of the standard projection $\pi : \mathbb{C}^n \to \mathbb{C}^{n-q}$ is finite and surjective. The above discussion shows that $H^i(\mathbb{C}^n \setminus A, \mathcal{F}) = 0$ for every $\mathcal{F} \in Coh(\mathbb{C}^n)$ and every $i \geq \hat{q}$. Hence we have just proved:

Theorem 1′. *Let $A \subset \mathbb{C}^n$ be a closed algebraic subvariety of pure codimension q. Then $H^i(\mathbb{C}^n \setminus A, \mathcal{F}) = 0$ for every $\mathcal{F} \in Coh(\mathbb{C}^n)$ and every $i \geq \hat{q}$.*

We will move now to the second main result of this paper. If X is a complex algebraic variety then there is a complex space, X_{an}, associated to X in a natural way and to each coherent (algebraic) sheaf $\mathcal{F}$ one can associate a coherent analytic sheaf $\mathcal{F}_{an}$ on X_{an} together with a canonical map $\alpha_q : H^q(X, \mathcal{F}) \to H^q(X_{an}, \mathcal{F}_{an})$, called comparison map. The main theorem of [18] is the following:

Proposition 1. *Let X be a Zariski open subset of $\mathbb{P}^n$, $\mathcal{F}$ a coherent sheaf on $\mathbb{P}^n$ and $\alpha : H^{n-2}(X, \mathcal{F}) \to H^{n-2}(X_{an}, \mathcal{F}_{an}) := T$ the comparison map. Then the image of α is dense in T with respect to the canonical topology of T. Hence α is surjective if T is finite dimensional.*

We will recall now the following important result due to Hunecke and Lyubeznik ([11], Theorem 5.1) for algebraic cohomology of $\mathbb{P}^n \setminus A$ when A is irreducible.

Proposition 2. *Let $A \subset \mathbb{P}^n$ be an irreducible closed analytic subset of codimension q and $\mathcal{F} \in Coh(\mathbb{P}^n)$. Then we have, for the algebraic cohomology spaces, $H^i(\mathbb{P}^n \setminus A, \mathcal{F}) = 0$ for every $i \geq n - \left[\frac{n-1}{q}\right]$.*

We can now prove:

Theorem 2. *Let $A \subset \mathbb{P}^n$ be an irreducible closed analytic subset of codimension q where q satisfies $n - 1 \geq 2q$. Then for every $\mathcal{F} \in Coh(\mathbb{P}^n)$ we have, for analytic cohomology spaces, $H^{n-2}(\mathbb{P}^n \setminus A, \mathcal{F}) = 0$.*

Proof. From Theorem 1 it follows that $H^{n-2}(\mathbb{P}^n \setminus A, \mathcal{F})$ is a finite dimensional vector space since $n - \left[\frac{n-1}{q}\right] \leq n - 2$. In particular its topology is separated (the topology of uniform convergence on compacts). It follows from Proposition 1 that the canonical comparison map in degree $n - 2$, between the algebraic and analytic cohomology, is surjective. As A is irreducible, Proposition 2 implies that the algebraic cohomology spaces in degree $n - 2$ vanish. We deduce that for the analytic cohomology $H^{n-2}(\mathbb{P}^n \setminus A, \mathcal{F}) = 0$ as well and the proof of the theorem is complete. □

Remark: The vanishing of the cohomology spaces $H^{n-1}(\mathbb{P}^n \setminus A, \mathcal{F})$ is studied in detail in [6], where A is a closed analytic subset of $\mathbb{P}^n$ of positive dimension and $\mathcal{F} \in Coh(\mathbb{P}^n \setminus A)$.

We will show now the optimality of $\hat{q}$ for every n and q. That means that for every n and q we will give an example of a closed analytic subset A of $\mathbb{P}^n$ of pure codimension q and such that $H^{\hat{q}-1}(\mathbb{P}^n \setminus A, \mathcal{F})$ is infinite dimensional for some $\mathcal{F} \in Coh(\mathbb{P}^n)$.

Proposition 3. *Suppose that $n, q \in \mathbb{N}$ with $n \geq 1$ and $0 \leq q \leq n$; Let $p = \left[\frac{n-1}{q}\right]$ and let $A_1, \ldots, A_{p+1}$ be $p+1$ closed analytic subsets of $\mathbb{P}^n$ of pure codimension q such that $A_1 \cap \cdots \cap A_{p+1} = \{x\}$ where x is some point in $\mathbb{P}^n$ and let $A = A_1 \cup \cdots \cup A_{p+1}$. Then $H^{\hat{q}-1}(\mathbb{P}^n \setminus A, \mathcal{O})$ is infinite dimensional where $\mathcal{O}$ stands for the sheaf of germs of holomorphic functions on $\mathbb{P}^n$.*

Proof. For $i = 1, \ldots, p+1$ let $D_i = \mathbb{P}^n \setminus A_i$. Hence we have to show that $H^{n-p-1}(\cap_{i=1}^{p+1} D_i, \mathcal{O})$ is infinite dimensional.

Step 1) We will prove by induction on $k = 1, \ldots, p+1$ that for every $J \subset \{k+1, \ldots, p+1\}$ and every $s \geq n-p+k-1$ the cohomology space $H^s\left((D_1 \cup \cdots \cup D_k) \cap \bigcap_{j \in J} D_j, \mathcal{O}\right)$ is finite dimensional.

For $k = 1$ we have to show that $H^{n-p}\left(\bigcap_{j \in J \cup \{1\}} D_j, \mathcal{O}\right)$ is finite dimensional. However $\bigcap_{j \in J \cup \{1\}} D_j = \mathbb{P}^n \setminus \bigcup_{j \in J \cup \{1\}} A_j$ and as $\bigcup_{j \in J \cup \{1\}} A_j$ is an analytic subset of $\mathbb{P}^n$ of pure codimension q the statement follows from Theorem 1.

We assume now that the statement is true for k and we prove it for $k+1$. So let $s \geq n-p+k$ and $J \subset \{k+2, \ldots, p+1\}$. We note $(D_1 \cup \cdots \cup D_{k+1}) \cap \bigcap_{j \in J} D_j = \left[(D_1 \cup \cdots \cup D_k) \cap \bigcap_{j \in J} D_j\right] \cup \left[\bigcap_{j \in J \cup \{k+1\}} D_j\right]$ and we write the Mayer-Vietoris sequence. We get that the sequence:

$$H^{s-1}\left((D_1 \cup \cdots \cup D_k) \cap \bigcap_{j \in J \cup \{k+1\}} D_j, \mathcal{O}\right) \to H^s\left((D_1 \cup \cdots \cup D_{k+1}) \cap \bigcap_{j \in J} D_j, \mathcal{O}\right)$$

$$\to H^s\left((D_1 \cup \cdots \cup D_k) \cap \bigcap_{j \in J} D_j, \mathcal{O}\right) \oplus H^s\left(\bigcap_{j \in J \cup \{k+1\}} D_j\right)$$

is exact. As $s-1 \geq n-p+k-1$ and $s \geq n-p+k-1$, it follows from the induction hypothesis that both $H^{s-1}\left((D_1 \cup \cdots \cup D_k) \cap \bigcap_{j \in J \cup \{k+1\}} D_j, \mathcal{O}\right)$

and $H^s\left((D_1 \cup \dots \cup D_k) \cap \bigcap_{j\in J} D_j, \mathcal{O}\right)$ are finite dimensional. On the other hand, as before, Theorem 1 implies that $H^s\left(\bigcap_{j\in J\cup\{k+1\}} D_j\right)$ is finite dimensional as well. We deduce that $H^s\left((D_1 \cup \dots \cup D_{k+1}) \cap \bigcap_{j\in J} D_j, \mathcal{O}\right)$ is finite dimensional and the proof by induction is complete.

Step 2) We will prove by descending induction on $k = p+1, p, \dots, 1$ that $H^{n-p+k-2}\left((\bigcup_{j=1}^{k} D_j) \cap (\bigcap_{j=k+1}^{p+1} D_j), \mathcal{O}\right)$ is infinite dimensional.

For $k = p+1$ we get $H^{n-1}(\cup_{j=1}^{p+1} D_j, \mathcal{O}) = H^{n-1}(\mathbb{P}^n \setminus \{x\}, \mathcal{O})$ which is infinite dimensional.

We assume now that the statement is true for $k+1$ and we will prove it for k. Using the identity

$$(\bigcup_{j=1}^{k+1} D_j) \cap (\bigcap_{j=k+2}^{p+1} D_j) = \left[(\bigcup_{j=1}^{k} D_j) \cap (\bigcap_{j=k+2}^{p+1} D_j)\right] \cup (\bigcap_{j=k+1}^{p+1} D_j)$$

we write the following Mayer-Vietoris sequence:

$$H^{n-p+k-2}\left((\bigcup_{j=1}^{k} D_j) \cap (\bigcap_{j=k+1}^{p+1} D_j), \mathcal{O}\right) \to H^{n-p+k-1}\left((\bigcup_{j=1}^{k+1} D_j) \cap (\bigcap_{j=k+2}^{p+1} D_j), \mathcal{O}\right)$$

$$\to H^{n-p+k-1}\left((\bigcup_{j=1}^{k} D_j) \cap (\bigcap_{j=k+2}^{p+1} D_j), \mathcal{O}\right) \oplus H^{n-p+k-1}(\bigcap_{j=k+1}^{p+1} D_j, \mathcal{O})$$

It follows from Step 1) that $H^{n-p+k-1}\left((\bigcup_{j=1}^{k} D_j) \cap (\bigcap_{j=k+2}^{p+1} D_j), \mathcal{O}\right)$ is finite dimensional and from Theorem 1 that $H^{n-p+k-1}(\bigcap_{j=k+1}^{p+1} D_j, \mathcal{O})$ is finite dimensional. At the same time, according to our induction hypothesis, $H^{n-p+k-1}\left((\bigcup_{j=1}^{k+1} D_j) \cap (\bigcap_{j=k+2}^{p+1} D_j), \mathcal{O}\right)$ is infinite dimensional. All these imply that $H^{n-p+k-2}\left((\bigcup_{j=1}^{k} D_j) \cap (\bigcap_{j=k+1}^{p+1} D_j), \mathcal{O}\right)$ is infinite dimensional and the induction is complete. For $k = 1$ we obtain exactly that $H^{n-p-1}(\cap_{i=1}^{p+1} D_i, \mathcal{O})$ is infinite dimensional.

□

Remark: As we mentioned before it follows from [7] that a finite intersection of q-complete open subsets of a complex manifold is $\tilde{q}$-complete. Matsumoto [13] raised the question of the optimality of $\tilde{q}$ for every q and every n. If M is a compact complex manifold of dimension n and X is a closed complex subvariety of M such that $M \setminus X$ is $(n-p)$-complete for some integer p, $0 \leq p \leq n-1$ then $H^j(M \setminus X, \mathbb{C})$ vanish for $j \geq 2n-p$ and then using a duality argument it follows easily that the morphism $H^j(M, \mathbb{C}) \longrightarrow H^j(X, \mathbb{C})$ induced by the inclusion $X \hookrightarrow M$ is an isomorphism for every $0 \leq j \leq p-1$ and a monomorphism for $j = p$. We set $M = \mathbb{P}^n$ and in $\mathbb{P}^n$ we consider $\left[\frac{n}{q}\right]+1$ linear subspaces of codimension q such that their intersection is empty. Let A be the union of this linear spaces. It follows from Theorem 10.9 of [12] that the topological condition mention above fails, hence $\mathbb{P}^n \setminus A$ is not $\tilde{q}-1$ complete. It turns out that the failure of this topological condition can be verified in an elementary fashion, without involving étale cohomology, using an argument similar to the one used in the proof of Proposition 3.

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