Chaotic dynamics of some rational maps

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1 Introduction

The iterations of real maps represent one of the easiest models of dynamical systems, but, despite its apparent simplicity, this one dimensional case proved to be rich in structure. Most of the efforts were concentrated to the understanding of the dynamics of maps from an interval to itself, like unimodal and S-unimodal cases and to the analysis of the behavior under iterations of the critical set.

In this paper we will study the dynamics of some rational maps $f(x) = \dfrac{P(x)}{Q(x)}$ where $P(x)$ and $Q(x)$ are polynomials in $\mathbb{R}[x]$. The dynamics of rational maps over the complex numbers has been extensively studied. For an introduction into the subject, one might consult [2]. In [13] J. Milnor describes the dynamics of quadratic rational maps ($\deg P$, $\deg Q \leq 2$) both with real and complex coefficients. In its full generality, the dynamical behavior of real rational maps remains an open problem.

As we mentioned before, we will place ourselves in the context of one dimensional dynamics and therefore we will regard f as a function on $\mathbb{R} \setminus A_1$, where A_1 (assumed $\neq \emptyset$) is the set of real zeros of Q. Since we will make use in an essential way of the order structure of $\mathbb{R}$ we will not pass to $S^1 = \mathbb{R} \cup \{\infty\}$. f^n will stand for the n-th iterate of f and the singular set of the system $\{f, f^2, \ldots\}$, i.e. the set of all backward iterates of A_1 via f, will be denoted by A. We will denote by $\mathrm{Fix}(g)$ the set of fixed points of g and by $\mathrm{Per}(g)$ the set of periodic points of g.

One of our main goals was to decide when $f : \mathbb{R} \setminus A \to \mathbb{R} \setminus A$ is Devaney-chaotic on the entire domain $\mathbb{R} \setminus A$. One well-known example of such a map is provided by $\dfrac{1}{2}(x - \dfrac{1}{x})$ which is treated in [10] by lifting the flow on S^1. In the second section we prove that if f is increasing and $\deg P > \deg Q$

then f is chaotic if and only if A is dense in $\mathbb{R}$. We also prove that $f(x) = x - \sum_{j=1}^{k} \frac{d_j}{(x-a_j)^{2n_j+1}}$, $d_j > 0$, is chaotic.

In the third section we study maps of the form $f(x) = cx - \sum_{j=1}^{k} \frac{d_j}{x^{2n_j+1}}$ with $c > 0$ and $d_j > 0$ and we prove that such a map is chaotic iff $c \leq 1$ (which can be viewed as a generalization of the above-mentioned example). More than that, we prove that any two maps in this class (with $c \in (0,1]$) are topologically conjugate. If $c > 1$ we show that the set of points with bounded orbit is a Cantor set.

The last section is devoted to functions $f(x) = cx - \frac{d}{x^{2n}}$ with $c, d > 0$, and we show that there are values of c for which the map is S-unimodal, providing in this way a new example of such maps additional to the well-known quadratic case.

2 Increasing rational maps

There are several definitions characterizing the chaotic behavior of a map, but we will refer to the following one (see Devaney [5]):

Definition 2.1. *Let (D, d) be a metric space. Then a map $g : D \to D$ is chaotic if and only if the following conditions are fulfilled:*
a) the periodic points of g are dense in D;
b) g is topologically transitive *(i.e. $\forall\, U,V$ open in D, $\exists\, n \in \mathbb{N}$ with $g^n(U) \cap V \neq \emptyset$);*

In his initial formulation, Devaney included a third condition (the sensitive dependence on initial conditions), which was later proven to be redundant (see [1]). From now on we will say that a map is *Devaney-chaotic* (or simply *chaotic*) if it satisfies the two conditions from Definition 2.1.

Let us consider now a $\mathcal{C}^\infty$ map f, with $f : \mathbb{R} \setminus A_1 \to \mathbb{R}$, where $A_1 \neq \emptyset$ is a finite set, such that $\lim_{x \to l} |f(x)| = \infty$ for all $l \in A_1 \cup \{-\infty, \infty\}$ and $\mathrm{Fix}(f^n)$ is finite for all $n \geq 1$. We set $A_n = \{x \in \mathbb{R} \mid \exists s \leq n-1,\ f^s(x) \in A_1\}$ and $A = \cup_{n=1}^{\infty} A_n$.

Proposition 2.2. *If $f' > 0$ and $\forall\, n \geq 1$ $\mathrm{Fix}(f^n)$ is finite, then:*
(i) If $\mathrm{Per}(f)$ is dense in $\mathbb{R}$, so is A.
(ii) If A is dense, then $f_{|\mathbb{R}\setminus A}$ chaotic.

Proof. Note that the sets A_n are finite, $\lim_{x\to\infty} f^n(x) = \infty$, $\lim_{x\to-\infty} f^n(x) = -\infty$, and, $\forall\, l \in A_n$, $\lim_{x\to l^-} f^n(x) = \infty$ and $\lim_{x\to l^+} f^n(x) = -\infty$.
(i) We show first that $\sup A = \infty$ and $\inf A = -\infty$. Suppose that $m := \sup A < \infty$.

If $m \in A$, then $m \in A_n$ for some n. Since $\lim_{x\to m^+} f^n(x) = -\infty$, we have $f^n((m,\infty)) = \mathbb{R}$ and therefore there exists $y \in (m,\infty)$ such that $f^n(y) \in A$. This implies that $y \in A$, which contradicts the definition of m.

If $m \notin A$, we consider two cases. If $f(m) \geq m$, then $f([m,\infty)) \subset [m,\infty)$ and since f is increasing on $[m,\infty]$, it follows that $\mathrm{Per}(f)\cap[m,\infty) = \mathrm{Fix}(f)\cap[m,\infty)$. We assumed that $\mathrm{Fix}(f)$ is finite, which implies that $\mathrm{Per}(f)$ cannot be dense in $\mathbb{R}$. If, however, $f(m) < m$, then $m \notin A$ implies that $\exists\; a \in A$ with $f(m) < a$. It follows that we can find $b \in (m,\infty)$ with $f(b) = a$, which implies that $b \in A$ and this again contradicts the definition of m. Therefore $\sup A = \infty$. Similarly $\inf A = -\infty$.

We prove now that A is dense. Assume that A is not dense, and let x_0 be a point in the interior of $\mathbb{R} \setminus A$. Let $\alpha = \sup\{x \in A | x < x_0\}$ and $\beta = \inf\{x \in A | x > x_0\}$. Then $-\infty < \alpha < x_0 < \beta < \infty$.

If $\alpha \in A$ then $\alpha \in A_n$ for some n. From this we get $f^n((\alpha, x_0)) = (-\infty, f^n(x_0))$, and we have $f^n((\alpha,x_0))\cap A \neq \emptyset$ because $\inf A = -\infty$. Therefore $(\alpha, x_0) \cap A \neq \emptyset$, which contradicts the definition of α. Similarly we obtain a contradiction if $\beta \in A$.

Suppose now that $\alpha, \beta \notin A$. It follows that $[\alpha,\beta]\cap A = \emptyset$ and both α and β are accumulation points for A. Observe also that for each $n \geq 1$, either $f^n([\alpha,\beta]) \subset [\alpha,\beta]$ or $f^n([\alpha,\beta]) \cap [\alpha,\beta] = \emptyset$. For example, if $f^n(\alpha) \leq \beta < f^n(\beta)$, then $A \cap [f^n(\alpha), f^n(\beta)] \neq \emptyset$ (since β is an accumulation point for A) and thus $[\alpha,\beta] \cap A \neq \emptyset$.

Because $[\alpha,\beta] \cap \mathrm{Per}(f) \neq \emptyset$, it follows that there exists $n \geq 1$ such that $[f^n(\alpha), f^n(\beta)] \cap [\alpha,\beta] \neq \emptyset$. Let n_0 be the smallest integer with this property. If $n_0 = 1$, then $f([\alpha,\beta]) \subset [\alpha,\beta]$, and since f is increasing on $[\alpha,\beta]$, $\mathrm{Per}(f)\cap[\alpha,\beta] = \mathrm{Fix}(f) \cap [\alpha,\beta]$. However, we assumed that $\mathrm{Fix}(f)$ is finite, and this contradicts the density of $\mathrm{Per}(f)$. If $n_0 > 1$ then $[\alpha,\beta] \cap \mathrm{Fix}(f^n) \neq \emptyset$ implies that $n_0|n$. Therefore $\mathrm{Per}(f) \cap [\alpha,\beta] = \mathrm{Per}(f^{n_0}) \cap [\alpha,\beta]$. Now we observe that, since $f^{n_0}([\alpha,\beta]) \subset [\alpha,\beta]$, f^{n_0} is increasing and $A \cap [\alpha,\beta] = \emptyset$, $\mathrm{Per}(f^{n_0}) \cap [\alpha,\beta]$=$\mathrm{Fix}(f^{n_0}) \cap [\alpha,\beta]$, which is finite, and we obtain again a contradiction with the density of $\mathrm{Per}(f)$.

(ii) Let I, J be two open intervals. We will prove that $\exists\, n \geq 1$ such that $I \cap \mathrm{Fix}(f^n) \neq \emptyset$ and $f^n(I \setminus A) \cap (J \setminus A) \neq \emptyset$. Since A is dense and $\{A_n\}$ is an increasing sequence whose union is A, it follows that $\exists\, n \geq 1$

such that $I \cap A_n$ has at least two elements. Choose such an n and let $a, b \in I \cap A_n$ be such that $a < b$ and $(a, b) \cap A_n = \emptyset$. It follows that $\lim_{x \to a^+} f^n(x) - x = -\infty$, $\lim_{x \to b^-} f^n(x) - x = \infty$ and $f^n((a, b)) = \mathbb{R}$. This implies that $(a, b) \cap \mathrm{Fix}(f^n) \neq \emptyset$ and $f^n((a, b)) \cap (J \setminus A) \neq \emptyset$. However if $f^n(x) \notin A$, then $x \notin A$. Hence $f^n((a, b) \setminus A) \cap (J \setminus A) \neq \emptyset$. □

As an immediate consequence of this proposition we obtain:

Corollary 2.3. *Suppose* $f(x) = \dfrac{P(x)}{Q(x)}$ *is a rational function with* $f' > 0$ *and* $\deg P > \deg Q$*. Then* f *is Devaney-chaotic if and only if* A *is dense.*

The next result provides an important example of chaotic rational maps based on the strategy suggested by the above corollary.

Proposition 2.4. *If* $f(x) = x - \displaystyle\sum_{j=1}^{k} \frac{d_j}{(x - a_j)^{2n_j + 1}}$ *where* $a_j \in \mathbb{R}$ *(not necessarily distinct),* $d_j > 0$ *and* $n_j \in \mathbb{N}$*, then* A *is dense (and hence* f *is chaotic).*

Proof. Let $x < y$, $x, y \in \mathbb{R} \setminus A$. We must prove that $[x, y] \cap A \neq \emptyset$. Note that if $[x, y] \cap A = \emptyset$, then $f([x, y]) = [f(x), f(y)]$ and it follows from the definition of A that $[f(x), f(y)] \cap A = \emptyset$. Thus it suffices to show that $\exists$ $n \in \mathbb{N}$ such that $[f^n(x), f^n(y)] \cap A \neq \emptyset$.

Suppose that $[f^n(x), f^n(y)] \cap A = \emptyset$ for all $n \in \mathbb{N}$. We can assume that $a_1 \leq \cdots \leq a_k$. Let $\alpha \in (-\infty, a_1)$ be such $f(\alpha) = a_1$, and let $\beta \in (a_k, \infty)$ be such that $f(\beta) = a_k$. Observe now that if $x > \beta$ for $x \in \mathbb{R} \setminus A$, then $a_k < f(x) < x$, and if $x < \alpha$, then $x < f(x) < a_1$. Since f has no fixed points it follows that the set $\mathcal{J}_z = \{n \in \mathbb{N} | f^n(z) \in [\alpha, \beta]\}$ is infinite for any $z \in \mathbb{R} \setminus A$. We also have $\mathcal{J}_x = \mathcal{J}_y$ because $\alpha, \beta \in A$. If $a, b \in \mathbb{R}$ and $[a, b] \cap A = \emptyset$, then

$$|f(a) - f(b)| \geq |a - b| \tag{1}$$

and if in addition $[a, b] \subset [\alpha, \beta]$, then

$$|f(a) - f(b)| \geq \rho |a - b|, \tag{2}$$

where $\rho = \inf\{f'(x) : x \in [\alpha, \beta] \setminus A\} > 1$. Set $\lambda_n = |f^n(x) - f^n(y)|$. It follows from our assumption and the previous argument that $\{\lambda_n\}$ is nondecreasing and, since $\mathcal{J}_x (= \mathcal{J}_y)$ is infinite and $\rho > 1$, we deduce that $\{\lambda_n\}$ diverges to ∞. On the other hand, $\lambda_n \leq \beta - \alpha$ for $n \in \mathcal{J}_x$, which obviously is a contradiction. □

As we will see later, the strategy of the above proof will be used to prove the chaotic behavior of other types of rational functions.

3 Maps of the form $f(x) = cx - \sum_{j=1}^{k} \dfrac{d_j}{x^{2n_j+1}}$

In this section we will discus functions $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ of the form $f(x) = cx - \sum_{j=1}^{k} \frac{d_j}{x^{2n_j+1}}$ where c, d_j $(j = 1 \dots k)$ are positive real numbers. Note that $f' > 0$ and therefore f is chaotic if and only if the singular set A is dense according to Corollary 2.3. We will show that A is dense if and only if $c \leq 1$. More than that, we will prove that if this is the case, f has a dense orbit and any two functions of this form (with $c \in (0, 1]$), when restricted to the complement of the singular set, are topologically conjugate. We note that it was proved in [6] that the topological transitivity of a function $g : D \to D$ implies the existence of a dense orbit if D is a complete metric space with a countable base and g is continuous. In our case $D = \mathbb{R} \setminus A$ which obviously is not complete.

In order to understand this type of function, we will briefly review a few facts from the symbolic dynamics theory. Consider the *symbol space* Σ_2 of all infinite sequences of 0's and 1's. If $a = a_0a_1a_2\dots$ and $b = b_0b_1b_2\dots$ are two points in Σ_2, one can define a distance:

$$d(a, b) = \sum_{i=0}^{\infty} \frac{|a_i - b_i|}{2^i},$$

which transforms the symbol space into a metric space. The *shift map*, which is the function $\sigma : \Sigma_2 \to \Sigma_2$ defined by $\sigma(a_0a_1a_2\dots) = a_1a_2\dots$, has some important dynamical properties: its periodic points are dense, the set of points which are neither periodic nor eventually periodic is dense in Σ_2 and there is an element of Σ_2, namely the Morse sequence, which is determined by the concatenation of all possible blocks of 0 and 1 of length k with $k \geq 1$, with a dense orbit (see [5], [8], [10]). This suggests that one might want to find a topological conjugacy with the shift map in order to establish the chaotic behavior of given function.

Kneading theory is a refined version of symbolic dynamics which was introduced in order to keep track of the orbits of other elements which characterize a function (like critical points). As a main reference we used [5].

In our case we will keep track of those points which under iteration reach the origin; in other words this technique will help describe the set A. For another recent application of symbolic dynamics to some (planar) systems with discontinuities, see [7].

Consider three symbols N, P and Z (our notation here differs from the standard one, but it makes the argument easier to follow). Define the following sets:

$$\begin{aligned} K_1 &= \{s_0 \dots s_{n-1}Z \mid s_i \in \{N,P\} \text{ for } 0 \le i < n\} \cup \{Z\} \\ K_2 &= \{s_0 s_1 \dots \mid s_i \in \{N,P\} \text{ for } i \ge 0\}, \ K = K_1 \cup K_2. \end{aligned}$$

For the function $f(x) = cx - \sum_{j=1}^{k} \frac{d_j}{x^{2n_j+1}}$ there are two kinds of iterative sequences of the form $a_0 = x \neq 0$ and $a_{n+1} = f(a_n)$. Either the sequence is finite because it eventually reaches the discontinuity at zero, or it is infinite because $f^n(x)$ is defined for all $n > 0$. Keeping this in mind, let $s(x) = N$ if $x < 0$, $s(x) = P$ if $x > 0$, and $s(0) = Z$. We construct a function $\pi : \mathbb{R} \to K$. Let $\pi(0) = Z$. If $x \neq 0$ and the iterative sequence beginning at x eventually reaches zero after n iterations, let

$$\pi(x) = s(x)s(f(x))s(f^2(x)) \dots s(f^{n-1}(x))Z.$$

Otherwise, let

$$\pi(x) = s(x)s(f(x))s(f^2(x)) \dots .$$

We will order K with the dictionary order induced by $N < Z < P$. If c and d are both positive the next lemma shows that π is increasing.

Lemma 3.1. *Suppose* $f(x) = cx - \sum_{j=1}^{k} \frac{d_j}{x^{2n_j+1}}$ *with* $c, d_j > 0$. *If* $x \le y$ *then* $\pi(x) \le \pi(y)$.

Proof. Consider $x, y \in \mathbb{R}$ such that $x < y$. If $\pi(x) \neq \pi(y)$, we must prove that $\pi(x) < \pi(y)$. Let $n_0 = \min\{n \in \mathbb{N} \mid s(f^n(x)) \neq s(f^n(y))\}$. If $n_0 = 0$ then $x \le 0 < y$ or $x < 0 \le y$, which implies that $s(x) \in \{N, Z\}$ and $s(y) = P$ or $s(x) = N$ and $s(y) \in \{Z, P\}$. Therefore $\pi(x) < \pi(y)$. Suppose now that $n_0 \ge 1$. Then, $\forall\, n \in \mathbb{N}$ with $n \le n_0 - 1$, $f^n(x)$ and $f^n(y)$ are not 0 and they have the same sign. The monotonicity of f shows that $\forall\, n \in \mathbb{N}$, $n \le n_0$, then $f^n(x) < f^n(y)$. For $n = n_0$ we then must have $f^{n_0}(x) \le 0 < f^{n_0}(y)$, or $f^{n_0}(x) < 0 \le f^{n_0}(y)$ which as before implies that $s(f^{n_0}(x)) \in \{N, Z\}$ and $s(f^{n_0}(y)) \in \{P\}$ or $s(f^{n_0}(x)) = N$ and $s(f^{n_0}(x)) \in \{Z, P\}$. This implies that $\pi(x) < \pi(y)$. $\square$

Proposition 3.2. *Let $f : \mathbb{R}\backslash\{0\} \to \mathbb{R}$ be defined by $f(x) = cx - \sum_{j=1}^{k} \dfrac{d_j}{x^{2n_j+1}}$ where $c, d_j \in (0, \infty)$, and $n_j \in \mathbb{Z}$, $n_j \ge 0$. If $0 < c \le 1$, then we have:*
(i) Let $s, t \in \mathbb{R} \setminus A$. There exists $n \in \mathbb{N}$ such that $(f^n(s), f^n(t)) \cap A \ne \emptyset$;
(ii) The set A is dense in $\mathbb{R}$.

If one tries to imitate the proof of Proposition 2.4 one sees that the only thing that fails is the inequalities (1) and (2). These inequalities will be replaced by (the less transparent ones) (3) and (4).

Proof. Notice that $f_{|(0,\infty)} : (0, \infty) \to \mathbb{R}$ and $f_{|(-\infty,0)} : (-\infty, 0) \to \mathbb{R}$ are increasing diffeomorphisms. Therefore $\forall\ a \in \mathbb{R}$ the equation $f(x) = a$ has two solutions: a positive one and a negative one. Define a sequence $\{a_n\}_{n\ge 0}$ as follows: $a_0 = 0$ and for $n \ge 1$, a_n is the positive solution of $f(a_n) = a_{n-1}$. Since $f(x) < x$ for $x > 0$, this sequence is increasing. As f has no fixed points $\{a_n\}$ is not convergent and therefore $\lim_{n\to\infty} a_n = \infty$. Obviously, $a_n \in A\ \forall\ n$, hence $\sup A = \infty$. Similarly, one can also show that $\inf A = -\infty$. If $b_n \in (-\infty, 0)$ is such that $f(b_n) = a_n$, then $b_n \in A$, $\{b_n\}$ is increasing and $\lim_{n\to\infty} b_n = 0$. Therefore $\sup (A \cap (-\infty, 0)) = 0$. Similarly we can show that $\inf (A \cap (0, \infty)) = 0$. For this, construct two sequences p_n and q_n as follows: $p_n < 0$, $p_0 = 0$ and $f(p_n) = p_{n-1}$ on one hand, and $q_n > 0$ with $f(q_n) = p_n$ on the other hand. Then $q_n \in A$ for all n, q_n is decreasing and $\lim_{n\to\infty} q_n = 0$.

Let $\epsilon \in (0, a_1)$, $\epsilon \in A$, such that $\forall\, x \in (-\epsilon, \epsilon) \setminus \{0\}$, $|f(x)| > a_2$. Observe now that for all $x \in \mathbb{R} \setminus A$ the set $\mathcal{J}_x = \{n \in \mathbb{N} | \epsilon \le |f^n(x)| \le a_2\}$ is infinite. This follows from the fact that f has no fixed point and noticing that if $x > a_2$, then $a_1 < f(x) < x$, while if $x < -a_2$, then $x < f(x) < -a_1$ (f is an odd function). Therefore for every $x \in \mathbb{R} \setminus A$ there exists $n \in \mathbb{N}$ such that $\epsilon \le |f^n(x)| \le a_2$.

The following fact is the main ingredient of the proof: if $x, y \in \mathbb{R}$, $xy > 0$ and $f(x) + f(y) \ne 0$, then

$$\left| \frac{f(x) - f(y)}{f(x) + f(y)} \right| \ge \left| \frac{x - y}{x + y} \right|. \tag{3}$$

Moreover, if in addition $\epsilon \leq |x| \leq a_2$ and $\epsilon \leq |y| \leq a_2$, then

$$\left|\frac{f(x)-f(y)}{f(x)+f(y)}\right| \geq \rho \left|\frac{x-y}{x+y}\right|, \tag{4}$$

for some $\rho > 1$. Indeed,

$$\left|\frac{f(x)-f(y)}{f(x)+f(y)}\right| = \left|\frac{x-y}{x+y}\right| \left|\frac{P(x,y)+Q(x,y)}{P(x,y)-Q(x,y)}\right|,$$

where

$$P(x,y) = c + \sum d_j \frac{x^{2n_j-1}y + \cdots + xy^{2n_j-1}}{x^{2n_j+1}y^{2n_j+1}},$$

$$Q(x,y) = \sum d_j \frac{x^{2n_j} + x^{2n_j-2}y^2 + \cdots + y^{2n_j}}{x^{2n_j+1}y^{2n_j+1}}.$$

If $xy > 0$, then $P(x,y) > 0$ and $Q(x,y) > 0$. Therefore $|P(x) + Q(x)| \geq |P(x) - Q(x)|$. For the second part we use the following inequality: if $a, b > 0$ and $a \neq b$, then $\left|\dfrac{a+b}{a-b}\right| \geq 1 + \dfrac{2\min\{a,b\}}{a+b}$, and we observe that if $\epsilon \leq |x| \leq a_2$ and $\epsilon \leq |y| \leq a_2$, then $P(x,y), Q(x,y) \in [m, M]$ for some real numbers $0 < m < M$.

Now suppose that $\forall\, n \in \mathbb{N}$, $(f^n(s), f^n(t)) \cap A = \emptyset$. In particular $\forall\, n \in \mathbb{N}$, $0 \notin (f^n(s), f^n(t))$. It follows that the sequence $\lambda_n = \left|\dfrac{f^n(s)-f^n(t)}{f^n(s)+f^n(t)}\right|$ is increasing. If $n \in \mathcal{J}_s$, since $\epsilon, \beta \in A$ and $(f^n(s), f^n(t)) \cap A = \emptyset$, then $n \in \mathcal{J}_t$ and hence $\rho\lambda_n \leq \lambda_{n+1}$. We noticed that $\mathcal{J}_s$ is infinite. Thus $\{\lambda_n\}$ diverges to ∞. However $f^n(s)$ and $f^n(t)$ have the same sign, hence $\lambda_n < 1, \forall\, n \in \mathbb{N}$ which is a contradiction.

As in the proof of Proposition 2.4, (ii) follows from (i). $\square$

Remark 3.3. *If* $f(x) = cx - \sum_{j=1}^{k} \dfrac{d_j}{(x-a_j)^{2n_j+1}}$ *and* $0 < c < 1$, *it is not necessarily true that* A *is dense. For example, if* $f(x) = \dfrac{x}{2} - \dfrac{1}{x-4}$, *then* $f((-\infty, \alpha]) \subset (-\infty, \alpha]$, *where* $\alpha = 2 - \sqrt{2}$, *and therefore* $A \cap (-\infty, \alpha] = \emptyset$.

Proposition 3.4. *If* $0 < c \leq 1$ *and* $d_j > 0\ \forall j$, *then* $\pi : \mathbb{R} \to K$ *is an order preserving injection onto* $K_1 \cup K_2' \subset K$, *where* $K_2' = \{\{s_k\} \in K_2 \mid \forall\, i \geq 0, \exists j > i,\ s_i \neq s_j\}$.

Proof. We notice first that it follows from the definition of π that $\pi(\mathbb{R} \setminus A) \subset K_2$ and $\pi(A) \subset K_1$. We prove first that π is one-to-one. Suppose $\pi(s) = \pi(t)$ for $s \neq t$, say $s < t$. Since A is dense in $\mathbb{R}$ (Proposition 3.2), there must be some $x \in A$ in the open interval (s,t). Also, because A is countable, $\mathbb{R} \setminus A$ is dense in $\mathbb{R}$ and therefore there exists $y \in (\mathbb{R} \setminus A) \cap (s,t)$. We have already proved that π is increasing, hence $\pi(s) \leq \pi(x) \leq \pi(t)$ and $\pi(s) \leq \pi(y) \leq \pi(t)$. Since $\pi(s) = \pi(t)$ we must have $\pi(x) = \pi(y)$. This is a contradiction because $\pi(x) \in K_1$ and $\pi(y) \in K_2$.

We will show now that $\pi(\mathbb{R} \setminus A) = K_2'$ and $\pi(A) = K_1$. Let $a_1 > 0$ be such that $f(a_1) = 0$. We have that $\pi(0) = Z$ and $\pi(a_1) = PZ$. Also $\pi(-a_1) = NZ$. As we noticed before, $\forall\ y \in \mathbb{R}$, the equation $f(x) = y$ has exactly two real solutions, one positive and the other negative, so, by induction, for each $\tau = s_0 \dots s_{n-1}Z$ in K_1, there is an $x \in \mathbb{R}$ such that $\pi(x) = \tau$. This proves that $\pi(A) = K_1$.

Now let x be a value such that $f^n(x)$ is never zero, that is, $\pi(x) = s_0 s_1 \ldots \in K_2$. Suppose there exists some n so that $s_i = P$ for all $i > n$. Since $f(y) < y$ for any $y > 0$, the sequence $f^{n+1}(x)$, $f^{n+2}(x)$, $\ldots$ would then be a decreasing sequence of positive numbers, which must converge, contradicting the fact that f has no fixed points. In like manner, since $f(y) > y$ for any $y < 0$, there cannot exist any n such that $s_i = N$ for all $i > n$. Therefore, $\pi(x) \in K_2'$.

Let $\tau \in K_2'$. There are two cases: $\tau = P^{p_1} N^{n_1} P^{p_2} \ldots$ or $\tau = N^{n_1} P^{p_1} N^{n_2} \ldots$, where $p_i, n_i \geq 1$ for all $i \geq 1$ (here N^n and P^n stand for n copies of the symbol N, respectively P). It suffices to consider only the first case. Define a sequence $\{a_i\}$ by $a_i = P^{p_1} N^{n_1} P^{p_2} \ldots N^{n_{i-1}} Z$ and a sequence $\{b_i\}$ by $b_i = P^{p_1} N^{n_1} P^{p_2} \ldots N^{n_{i-1}} P^{p_i} Z$. It follows that for every $i \geq 1$, $a_i, b_i \in K_1$, $a_i < a_{i+1} < \tau < b_{i+1} < b_i$ and τ is the only element of K with this last property. Let $\alpha_i, \beta_i \in A$ be such that $\pi(\alpha_i) = a_i$ and $\pi(\beta_i) = b_i$. They exist because $\pi(A) = K_1$. Since π is increasing and one-to-one, $\{\alpha_i\}$ is increasing, $\{\beta_i\}$ is decreasing and $\alpha_i < \beta_i$. Let $x \in \cap (\alpha_i, \beta_i)$. Then $\forall\, i \geq 1$, $a_i < \pi(x) < b_i$. From here we deduce that $\pi(x) = \tau$. $\square$

Remark 3.5. *Note that the density of A was used only to prove that $\pi_{|\mathbb{R} \setminus A}$ is 1-1.*

This result can be strengthened in the following way. Notice first that the shift map has two fixed points $\alpha = (000\ldots)$ and $\beta = (111\ldots)$. Define the set Ω of all negative trajectories of these two fixed points or, equivalently, the set of all sequences which become eventually constant. In other words,

$$\Omega = \{a \in \Sigma_2 \mid \exists\, k \in \mathbb{N} \text{ such that} \,\forall\, n \geq k,\ a_n = a_{n+1}\}.$$

Notice that Ω is dense in Σ_2. Define $\varepsilon : \mathbb{R} \setminus \{0\} \to \{0,1\}$ with $\varepsilon(x) = 0$ if $x < 0$ and $\varepsilon(x) = 1$ if $x > 0$. The restriction and corestriction of f to $\mathbb{R} \setminus A$ will also be denoted by f (i.e. we consider $f : \mathbb{R} \setminus A \to \mathbb{R} \setminus A$). Let: $\phi : \mathbb{R} \setminus A \longrightarrow \Sigma_2, \quad \phi(x)_n = \varepsilon(f^n(x)) \ \forall\ n \geq 0$. Notice that, using the notation of Proposition 3.4, Σ_2 is nothing else than the set K_2, $\Sigma_2 \setminus \Omega$ is K_2' and ϕ is π. Then we have the following result.

Theorem 3.6. *The map $\phi : \mathbb{R} \setminus A \longrightarrow \Sigma_2 \setminus \Omega$ defines a topological conjugacy between $f : \mathbb{R} \setminus A \longrightarrow \mathbb{R} \setminus A$ and $\sigma : \Sigma_2 \setminus \Omega \longrightarrow \Sigma_2 \setminus \Omega$.*

Proof. The commutativity $\phi \circ f = \sigma \circ \phi$ is obvious, so we only have to show that ϕ is a homeomorphism. From Proposition 3.4 it follows that ϕ is a bijection.
ϕ is continuous. Let $\phi(x) = a$. It suffices to show that there is an interval I such that $x \in I$ and $\phi(I \setminus A) \subset B\left(a, \frac{1}{2^n}\right)$. Choose I containing x such that $I \cap A_{n+1} = \emptyset$ and $y \in I \setminus A$. If there is a digit in position $k \leq n$ for which $\phi(x)$ and $\phi(y)$ differ, $f^k(x)$ and $f^k(y)$ will have different signs according to the definition of ϕ and hence there is a zero of f^k between x and y, which contradicts the choice of I. Hence the first $n+1$ digits of $\phi(x) = a$ and $\phi(y)$ are the same which implies the conclusion.
ϕ^{-1} is continuous. We show that if $\{x_k\}_{k \geq 0}$ is a sequence such that $\phi(x_k) \to \phi(x)$, then $x_k \to x$. Assume there is a sequence for which this is not true and hence by passing to a subsequence we may assume that there is a neighborhood of x which does not contain any element of $\{x_k\}$. Since A is dense and $\{A_n\}$ is an increasing sequence, there is an $n \in \mathbb{N}$ and $a, b \in A_n$ such that $a < x < b$, $a \cdot b > 0$ and (a, b) does not contain any element of the sequence $\{x_k\}$. Suppose we choose y not in (a, b), for example $y < a < x$ (the case $y > b$ is similar). If $f^j(x) \cdot f^j(y)$ were positive for every $j \leq n$, the restriction of f to the interval $(f^j(x), f^j(y))$ would be monotone, so $\forall j \leq n+1$, $f^j(y) < f^j(a) < f^j(x)$. But the definition of A_n implies that there is $j_0 \leq n$ with $f^{j_0}(a) = 0$ and hence $f^{j_0}(y) \cdot f^{j_0}(x) < 0$, which is a contradiction. Hence there is $j \leq n$ such that for any y which is not in (a, b), $\varepsilon(f^j(x))$ and $\varepsilon(f^j(y))$ are not equal, which in particular implies that for any $k \in \mathbb{N}$, $d(\phi(x_k), \phi(x)) \geq \frac{1}{2^j} \geq \frac{1}{2^n}$, contradicting the convergence of $\phi(x_k)$ to $\phi(x)$. $\square$

Since Ω does not contain either periodic orbits (of order greater than or equal to 2) or the Morse sequence described at the beginning of this section, we have:

Corollary 3.7. *For any $n \geq 1$, the map f has $2^n - 2$ periodic orbits of order n. The set of all periodic orbits is dense in $\mathbb{R}$ and there is a dense non-periodic orbit, hence f is chaotic.*

The approach based on kneading sequences has the advantage that also works if $c > 1$. On the other hand, Theorem 3.6 establishes an identical dynamical behavior between f and the shift map. In fact it shows the following not obvious pattern:

Corollary 3.8. *For any $0 < c \leq 1$ and $d_j > 0$, any two maps in the family of functions $f = cx - \sum_{j=1}^{k} \dfrac{d_j}{x^{2n_j+1}}$ are topological conjugate when restricted to $\mathbb{R}$ without the discontinuity set.*

We will consider now functions of the form $f(x) = cx - \sum_{j=1}^{k} \dfrac{d_j}{x^{2n_j+1}}$ where $c, d_j \in (0, \infty)$, $c > 1$. In this case we have two repelling fixed points for f. It is again the structure of the set A and the period orbits which present most of the interest. In fact, we will show that there are periodic orbits of any order. In what follows we denote by $|M|$ the cardinal of some set M. First we determine $|\operatorname{Fix}(f^n)|$.

Lemma 3.9. *Let $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ be defined by $f(x) = cx - \sum_{j=1}^{k} \dfrac{d_j}{x^{2n_j+1}}$ where $c, d_j \in \mathbb{R}$ and $n_j \in \mathbb{N}$. If $c > 1$ and $d_j > 0$, then $|\operatorname{Fix}(f^n)| = 2^n$.*

Proof. We compute first the cardinality of A_n. Since the equation $f(x) = a$ has exactly two solutions for any a, it follows that $|A_{n+1}| = |A_n| + 2|A_n \setminus A_{n-1}|$ for $n \geq 2$. As $|A_1| = 1$ and $|A_2| = 3$, we obtain $|A_n| = 2^n - 1$.

Consider now two consecutive elements u and v of A_n (consecutive in the sense that $u < v$ and $(u, v) \cap A_n = \emptyset$). Since $(f^n)'(x) > 1$ for every $x \in \mathbb{R} \setminus A_n$, $\lim_{x\to u^+} f^n(x) - x = -\infty$ and $\lim_{x\to v^-} f^n(x) - x = \infty$, it follows that $|Fix(f^n) \cap (u, v)| = 1$. If we denote $\alpha = \min A_n$ and $\beta = \max A_n$, since $\lim_{x\to\infty} f^n(x) - x = \infty$, $\lim_{x\to-\infty} f^n(x) - x = -\infty$, $\lim_{x\to\beta^+} f^n(x) - x = -\infty$, $\lim_{x\to\alpha^-} f^n(x) - x = \infty$, then $|Fix(f^n) \cap (-\infty, \alpha)| = 1$, $|Fix(f^n) \cap (\beta, \infty)| = 1$. We conclude that $|Fix(f^n)| = |A_n| + 1 = 2^n$. $\square$

Proposition 3.10. *The map f has periodic orbits of any order $n \geq 2$.*

Proof. Notice that the function has two fixed points. Using Lemma 3.9, we conclude that the number of solutions of $f^n(x) = x$ is greater than the number of fixed points of f which guarantees the existence of a periodic orbit of order n if n is prime. Consider now $n \in \mathbb{N}$ that is not prime. Let $\{d_1, ..., d_q\}$ be the set of all proper divisors of n. To prove that f has an orbit of order n we must prove that $Orb_n \not\subset \cup_{i=1}^q Orb_{d_i}$ (Orb_i stands for the set of points which determine an orbit of order i). From Lemma 3.9 it is sufficient to prove that $2^n > \sum_{i=1}^q 2^{d_i}$. This can be easily seen, for example using the inequalities $d_i \leq [\frac{n}{2}]$, $q \leq [\frac{n}{3}]$. $\square$

There is certainly no chaos for this case as the following proposition shows.

Proposition 3.11. *Let $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ be defined by $f(x) = cx - \sum_{j=1}^k \dfrac{d_j}{x^{2n_j+1}}$ where c and d_j are real numbers and n_j are integers, $n_j \geq 0$. If $c > 1$ and $d_j > 0$ then the periodic orbits of f are all repelling and they are uniformly bounded by the two repelling fixed points.*

Proof. Let $x_P > 0$ and $x_N < 0$ be such that $f(x_P) = x_P$, $f(x_N) = x_N$ (note that $x_N = -x_P$). If O is a periodic orbit of f we show that $O \subset [x_N, x_P]$. Let $a = \min O$ and $b = \max O$. Since $f(a) \in O$ and $f(b) \in O$ we have $a \leq f(a)$ and $b \geq f(b)$. These two inequalities imply that $x_N \leq a \leq b \leq x_P$. Indeed: if $b \leq 0$ obviously $b \leq x_P$. Suppose that $b > 0$. Since $f(x) - x$ is increasing in $(0, \infty)$ it follows that $f(x) < x$ if $x \in (0, x_P)$ and $f(x) > x$ if $x \in (x_P, \infty)$. Then $b \geq f(b)$ implies that $b \leq x_P$. The proof of $x_N \leq a$ is completely similar. Notice that, since $f'(x) > c > 1$, if p is a periodic point of period n we have from the chain rule that $(f^n)'(p) > c^n$, hence the periodic orbits are repelling. $\square$

Remark 3.12. *Note that since $[x_P, \infty)$ and $(-\infty, x_N]$ are invariant under f, $A \cap ((-\infty, x_N] \cup [x_P, \infty)) = \emptyset$, and since $f(x) > x$ $\forall x > x_P$ and $f(x) < x$ $\forall x < x_N$, we have $\forall x > x_P$, $\{f^n(x)\}$ diverges to ∞ and $\forall x < x_N$, $\{f^n(x)\}$ diverges to $-\infty$.*

The strategy based on kneading sequences can be also applied in this case and it leads to the following:

Proposition 3.13. *If $c > 1$ and $d_j > 0$, then $\pi : \mathbb{R} \to K$ is an order preserving surjection. Moreover, $\pi_{|\pi^{-1}(K_2')}$, where K_2' is defined as in Proposition 3.4, is injective.*

Proof. It follows in the same way as in the Proposition 3.4 (see Remark 3.5) that $\pi_{|A} : A \to K_1$ is bijective and $K_2' \subset \pi(\mathbb{R})$. Notice that $\pi(x_P) = \overline{P}$ and $\pi(x_N) = \overline{N}$ (the bar indicates infinite repetition of the symbol). For $x > x_P$ or $x < x_N$, the iteration sequence $f^n(x)$ diverges monotonically, so $\pi(x) = \overline{P}$ for all $x > x_P$ and $\pi(x) = \overline{P}$ for all $x < x_N$.

As we noted earlier, for any real number t, the equation $f(x) = t$ has exactly two solutions, one positive and the other negative. It follows by induction that $\exists x$ such that $\pi(x) = \tau N\overline{P}$ and $\exists x$ such that $\pi(x) = \tau P\overline{N}$, where τ is any finite sequence of characters N and P. This proves that π is surjective.

Now suppose $\pi(x) = \pi(y) \in K_2'$ for $x \neq y$. Then $f^n(x)$ and $f^n(y)$ have the same sign for all n. Thus, since $f'(t) > c$ for all $t \neq 0$, we have $|f^n(x) - f^n(y)| > c^n|x - y| > 2x_P$ for sufficiently large n, producing a contradiction. Therefore, $\pi(x) = \pi(y) \in K_2'$ implies $x = y$. □

Note that $\pi^{-1}(K_2')$ is the set of all points with bounded orbits. This enables us to improve the result in Proposition 3.11:

Corollary 3.14. *The set of points with bounded orbits is compact and totally disconnected, hence is a Cantor set.*

Proof. If $f^n(x)$ is defined for all n and $\{f^n(x)\}$ is unbounded, then there exists an n for which $|f^n(x)| > x_P$, so there exists an open interval about $f^n(x)$, and thus an open interval about x of points that diverge under iteration. Therefore, the set of points with unbounded orbits is an open set and the set of points with bounded orbits is a closed subset of $[x_N, x_P]$ and thus compact.

Now suppose x and y both have bounded orbits and $x < y$. It follows from Proposition 3.13 that points with bounded orbits have distinct kneading sequences, so $\pi(x) < \pi(y)$. Thus there exists a positive integer n and a sequence τ of length n consisting of symbols N and P so that $\pi(x) \leq \tau N\overline{P}$ and $\tau P\overline{N} \leq \pi(y)$. Let a be the unique value such that $\pi(a) = \tau N\overline{P}$ and $f^{n+1}(a) = x_P$ and let b be the unique value such that $\pi(b) = \tau P\overline{N}$ and $f^{n+1}(b) = x_N$. Then $x \leq a < b \leq y$ and there are no points with bounded orbits in the open interval (a, b). Therefore the set of points with bounded orbits is totally disconnected. □

The reader might compare Corollary 3.14 with the example in Section I.8 in [2].

4 Maps of the form $f(x) = cx - \dfrac{d}{x^{2k}}$

So far we have completely described the dynamics of all functions of the form $f(x) = cx - \dfrac{d}{x^n}$ for $c > 0, d > 0$ and n a positive *odd* integer. The next natural step would be the analysis of the case when the exponent n is a positive *even* integer. The main distinction between the two situations is that in this second case f has a critical point and hence the requirements of Proposition 2.2 are not satisfied.

Theorem 4.1. *Let $f(x) = cx - dx^{-2k}$, $c, d > 0$ and consider the sequence $a_{n+1} = f(a_n)$. For $c \neq 1$, define $L = \left(\frac{d}{c-1}\right)^{1/(2k+1)}$. Assume that a_0 is chosen such that $a_n \neq 0 \,\forall\, n$. Then we have:*
(i) If $c < 1$, L is the only fixed point. If we define $m = \left(\frac{-2kd}{c}\right)^{1/(2k+1)}$ and $M = f(m)$, then either $a_n \to L$, or $\exists\, n_0$ such that, for all $n \geq n_0$, $a_n \in [f(M), M]$.
(ii) If $c = 1$, f has no fixed points and a_n decreases, diverging to $-\infty$, hence there are no periodic orbits.
(iii) If $c > 1$, L is the unique (repelling) fixed point, and $a_n \to \infty$ if $a_0 > L$ or $a_n \to -\infty$ if $a_0 < L$, hence there are no periodic orbits in this case, as well.

Proof. (i) We start with three observations:
Observation 1. For any $x \in (-\infty, 0)$, $f(x) < 0$ and for any $x \in (0, \infty)$, $\exists\, n \in \mathbb{N}$ such that either $f^n(x) = 0$ or $f^n(x) < 0$. This shows that we need to prove our claim only for $x < 0$, and such is the subsequent goal of the proof.
Observation 2. The function f is increasing on $(-\infty, m]$ and decreasing on $(m, 0)$.
Observation 3. If $x < L$, then $x < f(x)$, and if $x > L$, then $x > f(x)$.

Observe that the equation $f(x) = L$ has at most two negative solutions, α and L ($\alpha = L$ if and only if $f'(L) = 0$). We consider separately the following two cases:
Case 1. Assume $\dfrac{2k}{2k+1} \leq c < 1$. It follows that $L \leq m \leq \alpha$. (L is an attractor in this case, as $0 \leq f'(L) < 1$.) If $x < L$, then $x < f(x) < L$ (by Observations 2, 3) and the sequence $f^n(x)$ is increasing and bounded from above by L. Therefore it is convergent and its limit must be L. If $x > \alpha$, then $f(x) < L$ (by Observation 2) and the above applies. If $L < x < \alpha$,

then $L < f(x) < x < \alpha$ (by Observations 2, 3) and the sequence $f^n(x)$ is decreasing and bounded from below by L. Therefore it is convergent and its limit must be L.

Case 2. Assume $0 < c < \dfrac{2k}{2k+1}$. It follows that $\alpha < m < L < M$ and $f(M) < L$.

First, observe that $[f(M), M]$ is invariant under f. Indeed, if $x \in [f(M), L]$ then $f(x) \geq x$ and therefore $f(M) \leq x \leq f(x) \leq M$. If $x \in [L, M]$ then, since on $[L, 0)$ f is decreasing, we get $f(M) \leq f(x) \leq f(L) = L$. Second, $\forall\, x \in (-\infty, 0)$, $\exists\, n \in \mathbb{N}$ such that $f^n(x) \in [f(M), M]$. If $x < f(M)$, then $\exists\, n_1 \in \mathbb{N}$ such that $f^{n_1}(x) > \alpha$. Otherwise, $f^n(x) < f^{n+1}(x) < \alpha$ for all $n \in \mathbb{N}$, which will imply that $f^n(x)$ is convergent to a limit $\leq \alpha < L$. That is a contradiction. If $f^{n_1}(x) > \alpha$, then either $f^{n_1}(x) > L$ and therefore $f(M) < L < f^{n_1}(x) \leq M$, or $\alpha < f^{n_1}(x) < L$ and therefore $f^{n_1+1}(x) > L$, which implies, as before, $f^{n_1+1}(x) \in [f(M), M]$. If $x > M$, then obviously $f(x) < f(M)$ and the above applies.

(ii) This case follows from the fact that $f(x) < x$.

(iii) In this situation $f(x) > x$ if $x > L$ and $f(x) < x$ if $x < L$ and it is straightforward that L is the unique fixed point from which the conclusion follows. □

Remark 4.2. *Notice that for* $\dfrac{2k-1}{2k+1} < c < 1$, *$L$ is a global attractor. Indeed, if* $\dfrac{2k}{2k+1} \leq c < 1$, *this follows from case 1 of the above proof. On the other hand, if* $\dfrac{2k-1}{2k+1} < c < \dfrac{2k}{2k+1}$, *notice that if $x < L$ we have that* $-1 < f'(L) < f'(x) < c < 1$ *hence the interval $[f(M), L]$ is in the basin of attraction and for $x > L$, $f(x) < x$. For* $c = \dfrac{2k-1}{2k+1}$ *a double-period bifurcation occurs since $f'(L) = -1$ and the Schwarzian derivative is negative at L.*

Compared to the odd case, the structure of the set A is easy to describe in this one. Notice that the equation $f(x) = 0$ has a unique positive solution. Also, if α is a positive number, the equation $f(x) = \alpha$ has a unique solution which is also positive because the above equation is equivalent to the polynomial equation $F(x) = cx^{2k+1} - \alpha x^{2k} - d = 0$. If $c > 1$, because $F(0) < 0$ and $F(L) > 0$ for any $\alpha < L$ and since $f(x) < x$ for $0 < x < L$, an easy induction shows that the pull-backs of the origin form an ascending sequence

a_n in $(0, L)$, hence $A_n = \{0, a_1, \ldots, a_n\}$ with $a_n \to L$. Similar arguments show that for $c \leq 1$ the sequence a_n diverges to ∞. Notice also that, if we denote by P the point at infinity, the system undergoes at $c = 1$ a bifurcation which resembles the features of a transcritical bifurcation: P repelling and L attracting for $\dfrac{2k}{2k+1} < c < 1$, P degenerate at $c = 1$ (meaning that there is a repelling direction from P and an attracting one towards P) and then P attracting and L repelling for $c > 1$. This exchange of stability is associated with a change in the structure of A, from unbounded when $c \leq 1$ to bounded in the case $c > 1$.

But case (i) in Theorem 4.1 can be described in much more detail. We first recall the definition of an unimodal and S-unimodal function as they are presented in [4].

Definition 4.3. *Suppose $f : [-1, 1] \to [-1, 1]$ is a continuous map. We will say that f is unimodal if the following conditions are fulfilled:*
(i) $f(0) = 1$;
(ii) f is strictly increasing on $[-1, 0]$ and strictly decreasing on $[0, 1]$.

For recent results regarding unimodal maps see for example [3].

Definition 4.4. *Suppose $f : [-1, 1] \to [-1, 1]$ is a unimodal function. We will say that f is S-unimodal if:*
(i) f is $\mathcal{C}^3$;
(ii) f has negative Schwarzian derivative, i.e. $Sf(x) = \frac{f'''(x)}{f'(x)} - \frac{3}{2}\left(\frac{f''(x)}{f'(x)}\right)^2 < 0$, for all $x \neq 0$ (at $x = 0$ we allow Sf to be $-\infty$);
iii) f maps $J(f) = [f(1), 1]$ onto *itself.*

The dynamics of S-unimodal maps are quite well understood (see, for example, [4] or [9]). The connection with the case described in this section is given by:

Proposition 4.5. *Suppose $f(x) = cx - d/x^{2k}$ with $0 < c < c_1 = \dfrac{2k}{2k+1}$ and $k \geq 1$. Then there is a value $c_0 < c_1$ (depending on k) such that for all $0 < c < c_0$ the function $f_{|[f(M),M]}$ is S-unimodal.*

Proof. Since we already showed in Theorem 4.1 that $[f(M), M]$ is invariant and that for $c < c_1$, $m < M$, we have to determine first a value c_0 of c such that $f(M) < m$ for all $0 < c < c_0$. It is not hard to see that this condition is equivalent to $P(c) > 0, c < c_1$, where

$$P(x) = (2k+1)^{2k+1}x^{2k+1} - 2k(2k+1)^{2k}x^{2k-1} + (2k)^{2k}.$$

For $x > 0$, this polynomial has a unique critical point $x_1 = \dfrac{\sqrt{(2k-1)2k}}{2k+1}$, decreases on $(0, x_1)$ and increases on (x_1, ∞), it has a root $\dfrac{2k}{2k+1} > x_1$ and $P(0) > 0$. Hence, one can choose c_0 as the other positive root of P.
For $c < c_0$ determined above, the map $f : [f(M), M] \to [f(M), M]$ is onto. Indeed, we have that $f(M) < m < L < M$. Hence if $y \in [f(M), f(L) = L]$, $\exists\, x \in [L, M]$ with $f(x) = y$. If $y \in [f(L) = L, f(m) = M]$, $\exists\, x \in [m, L]$ with $f(x) = y$. The only other condition to be checked is the negativity of the Schwarzian derivative. A direct computation shows that this is equivalent (for $x \neq m$) to $(cx^{2k+1} + 2kd)(2k+2) < 3k(2k+1)d$, which is obvious for $x < 0$ and $k \geq 1$. $\square$

As a final remark, we mention that the same ideas can be applied to some other rational functions. For example if $f(x) = cx - \dfrac{d}{x^n}$, n is odd, $-1 \leq c < 0$ and $d < 0$ one can prove that $f : \mathbb{R} \setminus A \to \mathbb{R} \setminus A$ is topologically conjugate to $\pi : \Sigma_2 \setminus \Omega' \to \Sigma_2 \setminus \Omega'$, where $\Omega' = \{a \in \Sigma_2 \mid \exists\, k \in \mathbb{N}\, \forall\, n \geq k,\ a_n \neq a_{n+1}\}$.
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