On Uniformly Runge Domains *

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1 Introduction

Approximation of holomorphic functions on compact sets in $\mathbb{C}^n$ (or in a Stein manifold or a Stein space) with polynomials (or globally defined functions) is a basic and well studied subject in Complex Analysis. However sometime in order to construct holomorphic functions with nice global properties one has to approximate on non-compact sets. Among other places where such phenomena appear see for example [2], [3] or [5].

In dimension one we have the following nice and useful result of R. Narasimhan [7]:
If D_n, $n \in \mathbb{N}$, are open sets in an open Riemann surface, $\mathcal{R}$, such that
- D_n are pairwise disjoint
- $\{D_n\}_{n\geq 1}$ is locally finite
- $\cup D_n$ is Runge in $\mathcal{R}$
then for every sequence of holomorphic functions $f_n \in \mathcal{O}(D_n)$, every sequence of compacts $K_n \subset D_n$ and every sequence of positive numbers $\epsilon_n > 0$ there exists $f \in \mathcal{O}(\mathcal{R})$ such that $\|f - f_n\|_{K_n} < \epsilon_n$ for every n.

The purpose of this note is to give an example showing that Narasimhan's result does not hold in higher dimension. We will give in fact two examples. For the first one we will use a sequence of complex curves in $\mathbb{C}^3$ that are perturbations of the curve given by $z_1z_2 = 1$, $(z_1 - 1)z_3 = 1$. This curve is smooth and has three ends. It was used by E. Kallin in [4] to construct three disjoint polydiscs with non-polynomially convex union. It turns out that in this first construction one cannot choose the Runge domains $\{\Delta_n\}$ to be polydiscs. The second example is not so clean from the computational point

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of view. We will use also a sequence of smooth complex curves with three ends in $\mathbb{C}^3$ however this sequence will not be convergent. The main point of this second example is that we obtain actually a locally finite sequence of disjoint polydiscs such that their union is Runge in $\mathbb{C}^3$ but the uniform approximation does not hold.

One more remark: it was proved in [3] that if one has a locally finite sequence of disjoint polydiscs such that their centers are on the same real line (in [3] this configuration is called *a ray of polydiscs*) then one *has* uniform approximation.

2 Preliminaries

We state all the results and definitions in this section for subsets of $\mathbb{C}^p$ but obviously everything can be done in any reduced Stein space.

If K is a compact subset of $\mathbb{C}^p$ we denote by $\widehat{K}$ its holomorphically convex hull. Namely $\widehat{K} = \{z \in \mathbb{C}^p : |f(z)| \leq \sup_K |f|, \forall f \in \mathcal{O}(\mathbb{C}^p)\}$. K is called holomorphically convex if $K = \widehat{K}$.

Definition 1.
1) If D is an open subset of $\mathbb{C}^p$, D is called Runge if it is Stein and for every compact set $K \subset D$, every $f \in \mathcal{O}(D)$ and every $\epsilon > 0$ there exist $\widetilde{f} \in \mathcal{O}(\mathbb{C}^p)$ such that $\sup_K |f - \widetilde{f}| < \epsilon$.
2) A sequence, $\{D_n\}_{n\geq 1}$, of Runge domains in $\mathbb{C}^p$ is called uniformly Runge if for every sequence of positive real numbers $\{\epsilon_n\}_{n\geq 1}$, every sequence of compact sets $\{K_n\}_{n\geq 1}$, $K_n \subset D_n$, and every sequence of holomorphic functions $\{f_n\}_{n\geq 1}$, $f_n \in \mathcal{O}(D_n)$, there exists $f \in \mathcal{O}(\mathbb{C}^p)$ such that for every $n \geq 1$ we have $\|f - f_n\|_{K_n} < \epsilon_n$.

The terminology "uniformly Runge" was used in [5].

Definition 2. *A sequence $\{K_n\}_{n\geq 1}$ of compact sets in $\mathbb{C}^p$ is called holomorphically separated if there exist $f \in \mathcal{O}(\mathbb{C}^p)$ and an increasing sequence of real numbers $\{\alpha_n\}_{n\geq 0}$ such that $\lim \alpha_n = \infty$ and for each $n \geq 1$ one has $\alpha_{n-1} < \Re(f_{|K_n}) < \alpha_n$.*

The following lemma was proved in [6] (see also [1]):

Lemma 1. *Let K_1 and K_2 be disjoint holomorphically convex compact subsets of $\mathbb{C}^p$. Then $K_1 \cup K_2$ is holomorphically convex if and only if there exists* *a holomorphic function f on $\mathbb{C}^p$ such that $\Re(f) < 0$ on K_1 and $\Re(f) > 0$ on K_2. ($\Re(f)$ stands for the real part of f.)*

Proposition 1. *A sequence $\{K_n\}_{n\geq 1}$ of holomorphically convex compact sets in $\mathbb{C}^p$ is holomorphically separated if and only if there exists an exhaustion $\{P_n\}_{n\geq 1}$ of $\mathbb{C}^p$ with holomorphically convex compact subsets such that*
a) $K_n \subset P_n$ for all n
b) For every j and n, $j > n$, $P_n \cap K_j = \emptyset$
c) For every finite set $S \subset \{n+1, n+2 \dots\}$, $P_n \cup \bigcup_{j\in S} K_j$ is holomorphically convex.

Proof. Suppose that $\{K_n\}$ is holomorphically separated and let $f \in \mathcal{O}(\mathbb{C}^p)$ and $\{\alpha_n\}$ as in the definition. Let $\{P_n\}$ be an exhaustion of $\mathbb{C}^p$ with holomorphically convex compact subsets such that $K_n \subset P_n$ and $P_n \subset \{\Re(f) < \alpha_n\}$. Note that it is possible to find such an exhaustion because, on one hand $\{\Re(f) < \alpha_n\}$ is Runge in $\mathbb{C}^p$ and, on the other hand, as $\lim \alpha_n = \infty$, $\cup\{\Re(f) < \alpha_n\} = \mathbb{C}^p$. Now Lemma 1 guarantees that for every finite set $S \subset \{n+1, n+2 \dots\}$, $P_n \cup \bigcup_{j\in S} K_j$ is holomorphically convex.

Conversely, let's assume the existence of $\{P_n\}$ with the three properties. We can construct inductively a sequence $\{g_n\}$ of holomorphic functions, $g_n \in \mathcal{O}(\mathbb{C}^p)$ such that $\|g_n\|_{P_{n-1}} < (\frac{1}{2})^{n+1}$ and $\|g_1 + g_2 \dots + g_n - n\|_{K_n} < \frac{1}{4}$. We set $f := \sum g_n$. It is easy to see that $f \in \mathcal{O}(\mathbb{C}^p)$ and $n - \frac{1}{2} < \Re(f_{|K_n}) < n + \frac{1}{2}$. □

Proposition 2. *Suppose that $\{D_n\}_{n\geq 1}$ is a sequence of Runge domains in $\mathbb{C}^p$. The following are equivalent:*
i) $\{D_n\}_{n\geq 1}$ is uniformly Runge
ii) Every sequence of compact sets $\{K_n\}_{n\geq 1}$, $K_n \subset D_n$, is holomorphically separated.

Proof.
i) $\Rightarrow$ *ii)* Let $\{K_n\}_{n\geq 1}$ be a sequence of compact sets, $K_n \subset D_n$. Because $\{D_n\}_{n\geq 1}$ is uniformly Runge we can find $f \in \mathcal{O}(\mathbb{C}^p)$ such that $\|f - 2n\|_{K_n} < 1$. Then we can choose $\alpha_n = 2n + 1$.
ii) $\Rightarrow$ *i)* We consider a sequence of compact sets $\{K_n\}_{n\geq 1}$, $K_n \subset D_n$, a sequence of positive numbers $\{\epsilon_n\}_{n\geq 1}$, and a sequence of holomorphic functions $\{f_n\}_{n\geq 1}$, $f_n \in \mathcal{O}(D_n)$. Since $\widehat{K}_n$ is a compact subset of D_n, replacing K_n by $\widehat{K}_n$ we can assume that K_n are holomorphically convex in $\mathbb{C}^p$. We use then Proposition 1 and we choose the sequence $\{P_n\}_{n\geq 1}$ satisfying a), b) and c). We construct inductively a sequence of holomorphic functions $\{g_n\}_{n\geq 1}$,

$g_n \in \mathcal{O}(\mathbb{C}^p)$, that satisfy, for every $n \geq 1$ the following two inequalities: $\|g_n\|_{P_{n-1}} < (\frac{1}{2})^n \min\{\epsilon_1, \cdots, \epsilon_{n-1}\}$ and $\|g_1 + g_2 \cdots + g_n - f_n\|_{K_n} < \frac{\epsilon_n}{2}$. We set $f = \sum_{n=1}^{\infty} g_n$.

□

Remark 1. Because the proof of $i) \Rightarrow ii)$ uses only the constant sequence $\{\epsilon_n = 1\}$ our notion of "uniformly Runge" is equivalent to the apparently weaker one in which we replace "any sequence of positive real numbers $\{\epsilon_n\}$" by "any constant sequence of positive real numbers $\{\epsilon_n\}$".

Proposition 3. *If $\{D_n\}_{n\geq 1}$ is a uniformly Runge sequence of pairwise disjoint open subsets of $\mathbb{C}^p$ then for every sequence of holomorphically convex compact subsets of $\mathbb{C}^p$, $\{K_n\}$, $K_n \subset D_n$, and every holomorphically convex compact subset $K \subset \mathbb{C}^p$ there exists $n_0 \in \mathbb{N}$ such that, for every $n \geq n_0$, $K \cup \bigcup_{j=n_0}^{n} K_j$ is holomorphically convex.*

Proof. Let $\{K_n\}$ be a sequence of holomorphically convex compact subsets of $\mathbb{C}^p$ with $K_n \subset D_n$, and $K \subset \mathbb{C}^p$ be a holomorphically convex compact set. According to Proposition 2 the sequence $\{K_n\}_{n\geq 1}$ is holomorphically separated. Then there exists $\{\alpha_n\}$ an increasing and unbounded sequence of real numbers and $f \in \mathcal{O}(\mathbb{C}^p)$ such that $\alpha_{n-1} < \Re(f_{K_n}) < \alpha_n$ for every n. We choose n_0 such that $\Re(f_{|K}) \leq \alpha_{n_0-1}$. The conclusion follows from Lemma 1. □

3 The Examples

Example 1:
We consider in $\mathbb{C}^3$ the following sequence of Riemann surfaces:

$$\Sigma_k = \{(z_1, z_2, z_3) : z_1(z_2 + \frac{1}{k}) = 1, z_3(z_1 - 1) = 1\}, \ k \geq 2$$

In Σ_k we consider:
$\gamma_k = \{(z_1, z_2, z_3) : |z_1| = 4k^2\}$,
$\eta_k = \{(z_1, z_2, z_3) : |z_1 - 1| = \frac{1}{4}\}$,
$\mu_k = \{(z_1, z_2, z_3) : |z_1| = \frac{1}{4}\}$.

Let $\pi_j : \mathbb{C}^3 \to \mathbb{C}$ be the projection $\pi_j(z_1, z_2, z_3) = z_j$.

Obviously γ_k are pairwise disjoint and $\{\gamma_k\}$ is locally finite. At the same time Σ_k is isomorphic to $\mathbb{C} \setminus \{0, 1\}$ via π_1. As $\pi_1(\gamma_k) = \{\zeta \in \mathbb{C} : |\zeta| = 4k^2\}$

which is holomorphically convex in $\mathbb{C}\setminus\{0,1\}$ it follows that γ_k is holomorphically convex in Σ_k and therefore in $\mathbb{C}^3$. Also $\pi_2(\gamma_k)\subset\{\zeta\in\mathbb{C}:|\zeta+\frac{1}{k}|<\frac{1}{4k^2-1}\}$ which are disjoint discs.

We choose for every k, Δ_k disjoint Runge open subset of $\mathbb{C}^3$ such that:

- $\Delta_k\supset\gamma_k$,
- $\{\Delta_k\}$ is locally finite
- $\pi_2(\Delta_k)\subset\{\zeta\in\mathbb{C}:|\zeta+\frac{1}{k}|<\frac{1}{4k^2-1}\}$.

Since the real part of π_2 separates Δ_k, from Lemma 1 it follows then that $\cup\Delta_k$ is Runge in $\mathbb{C}^3$.

Notice that $\eta_k\cup\mu_k$ is bounded. Indeed, if $z\in\eta_k\cup\mu_k$ it follows that $\frac{1}{4}\leq|z_1|\leq 2$, $|z_2|\leq 5$, $|z_3|\leq 4$ and so $\eta_k\cup\mu_k\subset\overline{B}:=\{z\in\mathbb{C}^3:|z|\leq 7\}$. On the other hand $(\gamma_k\cup\eta_k\cup\mu_k)\widehat{\ }$ contains the point $(2k^2,-\frac{1}{k}+\frac{1}{2k^2},\frac{1}{2k^2-1})\notin\overline{B}$ since the holomorphically convex hull of $\pi_1(\gamma_k\cup\eta_k\cup\mu_k)$ in $\mathbb{C}\setminus\{0,1\}$ is $\{\zeta\in\mathbb{C}:|\zeta|\leq 4k^2\}\setminus(\{\zeta\in\mathbb{C}:|\zeta|<\frac{1}{4}\}\cup\{\zeta\in\mathbb{C}:|\zeta-1|<\frac{1}{4}\})$.
Therefore $\gamma_k\cup\overline{B}$ is not holomorphically convex for any $k\geq 2$ and then Proposition 3 implies that $\{\Delta_n\}$ is not uniformly Runge.

Remark 2. This example shows also that the necessity of the condition $\lim\alpha_n=\infty$ in Definition 2.

Example 2:
For $s\in\mathbb{Z}$, $s\geq 1$ we consider the following two sequences of integers: $k_s=5^s$ and $\alpha_s=4k_s^2$. We define two sequences of points in $\mathbb{C}^3$, two sequences of polyradii and the associated polydiscs ($P(a,r)$ stands for the polydisc of center a and polyradius r) as follows:

$$z_s=(0,\frac{1}{k_s},k_s)\in\mathbb{C}^3,\ r_s=(\alpha_s+1,\frac{2}{\alpha_s},\frac{1}{2})\in(\mathbb{R}_{>0})^3,\ F_s=P(z_s,r_s)$$

$$w_s=(k_s,\frac{2}{k_s},k_s)\in\mathbb{C}^3,\ \rho_s=(\frac{2}{\alpha_s},\frac{1}{\alpha_s},\alpha_s k_s^2+1)\in(\mathbb{R}_{>0})^3,\ G_s=P(w_s,\rho_s).$$

Let $P_{2s}=F_s$, $P_{2s-1}=G_s$. We will show that P_s are pairwise disjoint, $\{P_s\}$ is locally finite, $\cup P_s$ is Runge in $\mathbb{C}^3$ and $\{P_s\}$ is not uniformly Runge.

It follows from the way we defined $\{k_s\}$ and $\{\alpha_s\}$ that

$$\frac{2}{k_{s+1}}+\frac{1}{\alpha_{s+1}}<\frac{2}{k_{s+1}}+\frac{2}{\alpha_{s+1}}<\frac{1}{k_s}-\frac{2}{\alpha_s}\text{ and }\frac{1}{k_s}+\frac{2}{\alpha_s}<\frac{1}{k_s}+\frac{3}{\alpha_s}<\frac{2}{k_s}-\frac{1}{\alpha_s}$$

This shows that

$$\Re(z_{2|G_{s+1}}) < \frac{2}{k_{s+1}} + \frac{2}{\alpha_{s+1}} < \Re(z_{2|F_s}) < \frac{1}{k_s} + \frac{3}{\alpha_s} < \Re(z_{2|G_s})$$

It follows on one hand that P_s are pairwise disjoint and on the other hand, using Lemma 1, that $\cup P_s$ is Runge in $\mathbb{C}^3$.

Since $\pi_3(F_s) = D(k_s, \frac{1}{2})$ and $\pi_1(G_s) = D(k_s, \frac{2}{\alpha_s})$ it follows that $\{P_s\}$ is locally finite.

It remains to be proved that $\{P_s\}$ is not uniformly Runge. For that we consider the following sequence of Riemann surfaces: $\{\Gamma_s\}$,

$$\Gamma_s = \{z = (z_1, z_2, z_3) \in \mathbb{C}^3 : z_1(z_2 - \frac{1}{k_s}) = 1 \text{ and } (z_1 - k_s)(z_3 - k_s) = k_s^2\}.$$

Each Γ_s is isomorphic, via π_1, to $\mathbb{C} \setminus \{0, k_s\}$. Let $\gamma_s, \mu_s, \eta_s \subset \Gamma_s$,

$$\gamma_s = \{z \in \Gamma_s : |z_1| = \alpha_s\}, \mu_s = \{z \in \Gamma_s : |z_1| = 1\}, \eta_s = \{z \in \Gamma_s : |z_1 - k_s| = \frac{1}{\alpha_s}\}$$

Using the isomorphism $\pi_2 : \Gamma_s \xrightarrow{\sim} \mathbb{C} \setminus \{0, k_s\}$ and the inequality $\alpha_s > k_s + \frac{1}{\alpha_s}$ it is clear that $\gamma_s \cup \mu_s \cup \eta_s$ is not holomorphically convex (in Γ_s and hence in $\mathbb{C}^3$) and in fact $\{(\gamma_s \cup \mu_s \cup \eta_s)^\wedge \setminus (\gamma_s \cup \mu_s \cup \eta_s)\}_s$ is unbounded. We will show that $\gamma_s \subset F_s$, $\eta_s \subset G_s$ and $\mu_s \subset P(0, (2, 2, 2))$. All these together with Proposition 3 will imply, as before that $\{P_s\}$ is not uniformly Runge.

- $\gamma_s \subset F_s$: if $z \in \gamma_s$ then

$$|z_1| = \alpha_s < \alpha_s + 1, \; |z_2 - \frac{1}{k_s}| = \frac{1}{\alpha_s} < \frac{2}{\alpha_s},$$

$$|z_3 - k_s| = \frac{k_s^2}{|z_1 - k_s|} \leq \frac{k_s^2}{\alpha_s - k_s} = \frac{k_s^2}{4k_s^2 - k_s} < \frac{1}{2}$$

- $\eta_s \subset G_s$: if $z \in \eta_s$ then

$$|z_1 - k_s| = \frac{1}{\alpha_s} < \frac{2}{\alpha_s}, \; |z_3 - k_s| = \alpha_s k_s^2 < \alpha_s k_s^2 + 1,$$

$$|z_2 - \frac{2}{k_s}| = \left| \frac{1}{z_s} - \frac{1}{k_s} \right| = \frac{|z_1 - k_s|}{|z_1 k_s|} \leq \frac{\frac{1}{\alpha_s}}{\left(k_s - \frac{1}{\alpha_s}\right) k_s} < \frac{1}{\alpha_s}$$

- $\mu_s \subset P(0,(2,2,2))$: if $z \in \mu_s$ then

$$|z_1| = 1 < 2,\ |z_2| = |\frac{1}{z_1} + \frac{1}{k_s}| \le 1 + \frac{1}{k_s} < 2,$$

$$|z_3| = |k_s + \frac{k_s^2}{z_1 - k_s}| = |\frac{z_1 k_s}{z_1 - k_s}| \le \frac{k_s}{k_s - 1} < 2$$

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