How to Prove Equivalence of Rewriting Systems without (Explicit) Induction

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Outline

1. Program transformation by templates
2. Verifying the equivalence of TRSs
   (a) Equivalent transformation of TRSs
   (b) Proof
3. Extending to a higher order setting
4. Conclusion & Future works

Program Transformation

- Compiler
- Optimization
- Refactoring
- Verification

How to Apply Program Transformation

1. Analizing the input program
2. Checking properties for the input
   (Correctness of transformation)
3. Applying a transformation rule
4. Checking the output

Program Transformation by Templates

Template

```
fun f x = if a x then b x
          else h (d x) (f (e x))
```

matching

```
fun sum x = if null x then 0
          else (hd x) + (sum (tl x))
```

```
fun f x = if a x then b x
          else g (e x) (d x)
and g x y = if a x then h y (b x)
            else g (e x) (h y (d x))
```

instantiation

```
fun sum x = if null x then 0
          else sum1 (tl x) (hd x)
and sum1 x y = if null x then y + 0
            else sum1 (tl x) (y + (hd x))
```

$\forall x, y, z.\ \mathtt{h}\ x\ (\mathtt{h}\ y\ z) = \mathtt{h}\ (\mathtt{h}\ x\ y)\ z$

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$\forall x, y, z.\ x + (y + z) = (x + y) + z$

Program Transformation by Templates

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Program Transformatin by Templates

- Pattern matching

- The correctness of transformations

Program Transformatin by Templates

Huet & Lang (1978): Lambda calculus

- Programs
  - lambda-terms + Y-combinators
- Pattern matching
  - substitution + β-reduction
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  - Denotational semantics

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Chiba et. al. (2010): Term Rewriting

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  - Operational semantics $\Leftarrow$ Hypothesis $\Leftarrow$ Automated Theorem Proving

Program Transformation by Templates

Template

$$\begin{array}{ll} \mathsf{f}(\mathsf{a}) & \to \mathsf{b} \\ \mathsf{f}(\mathsf{c}(u,v)) & \to \mathsf{g}(\mathsf{e}(u),\mathsf{f}(v)) \\ \mathsf{g}(\mathsf{b},u) & \to u \\ \mathsf{g}(\mathsf{d}(u,v),w) & \to \mathsf{d}(u,\mathsf{g}(v,w)) \end{array}$$

$$\begin{array}{ll} \mathsf{f}(u) & \to \mathsf{f}_1(u,\mathsf{b}) \\ \mathsf{f}_1(\mathsf{a},u) & \to u \\ \mathsf{f}_1(\mathsf{c}(u,v),w) & \to \mathsf{f}_1(v,\mathsf{g}(w,\mathsf{e}(u))) \\ \mathsf{g}(\mathsf{b},u) & \to u \\ \mathsf{g}(\mathsf{d}(u,v),w) & \to \mathsf{d}(u,\mathsf{g}(v,w)) \end{array}$$

$$\begin{array}{ll} \mathsf{g}(\mathsf{b},u) & \approx \mathsf{g}(u,\mathsf{b}) \\ \mathsf{g}(\mathsf{g}(u,v),w) & \approx \mathsf{g}(u,\mathsf{g}(v,w)) \end{array}$$

matching

$$\begin{array}{ll} \mathsf{sum}([\,]) & \to 0 \\ \mathsf{sum}(x:y) & \to +(x,\mathsf{sum}(y)) \\ +(0,x) & \to x \\ +(\mathsf{s}(x),y) & \to \mathsf{s}(+(x,y)) \end{array}$$

$$\begin{array}{ll} \mathsf{f} \mapsto \mathsf{sum}(\square_1), & \mathsf{b} \mapsto 0, \\ \mathsf{g} \mapsto +(\square_1,\square_2), & \mathsf{c} \mapsto \square_1 : \square_2, \\ \mathsf{f}_1 \mapsto \mathsf{sum1}(\square_1,\square_2), & \mathsf{d} \mapsto \mathsf{s}(\square_2), \\ \mathsf{a} \mapsto [\,], & \mathsf{e} \mapsto \square_1 \end{array}$$

instantiation

$$\begin{array}{ll} \mathsf{sum}(x) & \to \mathsf{sum1}(x,0) \\ \mathsf{sum1}([\,],x) & \to x \\ \mathsf{sum1}(x:y,z) & \to \mathsf{sum1}(y,+(z,x)) \\ +(0,x) & \to x \\ +(\mathsf{s}(x),y) & \to \mathsf{s}(+(x,y)) \end{array}$$

Program Transformation by Templates

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\begin{aligned}
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\mathsf{g}(\mathsf{d}(u,v),w) &\to \mathsf{d}(u,\mathsf{g}(v,w))
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\mathsf{f}(u) &\to \mathsf{f}_1(u,\mathsf{b})\\
\mathsf{f}_1(\mathsf{a},u) &\to u\\
\mathsf{f}_1(\mathsf{c}(u,v),w) &\to \mathsf{f}_1(v,\mathsf{g}(w,\mathsf{e}(u)))\\
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\begin{aligned}
\mathsf{g}(\mathsf{b},u) &\approx \mathsf{g}(u,\mathsf{b})\\
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\mathsf{sum}([\,]) &\to 0\\
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$$

$$
\begin{aligned}
&\mathsf{f} \mapsto \mathsf{sum}(\Box_1), && \mathsf{b} \mapsto 0,\\
&\mathsf{g} \mapsto +(\Box_1,\Box_2), && \mathsf{c} \mapsto \Box_1{:}\Box_2,\\
&\mathsf{f}_1 \mapsto \mathsf{sum1}(\Box_1,\Box_2), && \mathsf{d} \mapsto \mathsf{s}(\Box_2),\\
&\mathsf{a} \mapsto [\,], && \mathsf{e} \mapsto \Box_1
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\mathsf{sum}(x) &\to \mathsf{sum1}(x,0)\\
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Program Transformation by Templates

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RAPT

- Rewriting-based Automated Program Transformation System
- SML#
- Automated verification of the correctness.
- `http://www.jaist.ac.jp/~chiba/RAPT/`

Demo

The Correctness of Transformations

A program transformation is correct.

$$\Downarrow$$

Input and output programs are equivalent.

Equivalence of Programs

Program `P` and `P'` are equivalent.

$$\Downarrow$$

$\forall$`d. P(d)=P'(d)`

Equivalence of TRSs

TRS $\mathcal{R}$ and $\mathcal{R}'$ are equivalnet for $\mathscr{G}$. $(\mathcal{R} \simeq_{\mathscr{G}} \mathcal{R}')$

$$\Downarrow$$

$$\forall s \in \mathrm{T}(\mathscr{G}), \forall t \in \mathrm{T}(\mathscr{C}).\ s \xrightarrow{*}_{\mathcal{R}} t \text{ iff } s \xrightarrow{*}_{\mathcal{R}'} t$$

$$\left\{ \begin{array}{ll} \mathsf{sum}([\,]) & \to \mathsf{0} \\ \mathsf{sum}(x : y) & \to +(x, \mathsf{sum}(y)) \\ +(\mathsf{0}, x) & \to x \\ +(\mathsf{s}(x), y) & \to \mathsf{s}(+(x, y)) \end{array} \right\}$$

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Equivalent Transformation of TRSs

- Toyama (1991)
- Without a help of explicit induction

Equivalent Transformation

$\mathcal{R}_0$: left-linear CS over $\mathscr{F}_0$, $\mathcal{E}$: set of equations over $\mathscr{F}_0$

- Introduction

  $\mathcal{R}_k \underset{I}{\Rightarrow} \mathcal{R}_k \cup \{f(x_1,\ldots,x_n) \to r\}$ $(= \mathcal{R}_{k+1})$

  $\mathscr{F}_{k+1} = \mathscr{F}_k \cup \{f\}$

  - $f(x_1,\ldots,x_n)$ is linear
  - $f \notin \mathscr{F}_k$, and
  - $r \in T(\mathscr{F}_k, \mathscr{V})$

- Addition

  $\mathcal{R}_k \underset{A}{\Rightarrow} \mathcal{R}_k \cup \{l \to r\}$ $(= \mathcal{R}_{k+1})$

  - $l \overset{*}{\leftrightarrow}_{\mathcal{R}_k \cup \mathcal{E}} r$

- Elimination

  $\mathcal{R}_k \underset{E}{\Rightarrow} \mathcal{R}_k \setminus \{l \to r\}$ $(= \mathcal{R}_{k+1})$

Example

$$
\begin{aligned}
\mathsf{factlist}(0) &\rightarrow [\,] \\
\mathsf{factlist}(\mathsf{s}(x)) &\rightarrow \mathsf{fact}(x){:}\mathsf{factlist}(x) \\
\mathsf{fact}(0) &\rightarrow \mathsf{s}(0) \\
\mathsf{fact}(\mathsf{s}(x)) &\rightarrow \times(\mathsf{s}(x), \mathsf{fact}(x)) \\
\pi_1(\langle x, y\rangle) &\rightarrow x \\
\pi_2(\langle x, y\rangle) &\rightarrow y
\end{aligned}
$$

$$
\begin{aligned}
\mathsf{factlist}(x) &\rightarrow \pi_1(\mathsf{step}(x)) \\
\mathsf{fact}(x) &\rightarrow \pi_2(\mathsf{step}(x)) \\
\mathsf{factpair}(0) &\rightarrow \langle [\,], \mathsf{s}(0)\rangle \\
\mathsf{factpair}(\mathsf{s}(x)) &\rightarrow \mathsf{step}(\mathsf{s}(x), \mathsf{factpair}(x)) \\
\mathsf{step}(x, y) &\rightarrow \langle \pi_2(y){:}\pi_1(y), \times(x, \pi_2(y))\rangle \\
\pi_1(\langle x, y\rangle) &\rightarrow x \\
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$$

Example

$$\begin{array}{ll} \mathsf{factlist}(0) & \to [\,] \\ \mathsf{factlist}(\mathsf{s}(x)) & \to \mathsf{fact}(x){:}\mathsf{factlist}(x) \\ \mathsf{fact}(0) & \to \mathsf{s}(0) \\ \mathsf{fact}(\mathsf{s}(x)) & \to \times(\mathsf{s}(x), \mathsf{fact}(x)) \\ \pi_1(\langle x, y\rangle) & \to x \\ \pi_2(\langle x, y\rangle) & \to y \end{array}$$

- $\mathcal{R}_0 = \mathcal{R}_{\mathsf{factlist}}$

$$\begin{array}{ll} \mathsf{factlist}(x) & \to \pi_1(\mathsf{step}(x)) \\ \mathsf{fact}(x) & \to \pi_2(\mathsf{step}(x)) \\ \mathsf{factpair}(0) & \to \langle [\,], \mathsf{s}(0)\rangle \\ \mathsf{factpair}(\mathsf{s}(x)) & \to \mathsf{step}(\mathsf{s}(x), \mathsf{factpair}(x)) \\ \mathsf{step}(x, y) & \to \langle \pi_2(y){:}\pi_1(y), \times(x, \pi_2(y))\rangle \\ \pi_1(\langle x, y\rangle) & \to x \\ \pi_2(\langle x, y\rangle) & \to y \end{array}$$

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$$

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- $\mathcal{R}_1 \underset{I}{\Rightarrow} \mathcal{R}_0 \cup \{\mathsf{factpair}(x) \to \langle \mathsf{factlist}(x), \mathsf{fact}(x) \rangle\}$

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- $\mathcal{R}_1 \underset{I}{\Rightarrow} \mathcal{R}_0 \cup \{\mathsf{factpair}(x) \to \langle \mathsf{factlist}(x), \mathsf{fact}(x)\rangle\}$
- $\mathcal{R}_2 \underset{I}{\Rightarrow} \mathcal{R}_1 \cup \{\mathsf{step}(x, y) \to \langle \pi_1(y):\pi_2(y), x \times \pi_2(y)\rangle\}$

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\mathsf{step}(x, y) &\to \langle \pi_2(y):\pi_1(y), \times(x, \pi_2(y))\rangle \\
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- $\mathcal{R}_3 \underset{A}{\Rightarrow} \mathcal{R}_2 \cup \{\mathsf{factpair}(0) \to \langle [\,], \mathsf{s}(0)\rangle\}$

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\mathsf{factlist}(x) &\to \pi_1(\mathsf{step}(x)) \\
\mathsf{fact}(x) &\to \pi_2(\mathsf{step}(x)) \\
\mathsf{factpair}(0) &\to \langle [\,], \mathsf{s}(0)\rangle \\
\mathsf{factpair}(\mathsf{s}(x)) &\to \mathsf{step}(\mathsf{s}(x), \mathsf{factpair}(x)) \\
\mathsf{step}(x, y) &\to \langle \pi_2(y):\pi_1(y), \times(x, \pi_2(y))\rangle \\
\pi_1(\langle x, y\rangle) &\to x \\
\pi_2(\langle x, y\rangle) &\to y
\end{aligned}$$

Example

$$
\begin{array}{ll}
\mathsf{factlist}(0) & \to [\,] \\
\mathsf{factlist}(\mathsf{s}(x)) & \to \mathsf{fact}(x){:}\mathsf{factlist}(x) \\
\mathsf{fact}(0) & \to \mathsf{s}(0) \\
\mathsf{fact}(\mathsf{s}(x)) & \to \times(\mathsf{s}(x), \mathsf{fact}(x)) \\
\pi_1(\langle x, y \rangle) & \to x \\
\pi_2(\langle x, y \rangle) & \to y
\end{array}
$$

- $\mathcal{R}_0 = \mathcal{R}_{\mathsf{factlist}}$
- $\mathcal{R}_1 \underset{I}{\Rightarrow} \mathcal{R}_0 \cup \{\mathsf{factpair}(x) \to \langle \mathsf{factlist}(x), \mathsf{fact}(x) \rangle\}$
- $\mathcal{R}_2 \underset{I}{\Rightarrow} \mathcal{R}_1 \cup \{\mathsf{step}(x, y) \to \langle \pi_1(y){:}\pi_2(y), x \times \pi_2(y) \rangle\}$
- $\mathcal{R}_3 \underset{A}{\Rightarrow} \mathcal{R}_2 \cup \{\mathsf{factpair}(0) \to \langle [\,], \mathsf{s}(0) \rangle\}$
- $\mathcal{R}_4 \underset{A}{\Rightarrow} \mathcal{R}_3 \cup \{\mathsf{factpair}(\mathsf{s}(x)) \to \mathsf{step}(\mathsf{s}(x), \mathsf{factpair}(x))\}$

$$
\begin{array}{lll}
\mathsf{factpair}(\mathsf{s}(x)) & \to_{\mathcal{R}_3} & \langle \mathsf{factlist}(\mathsf{s}(x)), \mathsf{fact}(\mathsf{s}(x)) \rangle \\
& \overset{*}{\to}_{\mathcal{R}_3} & \langle \mathsf{fact}(x){:}\mathsf{factlist}(x), \mathsf{s}(x) \times \mathsf{fact}(x) \rangle \\
& \overset{*}{\leftarrow}_{\mathcal{R}_3} & \langle \pi_2(\langle \mathsf{factlist}(x), \mathsf{fact}(x) \rangle){:}\pi_1(\langle \mathsf{factlist}(x), \mathsf{fact}(x) \rangle), \\
& & \quad \mathsf{s}(x) \times \pi_2(\langle \mathsf{factlist}(x), \mathsf{fact}(x) \rangle) \\
& \overset{*}{\leftarrow}_{\mathcal{R}_3} & \langle \pi_2(\mathsf{factpair}(x)){:}\pi_1(\mathsf{factpair}(x)), \mathsf{s}(x) \times \pi_2(\mathsf{factpair}(x) \\
& \leftarrow_{\mathcal{R}_3} & \mathsf{step}(\mathsf{s}(x), \mathsf{factpair}(x))
\end{array}
$$

$$
\begin{array}{ll}
\mathsf{factlist}(x) & \to \pi_1(\mathsf{step}(x)) \\
\mathsf{fact}(x) & \to \pi_2(\mathsf{step}(x)) \\
\mathsf{factpair}(0) & \to \langle [\,], \mathsf{s}(0) \rangle \\
\mathsf{factpair}(\mathsf{s}(x)) & \to \mathsf{step}(\mathsf{s}(x), \mathsf{factpair}(x)) \\
\mathsf{step}(x, y) & \to \langle \pi_2(y){:}\pi_1(y), \times(x, \pi_2(y)) \rangle \\
\pi_1(\langle x, y \rangle) & \to x \\
\pi_2(\langle x, y \rangle) & \to y
\end{array}
$$

Example

$$
\begin{array}{ll}
\mathsf{factlist}(0) & \to [\,] \\
\mathsf{factlist}(\mathsf{s}(x)) & \to \mathsf{fact}(x){:}\mathsf{factlist}(x) \\
\mathsf{fact}(0) & \to \mathsf{s}(0) \\
\mathsf{fact}(\mathsf{s}(x)) & \to \times(\mathsf{s}(x), \mathsf{fact}(x)) \\
\pi_1(\langle x, y\rangle) & \to x \\
\pi_2(\langle x, y\rangle) & \to y
\end{array}
$$

- $\mathcal{R}_0 = \mathcal{R}_{\mathsf{factlist}}$
- $\mathcal{R}_1 \underset{I}{\Rightarrow} \mathcal{R}_0 \cup \{\mathsf{factpair}(x) \to \langle \mathsf{factlist}(x), \mathsf{fact}(x)\rangle\}$
- $\mathcal{R}_2 \underset{I}{\Rightarrow} \mathcal{R}_1 \cup \{\mathsf{step}(x, y) \to \langle \pi_1(y){:}\pi_2(y), x \times \pi_2(y)\rangle\}$
- $\mathcal{R}_3 \underset{A}{\Rightarrow} \mathcal{R}_2 \cup \{\mathsf{factpair}(0) \to \langle [\,], \mathsf{s}(0)\rangle\}$
- $\mathcal{R}_4 \underset{A}{\Rightarrow} \mathcal{R}_3 \cup \{\mathsf{factpair}(\mathsf{s}(x)) \to \mathsf{step}(\mathsf{s}(x), \mathsf{factpair}(x))\}$

$$
\begin{array}{rl}
\mathsf{factpair}(\mathsf{s}(x)) & \to_{\mathcal{R}_3} \langle \mathsf{factlist}(\mathsf{s}(x)), \mathsf{fact}(\mathsf{s}(x))\rangle \\
& \overset{*}{\to}_{\mathcal{R}_3} \langle \mathsf{fact}(x){:}\mathsf{factlist}(x), \mathsf{s}(x) \times \mathsf{fact}(x)\rangle \\
& \overset{*}{\leftarrow}_{\mathcal{R}_3} \langle \pi_2(\langle \mathsf{factlist}(x), \mathsf{fact}(x)\rangle){:}\pi_1(\langle \mathsf{factlist}(x), \mathsf{fact}(x)\rangle), \\
& \qquad \mathsf{s}(x) \times \pi_2(\langle \mathsf{factlist}(x), \mathsf{fact}(x)\rangle) \\
& \overset{*}{\leftarrow}_{\mathcal{R}_3} \langle \pi_2(\mathsf{factpair}(x)){:}\pi_1(\mathsf{factpair}(x)), \mathsf{s}(x) \times \pi_2(\mathsf{factpair}(x) \\
& \leftarrow_{\mathcal{R}_3} \mathsf{step}(\mathsf{s}(x), \mathsf{factpair}(x))
\end{array}
$$

$$
\begin{array}{ll}
\mathsf{factlist}(x) & \to \pi_1(\mathsf{step}(x)) \\
\mathsf{fact}(x) & \to \pi_2(\mathsf{step}(x)) \\
\mathsf{factpair}(0) & \to \langle [\,], \mathsf{s}(0)\rangle \\
\mathsf{factpair}(\mathsf{s}(x)) & \to \mathsf{step}(\mathsf{s}(x), \mathsf{factpair}(x)) \\
\mathsf{step}(x, y) & \to \langle \pi_2(y){:}\pi_1(y), \times(x, \pi_2(y))\rangle \\
\pi_1(\langle x, y\rangle) & \to x \\
\pi_2(\langle x, y\rangle) & \to y
\end{array}
$$

- $\mathcal{R}_5 \underset{A}{\Rightarrow} \mathcal{R}_4 \cup \{\mathsf{factlist}(x) \to \pi_1(\mathsf{factpair}(x))\}$
- $\mathcal{R}_6 \underset{A}{\Rightarrow} \mathcal{R}_5 \cup \{\mathsf{fact}(x) \to \pi_2(\mathsf{factpair}(x))\}$

Example

$$
\begin{array}{ll}
\mathsf{factlist}(0) & \to [\,] \\
\mathsf{factlist}(\mathsf{s}(x)) & \to \mathsf{fact}(x){:}\mathsf{factlist}(x) \\
\mathsf{fact}(0) & \to \mathsf{s}(0) \\
\mathsf{fact}(\mathsf{s}(x)) & \to \times(\mathsf{s}(x), \mathsf{fact}(x)) \\
\pi_1(\langle x, y\rangle) & \to x \\
\pi_2(\langle x, y\rangle) & \to y
\end{array}
$$

$$
\begin{array}{ll}
\mathsf{factlist}(x) & \to \pi_1(\mathsf{step}(x)) \\
\mathsf{fact}(x) & \to \pi_2(\mathsf{step}(x)) \\
\mathsf{factpair}(0) & \to \langle [\,], \mathsf{s}(0)\rangle \\
\mathsf{factpair}(\mathsf{s}(x)) & \to \mathsf{step}(\mathsf{s}(x), \mathsf{factpair}(x)) \\
\mathsf{step}(x, y) & \to \langle \pi_2(y){:}\pi_1(y), \times(x, \pi_2(y))\rangle \\
\pi_1(\langle x, y\rangle) & \to x \\
\pi_2(\langle x, y\rangle) & \to y
\end{array}
$$

- $\mathcal{R}_0 = \mathcal{R}_{\mathsf{factlist}}$
- $\mathcal{R}_1 \underset{I}{\Rightarrow} \mathcal{R}_0 \cup \{\mathsf{factpair}(x) \to \langle \mathsf{factlist}(x), \mathsf{fact}(x)\rangle\}$
- $\mathcal{R}_2 \underset{I}{\Rightarrow} \mathcal{R}_1 \cup \{\mathsf{step}(x, y) \to \langle \pi_1(y){:}\pi_2(y), x \times \pi_2(y)\rangle\}$
- $\mathcal{R}_3 \underset{A}{\Rightarrow} \mathcal{R}_2 \cup \{\mathsf{factpair}(0) \to \langle [\,], \mathsf{s}(0)\rangle\}$
- $\mathcal{R}_4 \underset{A}{\Rightarrow} \mathcal{R}_3 \cup \{\mathsf{factpair}(\mathsf{s}(x)) \to \mathsf{step}(\mathsf{s}(x), \mathsf{factpair}(x))\}$

$$
\begin{array}{lll}
\mathsf{factpair}(\mathsf{s}(x)) & \to_{\mathcal{R}_3} & \langle \mathsf{factlist}(\mathsf{s}(x)), \mathsf{fact}(\mathsf{s}(x))\rangle \\
 & \overset{*}{\to}_{\mathcal{R}_3} & \langle \mathsf{fact}(x){:}\mathsf{factlist}(x), \mathsf{s}(x) \times \mathsf{fact}(x)\rangle \\
 & \overset{*}{\leftarrow}_{\mathcal{R}_3} & \langle \pi_2(\langle \mathsf{factlist}(x), \mathsf{fact}(x)\rangle){:}\pi_1(\langle \mathsf{factlist}(x), \mathsf{fact}(x)\rangle), \\
 & & \quad \mathsf{s}(x) \times \pi_2(\langle \mathsf{factlist}(x), \mathsf{fact}(x)\rangle \\
 & \overset{*}{\leftarrow}_{\mathcal{R}_3} & \langle \pi_2(\mathsf{factpair}(x)){:}\pi_1(\mathsf{factpair}(x)), \mathsf{s}(x) \times \pi_2(\mathsf{factpair}(x) \\
 & \leftarrow_{\mathcal{R}_3} & \mathsf{step}(\mathsf{s}(x), \mathsf{factpair}(x))
\end{array}
$$

- $\mathcal{R}_5 \underset{A}{\Rightarrow} \mathcal{R}_4 \cup \{\mathsf{factlist}(x) \to \pi_1(\mathsf{factpair}(x))\}$
- $\mathcal{R}_6 \underset{A}{\Rightarrow} \mathcal{R}_5 \cup \{\mathsf{fact}(x) \to \pi_2(\mathsf{factpair}(x))\}$
- $\mathcal{R}_6 \overset{*}{\underset{E}{\Rightarrow}} \mathcal{R}'_{\mathsf{factlist}}$.

Verifying Equivalence of TRSs

Theorem

- $\mathcal{R}$ is a left-linear CS over $\mathscr{G}$
- $\mathcal{R}'$ is a TRS over $\mathscr{G}'$
- $\mathcal{E}$ is a set of equations over $\mathscr{G}$
- $\mathcal{R} \overset{*}{\underset{I}{\Rightarrow}} \cdot \overset{*}{\underset{A}{\Rightarrow}} \cdot \overset{*}{\underset{E}{\Rightarrow}} \mathcal{R}'$ under $\mathcal{E}$
- $\mathcal{R}, \mathscr{G} \vdash_{ind} \mathcal{E}$
- $CR(\mathcal{R})$, $SC(\mathcal{R}, \mathscr{G})$
- $SC(\mathcal{R}', \mathscr{G}')$

$$\Longrightarrow \quad \mathcal{R} \simeq_{\mathscr{G} \cap \mathscr{G}'} \mathcal{R}'$$

Abstruct Reduction System

Definition

- $\mathcal{A} = \langle A, \rightarrow \rangle$
  - TRS: $\langle \mathrm{T}(\mathscr{F}, \mathscr{V}), \rightarrow_{\mathcal{R}} \rangle$
  - EQL: $\langle \mathrm{T}(\mathscr{F}, \mathscr{V}), =_{\mathcal{E}} \rangle$
  - lambda-calculus: $\langle \{s | s : \text{lambda term}\}, \rightarrow_{\beta} \rangle$
  - SM: $\langle S, \rightarrow \rangle$
  - $\ldots$
- $\mathrm{NF}(\mathcal{A}) \stackrel{\text{def}}{=} \{a \in A \mid \neg \exists b \in A . a \rightarrow b\}$

Confluence

Definition

- $\mathcal{A} = \langle A, \to \rangle$
- $\mathrm{CR}(\mathcal{A}) \stackrel{\mathsf{def}}{=} \stackrel{*}{\leftarrow} \cdot \stackrel{*}{\to} \subseteq \stackrel{*}{\to} \cdot \stackrel{*}{\leftarrow}$

Confluence

Definition

- $\mathcal{A} = \langle A, \rightarrow \rangle$
- $\mathrm{CR}(\mathcal{A}) \overset{\text{def}}{=} \overset{*}{\leftarrow} \cdot \overset{*}{\rightarrow} \subseteq \overset{*}{\rightarrow} \cdot \overset{*}{\leftarrow}$

Proposition

- $\mathrm{CR}(\mathcal{A})$ iff $\overset{*}{\leftrightarrow} \subseteq \overset{*}{\rightarrow} \cdot \overset{*}{\leftarrow}$

Confluence

Definition

- $\mathcal{A} = \langle A, \rightarrow \rangle$
- $\mathrm{CR}(\mathcal{A}) \stackrel{\mathsf{def}}{=} \stackrel{*}{\leftarrow} \cdot \stackrel{*}{\rightarrow} \subseteq \stackrel{*}{\rightarrow} \cdot \stackrel{*}{\leftarrow}$

Proposition

- $\mathrm{CR}(\mathcal{A})$ iff $\stackrel{*}{\leftrightarrow} \subseteq \stackrel{*}{\rightarrow} \cdot \stackrel{*}{\leftarrow}$
- $\mathrm{CR}(\mathcal{A}) \wedge a \stackrel{*}{\leftrightarrow} b \wedge a, b \in \mathrm{NF}(\mathcal{A})$ imply $a = b$

Reachability

Definition

A' is reachable from A by $\to$ ($\mathrm{Reach}(A, \to, A')$) $\stackrel{\mathsf{def}}{=} \forall a \in A. \exists b \in A'. a \to b$

- $\mathrm{SC}(\mathcal{R}, \mathscr{G}) \stackrel{\mathsf{def}}{=} \mathrm{Reach}(\mathrm{T}(\mathscr{G}), \xrightarrow{*}_{\mathcal{R}}, \mathrm{T}(\mathscr{C}))$
- $\mathrm{SC}(SP) \stackrel{\mathsf{def}}{=} \mathrm{Reach}(\mathrm{T}(\mathscr{F}, Y), \approx_{\mathcal{E}}, \mathrm{T}(\mathscr{C}, Y))$

Commutativity of confluence

$\mathcal{A}_1 = \langle A, \rightarrow_1 \rangle$ and $\mathcal{A}_2 = \langle A, \rightarrow_2 \rangle$

Definition

$\mathcal{A}_1$ commutes with $\mathcal{A}_2$ $(\mathrm{COMM}(\mathcal{A}_1, \mathcal{A}_2)) \stackrel{\mathsf{def}}{=} \stackrel{*}{\leftarrow}_1 \cdot \stackrel{*}{\rightarrow}_2 \subseteq \stackrel{*}{\rightarrow}_2 \cdot \stackrel{*}{\leftarrow}_1$

Proposition

$\mathrm{CR}(\mathcal{A}_1) \wedge \mathrm{CR}(\mathcal{A}_2) \wedge \mathrm{COMM}(\mathcal{A}_1, \mathcal{A}_2)$ imply $\mathrm{CR}(\mathcal{A}_1 \sqcup \mathcal{A}_2)$

Commutativity of confluence

$\mathcal{A}_1 = \langle A, \to_1 \rangle$ and $\mathcal{A}_2 = \langle A, \to_2 \rangle$

Definition

$\mathcal{A}_1$ commutes with $\mathcal{A}_2$ $(\mathrm{COMM}(\mathcal{A}_1, \mathcal{A}_2)) \stackrel{\mathsf{def}}{=} \stackrel{*}{\leftarrow}_1 \cdot \stackrel{*}{\rightarrow}_2 \subseteq \stackrel{*}{\rightarrow}_2 \cdot \stackrel{*}{\leftarrow}_1$

Proposition

$\mathrm{CR}(\mathcal{A}_1) \wedge \mathrm{CR}(\mathcal{A}_2) \wedge \mathrm{COMM}(\mathcal{A}_1, \mathcal{A}_2)$ imply $\mathrm{CR}(\mathcal{A}_1 \sqcup \mathcal{A}_2)$

Theorem (Toyama 1988)

If left-linear term rewriting systems $\mathcal{R}_1$ and $\mathcal{R}_2$ are non-overlapping with each other, then $\mathcal{R}_1$ commutes with $\mathcal{R}_2$.

A Principle of Inductionless Induction

$\mathcal{A}_1 = \langle A, \rightarrow_1 \rangle$ and $\mathcal{A}_2 = \langle A, \rightarrow_2 \rangle$

Theorem (Toyama 1986, Koike and Toyama 2000)

1. $\rightarrow_1 \subseteq \rightarrow_2$,
2. $\mathrm{CR}(\mathcal{A}_2)$,
3. $A'' \subseteq \mathrm{NF}(\mathcal{A}_2)$, and
4. $\mathrm{Reach}(A', \overset{*}{\leftrightarrow}_1, A'')$

imply
$\overset{*}{\leftrightarrow}_1 = \overset{*}{\leftrightarrow}_2$ on A'.

Equivalent Transformation

$\mathcal{R}_0$: left-linear CS over $\mathscr{F}_0$, $\mathcal{E}$: set of equations over $\mathscr{F}_0$

- Introduction

  $$\mathcal{R}_k \underset{I}{\Rightarrow} \mathcal{R}_k \cup \{f(x_1,\ldots,x_n) \to r\} \; (= \mathcal{R}_{k+1})$$

  $$\mathscr{F}_{k+1} = \mathscr{F}_k \cup \{f\}$$

  - $f(x_1,\ldots,x_n)$ is linear
  - $f \notin \mathscr{F}_k$, and
  - $r \in T(\mathscr{F}_k, \mathscr{V})$

- Addition

  $$\mathcal{R}_k \underset{A}{\Rightarrow} \mathcal{R}_k \cup \{l \to r\} \; (= \mathcal{R}_{k+1})$$

  - $l \overset{*}{\leftrightarrow}_{\mathcal{R}_k \cup \mathcal{E}} r$

- Elimination

  $$\mathcal{R}_k \underset{E}{\Rightarrow} \mathcal{R}_k \setminus \{l \to r\} \; (= \mathcal{R}_{k+1})$$

Verifying Equivalence of TRSs

Theorem

- $\mathcal{R}$ is a left-linear CS over $\mathscr{G}$
- $\mathcal{R}'$ is a TRS over $\mathscr{G}'$
- $\mathcal{E}$ is a set of equations over $\mathscr{G}$
- $\mathcal{R} \overset{*}{\underset{I}{\Rightarrow}} \cdot \overset{*}{\underset{A}{\Rightarrow}} \cdot \overset{*}{\underset{E}{\Rightarrow}} \mathcal{R}'$ under $\mathcal{E}$
- $\mathcal{R}, \mathscr{G} \vdash_{ind} \mathcal{E}$
- $CR(\mathcal{R})$, $SC(\mathcal{R}, \mathscr{G})$
- $SC(\mathcal{R}', \mathscr{G}')$

$$\Longrightarrow \quad \mathcal{R} \simeq_{\mathscr{G} \cap \mathscr{G}'} \mathcal{R}'$$

Proof

$$\mathcal{R} \overset{*}{\underset{I}{\Rightarrow}} \mathcal{R}_I \overset{*}{\underset{A}{\Rightarrow}} \mathcal{R}_A \overset{*}{\underset{E}{\Rightarrow}} \mathcal{R}'$$

1. $\mathrm{CR}(\mathcal{R}_I)$ and $\forall s \in \mathrm{T}(\mathscr{F}).\exists s' \in \mathrm{T}(\mathscr{G}).s \xrightarrow{*}_{\mathcal{R}_I} s'$
2. $\overset{*}{\leftrightarrow}_{\mathcal{R}} = \overset{*}{\leftrightarrow}_{\mathcal{R}_I}$ on $\mathrm{T}(\mathscr{G})$
3. $\overset{*}{\leftrightarrow}_{\mathcal{R}_I} = \overset{*}{\leftrightarrow}_{\mathcal{R}_A}$ on $\mathrm{T}(\mathscr{F})$
4. $\overset{*}{\leftrightarrow}_{\mathcal{R}_I} = \overset{*}{\leftrightarrow}_{\mathcal{R}'}$ on $\mathrm{T}(\mathscr{G}')$
5. $\mathcal{R} \simeq_{\mathscr{G} \cap \mathscr{G}'} \mathcal{R}'$

$\mathrm{CR}(\mathcal{R}_I)$ **and** $\forall t \in \mathrm{T}(\mathscr{F}).\exists t' \in \mathrm{T}(\mathscr{G}).t \xrightarrow{*}_{\mathcal{R}_{\underline{I}}} t'$

$\mathcal{R} \underset{I}{\overset{*}{\Rightarrow}} \mathcal{R}_I \underset{A}{\overset{*}{\Rightarrow}} \mathcal{R}_A \underset{E}{\overset{*}{\Rightarrow}} \mathcal{R}'$

- Introduction

  $\mathcal{R}_k \underset{I}{\Rightarrow} \mathcal{R}_k \cup \{f(x_1, \ldots, x_n) \to r\}$ $(= \mathcal{R}_{k+1})$

  $\mathscr{F}_{k+1} = \mathscr{F}_k \cup \{f\}$

  - $f(x_1, \ldots, x_n)$ is linear
  - $f \notin \mathscr{F}_k$, and
  - $r \in T(\mathscr{F}_k, \mathscr{V})$

$\mathrm{CR}(\mathcal{R}_I)$ **and** $\forall t \in \mathrm{T}(\mathscr{F}).\exists t' \in \mathrm{T}(\mathscr{G}).t \xrightarrow{*}_{\mathcal{R}_I} t'$

$\mathcal{R} \overset{*}{\underset{I}{\Rightarrow}} \mathcal{R}_I \overset{*}{\underset{A}{\Rightarrow}} \mathcal{R}_A \overset{*}{\underset{E}{\Rightarrow}} \mathcal{R}'$

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  $\mathcal{R}_k \underset{I}{\Rightarrow} \mathcal{R}_k \cup \{f(x_1, \ldots, x_n) \to r\}$ $(= \mathcal{R}_{k+1})$

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- $\forall s \in \mathrm{T}(\mathscr{F})$

  s

$\mathrm{CR}(\mathcal{R}_I)$ and $\forall t \in \mathrm{T}(\mathscr{F}).\exists t' \in \mathrm{T}(\mathscr{G}).t \xrightarrow{*}_{\mathcal{R}_I} t'$

$\mathcal{R} \overset{*}{\underset{I}{\Rightarrow}} \mathcal{R}_I \overset{*}{\underset{A}{\Rightarrow}} \mathcal{R}_A \overset{*}{\underset{E}{\Rightarrow}} \mathcal{R}'$

- Introduction

  $\mathcal{R}_k \underset{I}{\Rightarrow} \mathcal{R}_k \cup \{f(x_1, \ldots, x_n) \to r\}$ ($= \mathcal{R}_{k+1}$)

  $\mathscr{F}_{k+1} = \mathscr{F}_k \cup \{f\}$

  - $f(x_1, \ldots, x_n)$ is linear
  - $f \notin \mathscr{F}_k$, and
  - $r \in T(\mathscr{F}_k, \mathscr{V})$

- $\forall s \in \mathrm{T}(\mathscr{F}),\ \exists s' \in \mathrm{T}(\mathscr{G})$

  $s \xrightarrow{*}_{\mathcal{R}_I} s'$

$\mathrm{CR}(\mathcal{R}_I)$ and $\forall t \in \mathrm{T}(\mathscr{F}).\exists t' \in \mathrm{T}(\mathscr{G}).t \xrightarrow{*}_{\mathcal{R}_I} t'$

$$\mathcal{R} \underset{I}{\overset{*}{\Rightarrow}} \mathcal{R}_I \underset{A}{\overset{*}{\Rightarrow}} \mathcal{R}_A \underset{E}{\overset{*}{\Rightarrow}} \mathcal{R}'$$

- Introduction

  $\mathcal{R}_k \underset{I}{\Rightarrow} \mathcal{R}_k \cup \{f(x_1, \ldots, x_n) \to r\}$ $(= \mathcal{R}_{k+1})$

  $\mathscr{F}_{k+1} = \mathscr{F}_k \cup \{f\}$

  - $f(x_1, \ldots, x_n)$ is linear
  - $f \notin \mathscr{F}_k$, and
  - $r \in T(\mathscr{F}_k, \mathscr{V})$

- $\forall s \in \mathrm{T}(\mathscr{F}),\ \exists s' \in \mathrm{T}(\mathscr{G})$

  $s \xrightarrow{*}_{\mathcal{R}_I} s'$

- $\mathcal{R}$: left-linear,

$\mathrm{CR}(\mathcal{R}_I)$ **and** $\forall t \in \mathrm{T}(\mathscr{F}).\exists t' \in \mathrm{T}(\mathscr{G}).t \xrightarrow{*}_{\mathcal{R}_I} t'$

$$\mathcal{R} \underset{I}{\overset{*}{\Rightarrow}} \mathcal{R}_I \underset{A}{\overset{*}{\Rightarrow}} \mathcal{R}_A \underset{E}{\overset{*}{\Rightarrow}} \mathcal{R}'$$

- Introduction

  $\mathcal{R}_k \underset{I}{\Rightarrow} \mathcal{R}_k \cup \{f(x_1, \ldots, x_n) \to r\}$ $(= \mathcal{R}_{k+1})$

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- $\forall s \in \mathrm{T}(\mathscr{F}),\ \exists s' \in \mathrm{T}(\mathscr{G})$

  $s \xrightarrow{*}_{\mathcal{R}_I} s'$

- $\mathcal{R}$: left-linear, $\mathrm{CR}(\mathcal{R})$,

$\mathrm{CR}(\mathcal{R}_I)$ and $\forall t \in \mathrm{T}(\mathscr{F}).\exists t' \in \mathrm{T}(\mathscr{G}).t \xrightarrow{*}_{\mathcal{R}_I} t'$

$$\mathcal{R} \underset{I}{\overset{*}{\Rightarrow}} \mathcal{R}_I \underset{A}{\overset{*}{\Rightarrow}} \mathcal{R}_A \underset{E}{\overset{*}{\Rightarrow}} \mathcal{R}'$$

- Introduction

  $\mathcal{R}_k \underset{I}{\Rightarrow} \mathcal{R}_k \cup \{f(x_1, \ldots, x_n) \to r\}$ $(= \mathcal{R}_{k+1})$

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  $s \xrightarrow{*}_{\mathcal{R}_I} s'$

- $\mathcal{R}$: left-linear, $\mathrm{CR}(\mathcal{R})$, $\mathcal{R}_I \setminus \mathcal{R}$: left-linear,

$\mathrm{CR}(\mathcal{R}_I)$ and $\forall t \in \mathrm{T}(\mathscr{F}).\exists t' \in \mathrm{T}(\mathscr{G}).t \xrightarrow{*}_{\mathcal{R}_I} t'$

$\mathcal{R} \underset{I}{\overset{*}{\Rightarrow}} \mathcal{R}_I \underset{A}{\overset{*}{\Rightarrow}} \mathcal{R}_A \underset{E}{\overset{*}{\Rightarrow}} \mathcal{R}'$

- Introduction

  $\mathcal{R}_k \underset{I}{\Rightarrow} \mathcal{R}_k \cup \{f(x_1, \ldots, x_n) \to r\}$ $(= \mathcal{R}_{k+1})$

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  - $f(x_1, \ldots, x_n)$ is linear
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  - $r \in T(\mathscr{F}_k, \mathscr{V})$

- $\forall s \in \mathrm{T}(\mathscr{F}),\ \exists s' \in \mathrm{T}(\mathscr{G})$

  $s \xrightarrow{*}_{\mathcal{R}_I} s'$

- $\mathcal{R}$: left-linear, $\mathrm{CR}(\mathcal{R})$, $\mathcal{R}_I \setminus \mathcal{R}$: left-linear, $\mathrm{CR}(\mathcal{R}_I \setminus \mathcal{R})$,

$\mathrm{CR}(\mathcal{R}_I)$ **and** $\forall t \in \mathrm{T}(\mathscr{F}).\exists t' \in \mathrm{T}(\mathscr{G}).t \xrightarrow{*}_{\mathcal{R}_I} t'$

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- Introduction

  $\mathcal{R}_k \underset{I}{\Rightarrow} \mathcal{R}_k \cup \{f(x_1, \ldots, x_n) \to r\}$ $(= \mathcal{R}_{k+1})$

  $\mathscr{F}_{k+1} = \mathscr{F}_k \cup \{f\}$

  - $f(x_1, \ldots, x_n)$ is linear
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- $\forall s \in \mathrm{T}(\mathscr{F}),\ \exists s' \in \mathrm{T}(\mathscr{G})$

  $s \xrightarrow{*}_{\mathcal{R}_I} s'$

- $\mathcal{R}$: left-linear, $\mathrm{CR}(\mathcal{R})$, $\mathcal{R}_I \setminus \mathcal{R}$: left-linear, $\mathrm{CR}(\mathcal{R}_I \setminus \mathcal{R})$, and $\mathcal{R}$ and $\mathcal{R}_I \setminus \mathcal{R}$ do not overlap to each other

$\mathrm{CR}(\mathcal{R}_I)$ and $\forall t \in \mathrm{T}(\mathscr{F}).\exists t' \in \mathrm{T}(\mathscr{G}).t \xrightarrow{*}_{\mathcal{R}_I} t'$

$\mathcal{R} \overset{*}{\underset{I}{\Rightarrow}} \mathcal{R}_I \overset{*}{\underset{A}{\Rightarrow}} \mathcal{R}_A \overset{*}{\underset{E}{\Rightarrow}} \mathcal{R}'$

- Introduction

  $\mathcal{R}_k \underset{I}{\Rightarrow} \mathcal{R}_k \cup \{f(x_1, \ldots, x_n) \to r\}$ $(= \mathcal{R}_{k+1})$

  $\mathscr{F}_{k+1} = \mathscr{F}_k \cup \{f\}$

  - $f(x_1, \ldots, x_n)$ is linear
  - $f \notin \mathscr{F}_k$, and
  - $r \in T(\mathscr{F}_k, \mathscr{V})$

- $\forall s \in \mathrm{T}(\mathscr{F}),\ \exists s' \in \mathrm{T}(\mathscr{G})$

  $s \xrightarrow{*}_{\mathcal{R}_I} s'$

- $\mathcal{R}$: left-linear, $\mathrm{CR}(\mathcal{R})$, $\mathcal{R}_I \setminus \mathcal{R}$: left-linear, $\mathrm{CR}(\mathcal{R}_I \setminus \mathcal{R})$, and $\mathcal{R}$ and $\mathcal{R}_I \setminus \mathcal{R}$ do not overlap to each other

  $\Rightarrow \mathrm{CR}(\mathcal{R}_I)$

$\overset{*}{\leftrightarrow}_{\mathcal{R}} = \overset{*}{\leftrightarrow}_{\mathcal{R}_I}$ on $\mathrm{T}(\mathscr{G})$

$\mathcal{R} \overset{*}{\underset{I}{\Rightarrow}} \mathcal{R}_I \overset{*}{\underset{A}{\Rightarrow}} \mathcal{R}_A \overset{*}{\underset{E}{\Rightarrow}} \mathcal{R}'$

- Introduction

  $\mathcal{R}_k \underset{I}{\Rightarrow} \mathcal{R}_k \cup \{f(x_1, \ldots, x_n) \to r\}$ $(= \mathcal{R}_{k+1})$

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  - $f(x_1, \ldots, x_n)$ is linear
  - $f \notin \mathscr{F}_k$, and
  - $r \in T(\mathscr{F}_k, \mathscr{V})$

$\mathrm{T}(\mathscr{G}) \ni s \overset{*}{\underset{\mathcal{R}_I}{\longleftrightarrow}} t \in \mathrm{T}(\mathscr{G})$

$* \; \mathcal{R}_I$ $\quad * \; \mathcal{R}$ $\qquad * \; \mathcal{R}$ $\quad * \; \mathcal{R}_I$

$\mathrm{T}(\mathscr{C}) \ni s' \quad = \quad t' \in \mathrm{T}(\mathscr{C})$

- $\mathrm{SC}(\mathcal{R}, \mathscr{G})$
- $\mathcal{R} \subseteq \mathcal{R}_I$
- $\mathrm{CR}(\mathcal{R}_I)$ and $\mathrm{T}(\mathscr{C}) \subseteq \mathrm{NF}(\mathcal{R}_I)$

$\overset{*}{\leftrightarrow}_{\mathcal{R}} = \overset{*}{\leftrightarrow}_{\mathcal{R}_I}$ on $\mathrm{T}(\mathscr{G})$

$\mathcal{R} \underset{I}{\overset{*}{\Rightarrow}} \mathcal{R}_I \underset{A}{\overset{*}{\Rightarrow}} \mathcal{R}_A \underset{E}{\overset{*}{\Rightarrow}} \mathcal{R}'$

- Introduction

  $\mathcal{R}_k \underset{I}{\Rightarrow} \mathcal{R}_k \cup \{f(x_1, \ldots, x_n) \to r\}$ $(= \mathcal{R}_{k+1})$

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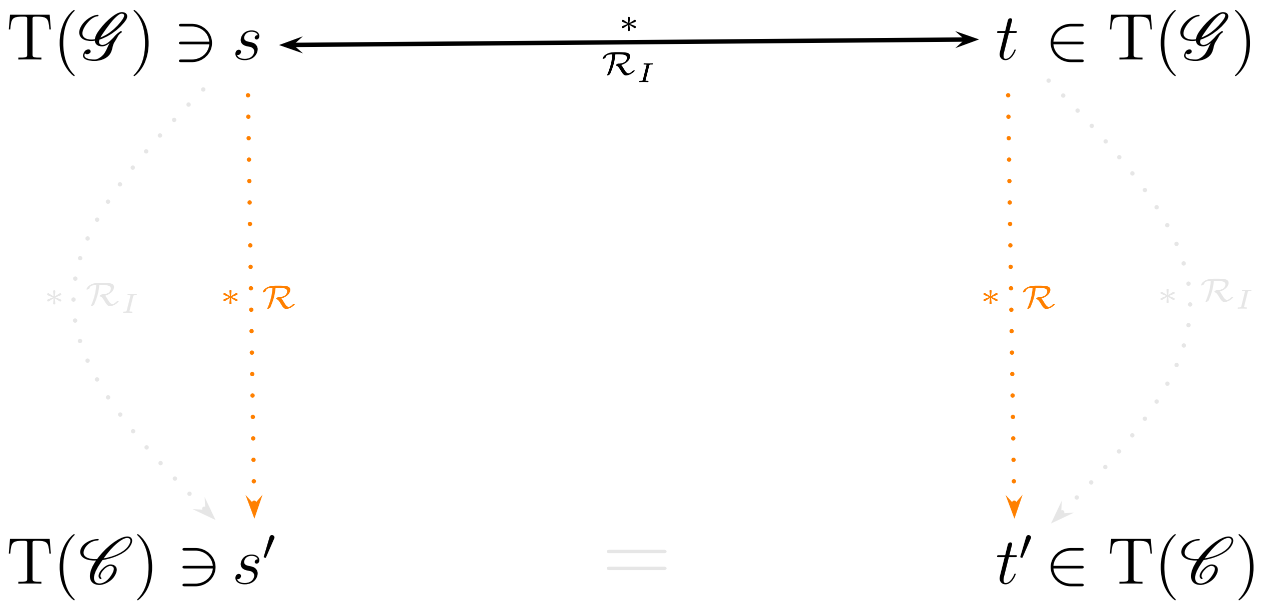
  - $r \in T(\mathscr{F}_k, \mathscr{V})$



- $\mathrm{SC}(\mathcal{R}, \mathscr{G})$
- $\mathcal{R} \subseteq \mathcal{R}_I$
- $\mathrm{CR}(\mathcal{R}_I)$ and $\mathrm{T}(\mathscr{C}) \subseteq \mathrm{NF}(\mathcal{R}_I)$

$\overset{*}{\leftrightarrow}_{\mathcal{R}} = \overset{*}{\leftrightarrow}_{\mathcal{R}_I}$ on $\mathrm{T}(\mathscr{G})$

$\mathcal{R} \underset{I}{\overset{*}{\Rightarrow}} \mathcal{R}_I \underset{A}{\overset{*}{\Rightarrow}} \mathcal{R}_A \underset{E}{\overset{*}{\Rightarrow}} \mathcal{R}'$

- Introduction

  $\mathcal{R}_k \underset{I}{\Rightarrow} \mathcal{R}_k \cup \{f(x_1, \ldots, x_n) \to r\}$ $(= \mathcal{R}_{k+1})$

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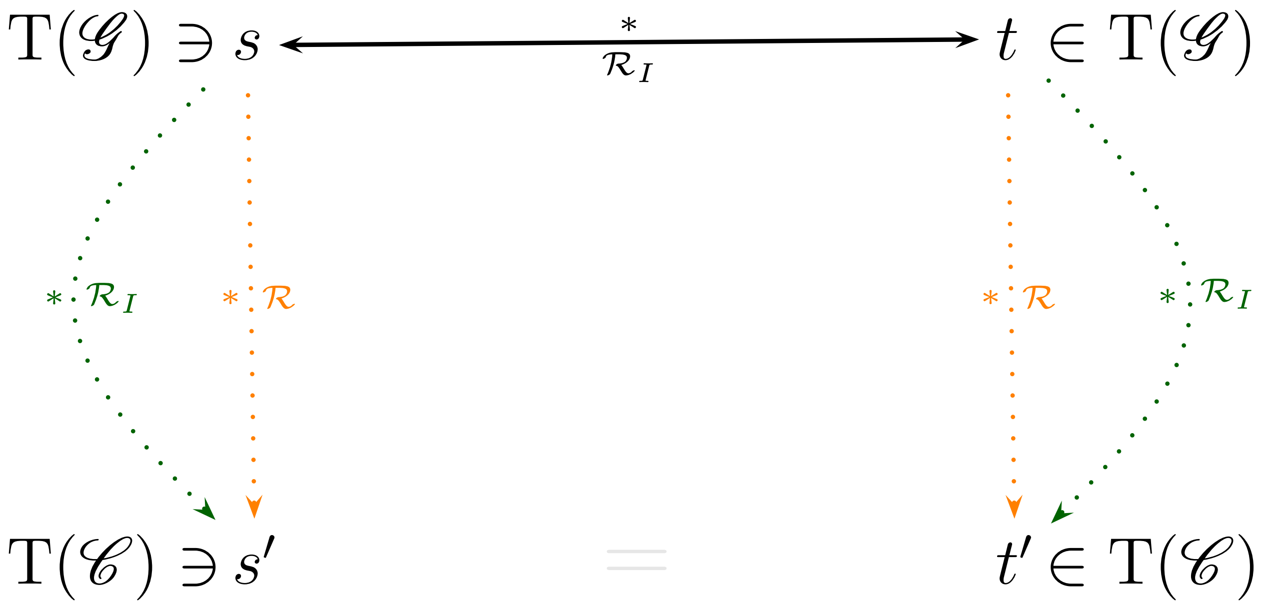

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$\overset{*}{\leftrightarrow}_{\mathcal{R}} = \overset{*}{\leftrightarrow}_{\mathcal{R}_I}$ on $\mathrm{T}(\mathscr{G})$

$\mathcal{R} \overset{*}{\underset{I}{\Rightarrow}} \mathcal{R}_I \overset{*}{\underset{A}{\Rightarrow}} \mathcal{R}_A \overset{*}{\underset{E}{\Rightarrow}} \mathcal{R}'$

- Introduction

  $\mathcal{R}_k \underset{I}{\Rightarrow} \mathcal{R}_k \cup \{f(x_1, \ldots, x_n) \to r\}$ $(= \mathcal{R}_{k+1})$

  $\mathscr{F}_{k+1} = \mathscr{F}_k \cup \{f\}$

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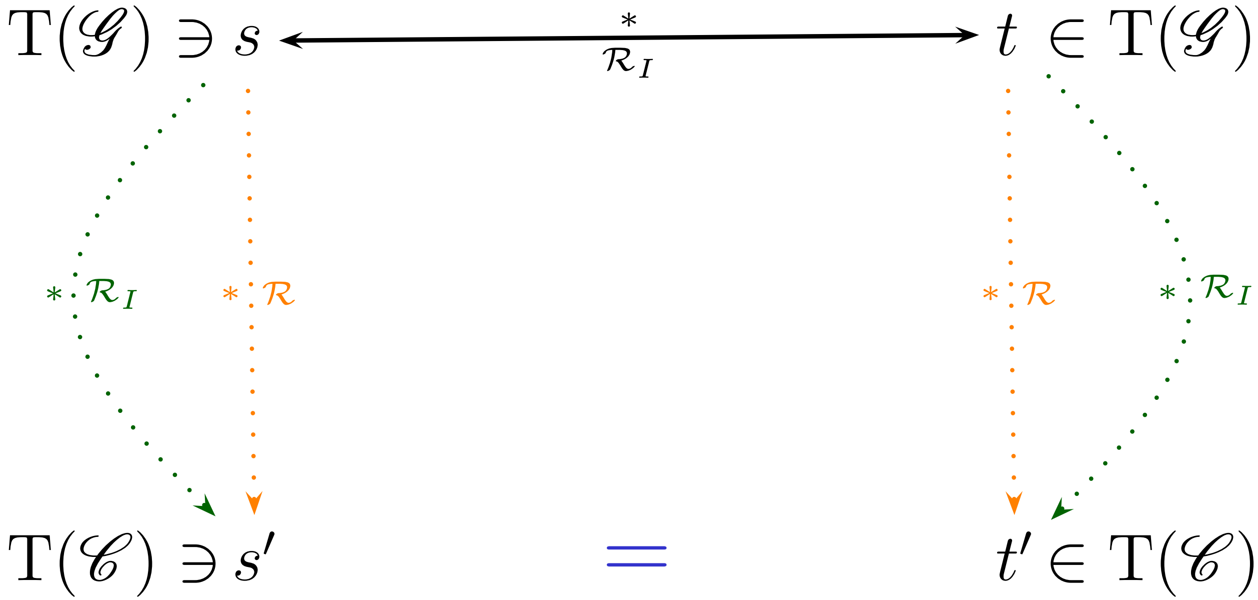


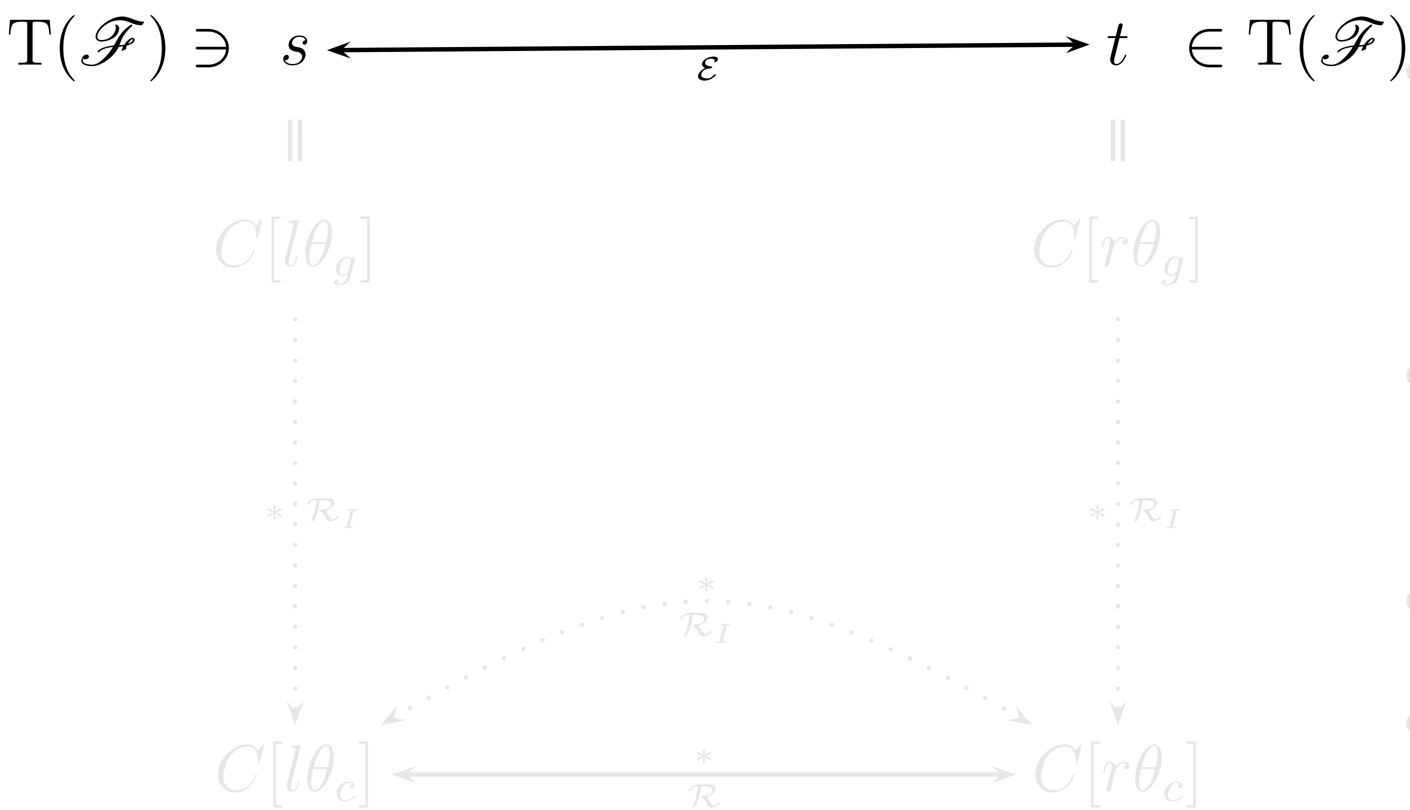
- $\mathrm{SC}(\mathcal{R}, \mathscr{G})$
- $\mathcal{R} \subseteq \mathcal{R}_I$
- $\mathrm{CR}(\mathcal{R}_I)$ and $\mathrm{T}(\mathscr{C}) \subseteq \mathrm{NF}(\mathcal{R}_I)$

$\overset{*}{\leftrightarrow}_{\mathcal{R}_I} = \overset{*}{\leftrightarrow}_{\mathcal{R}_A}$ on $\mathrm{T}(\mathscr{F})$

$\mathcal{R} \overset{*}{\underset{I}{\Rightarrow}} \mathcal{R}_I \overset{*}{\underset{A}{\Rightarrow}} \mathcal{R}_A \overset{*}{\underset{E}{\Rightarrow}} \mathcal{R}'$

- Addition

  $\mathcal{R}_k \underset{A}{\Rightarrow} \mathcal{R}_k \cup \{l \to r\}$ $(= \mathcal{R}_{k+1})$
  - $l \overset{*}{\leftrightarrow}_{\mathcal{R}_k \cup \mathcal{E}} r$



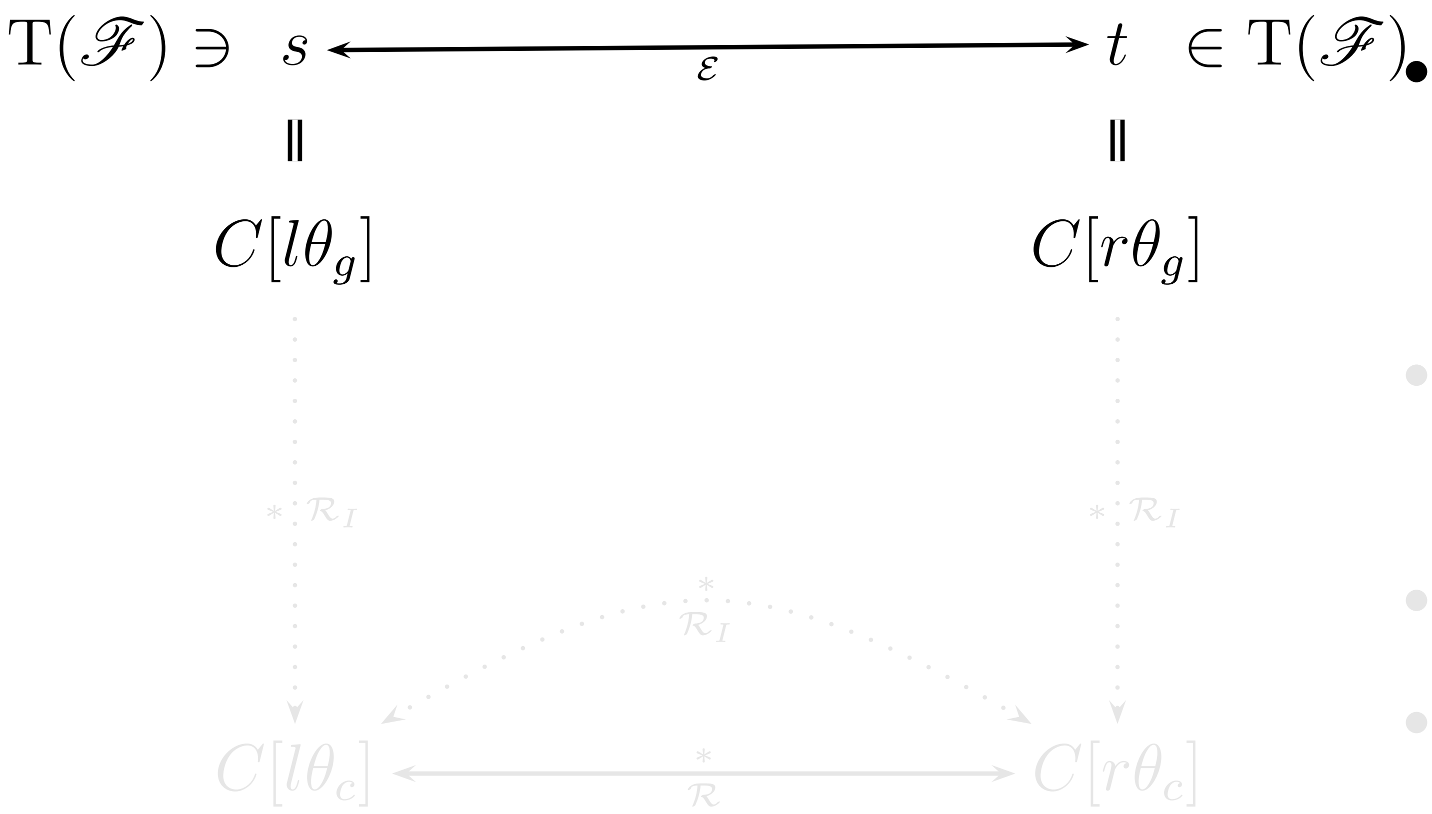
- The definition of reduction relation where
  - $\theta_g : \mathscr{V} \to \mathrm{T}(\mathscr{F})$
- From 1 where
  - $\theta_c : \mathscr{V} \to \mathrm{T}(\mathscr{G})$
- $\mathcal{R}, \mathscr{G} \vdash_{ind} \mathcal{E}$
- $\mathcal{R} \subseteq \mathcal{R}_I$

$\overset{*}{\leftrightarrow}_{\mathcal{R}_I} = \overset{*}{\leftrightarrow}_{\mathcal{R}_A}$ on $\mathrm{T}(\mathscr{F})$

$\mathcal{R} \underset{I}{\overset{*}{\Rightarrow}} \mathcal{R}_I \underset{A}{\overset{*}{\Rightarrow}} \mathcal{R}_A \underset{E}{\overset{*}{\Rightarrow}} \mathcal{R}'$

- Addition

  $\mathcal{R}_k \underset{A}{\Rightarrow} \mathcal{R}_k \cup \{l \to r\}$ $(= \mathcal{R}_{k+1})$ – $l \overset{*}{\leftrightarrow}_{\mathcal{R}_k \cup \mathcal{E}} r$



- The definition of reduction relation where
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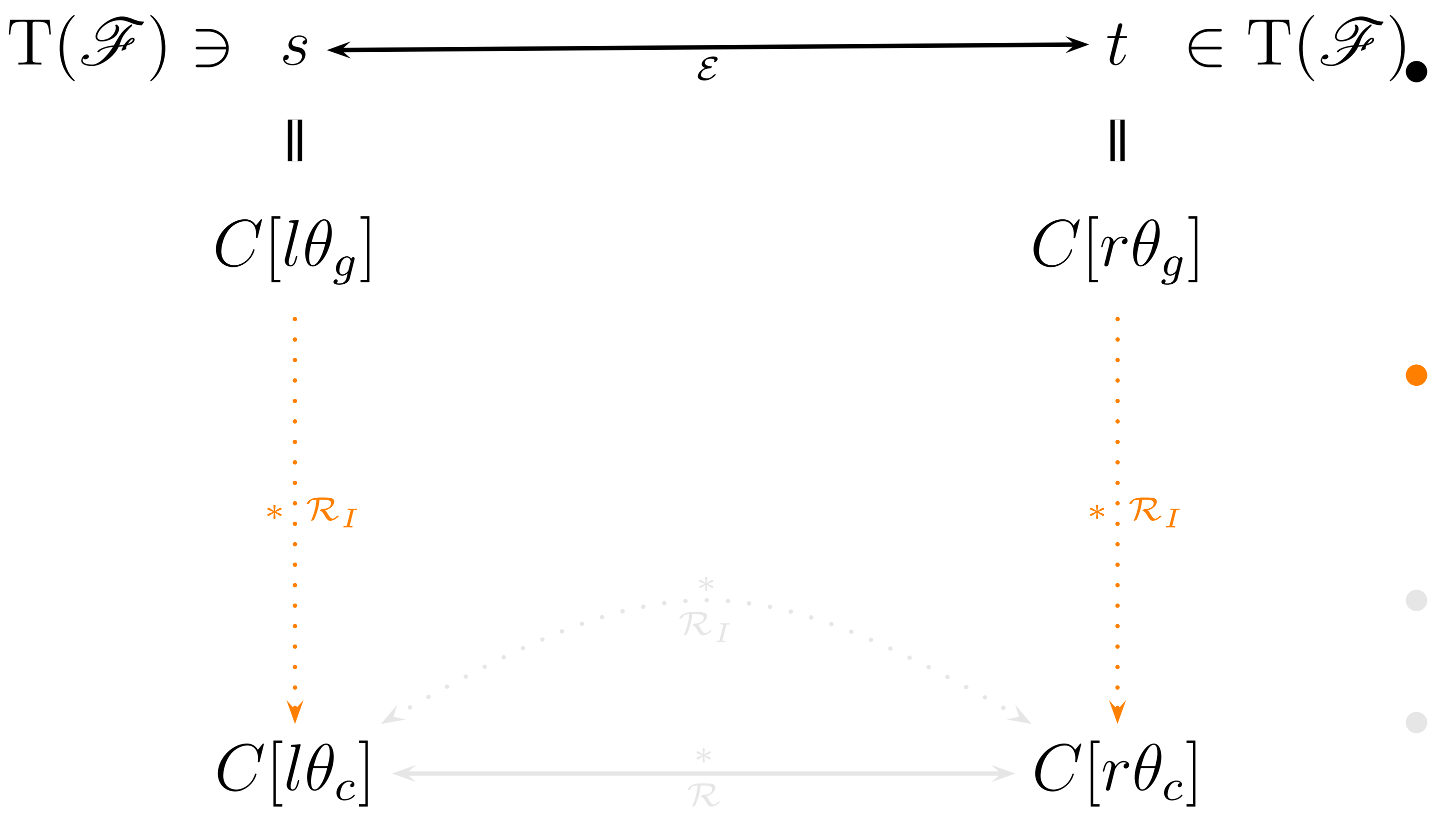
- $\mathcal{R} \subseteq \mathcal{R}_I$

$\overset{*}{\leftrightarrow}_{\mathcal{R}_I} = \overset{*}{\leftrightarrow}_{\mathcal{R}_A}$ on $\mathrm{T}(\mathscr{F})$

$\mathcal{R} \underset{I}{\overset{*}{\Rightarrow}} \mathcal{R}_I \underset{A}{\overset{*}{\Rightarrow}} \mathcal{R}_A \underset{E}{\overset{*}{\Rightarrow}} \mathcal{R}'$

- Addition

  $\mathcal{R}_k \underset{A}{\Rightarrow} \mathcal{R}_k \cup \{l \to r\}$ $(= \mathcal{R}_{k+1})$ – $l \overset{*}{\leftrightarrow}_{\mathcal{R}_k \cup \mathcal{E}} r$



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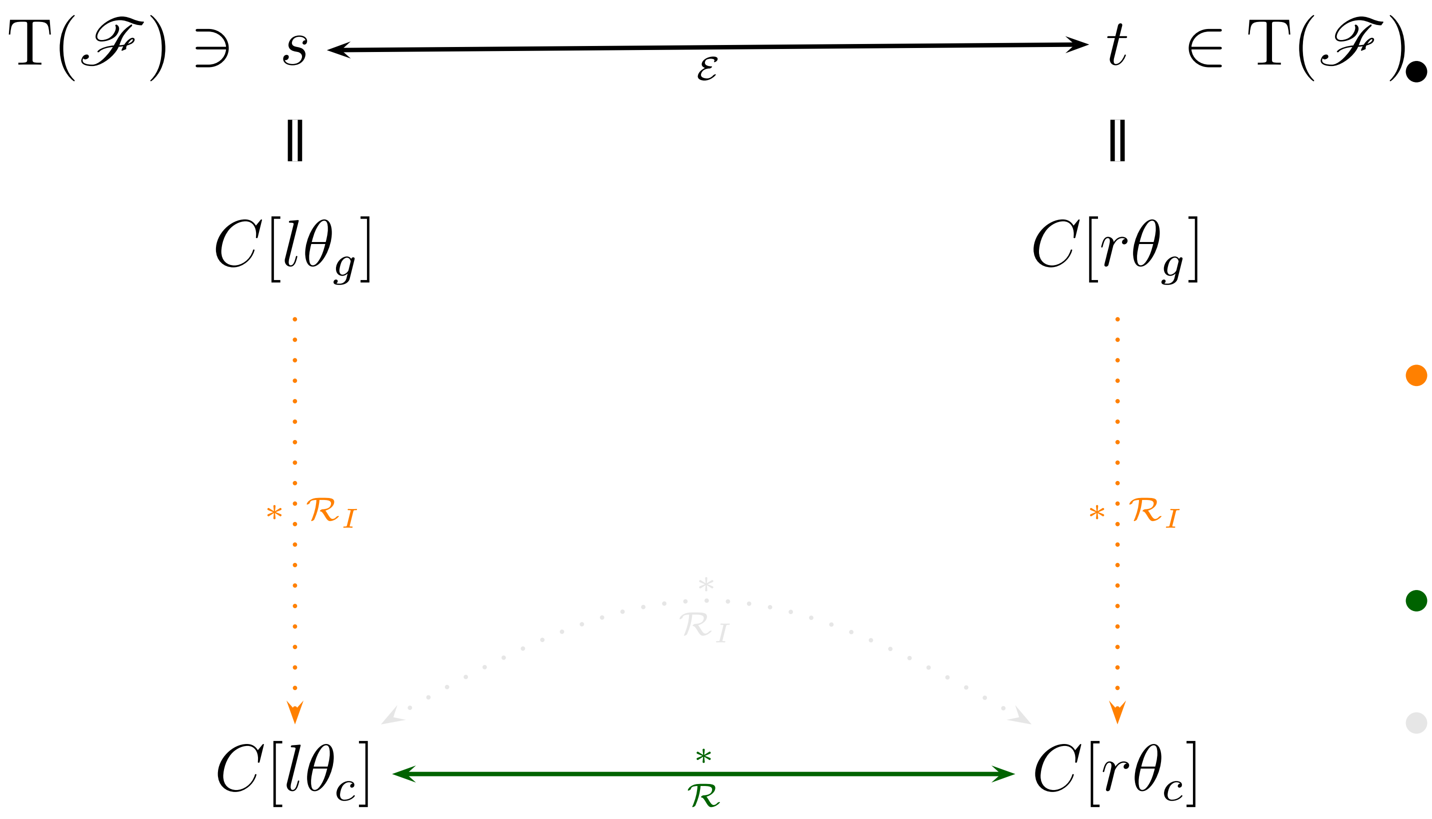
- $\mathcal{R}, \mathscr{G} \vdash_{ind} \mathcal{E}$
- $\mathcal{R} \subseteq \mathcal{R}_I$

$\overset{*}{\leftrightarrow}_{\mathcal{R}_I} = \overset{*}{\leftrightarrow}_{\mathcal{R}_A}$ on $\mathrm{T}(\mathscr{F})$

$\mathcal{R} \underset{I}{\overset{*}{\Rightarrow}} \mathcal{R}_I \underset{A}{\overset{*}{\Rightarrow}} \mathcal{R}_A \underset{E}{\overset{*}{\Rightarrow}} \mathcal{R}'$

- Addition

  $\mathcal{R}_k \underset{A}{\Rightarrow} \mathcal{R}_k \cup \{l \to r\}$ $(= \mathcal{R}_{k+1})$ – $l \overset{*}{\leftrightarrow}_{\mathcal{R}_k \cup \mathcal{E}} r$



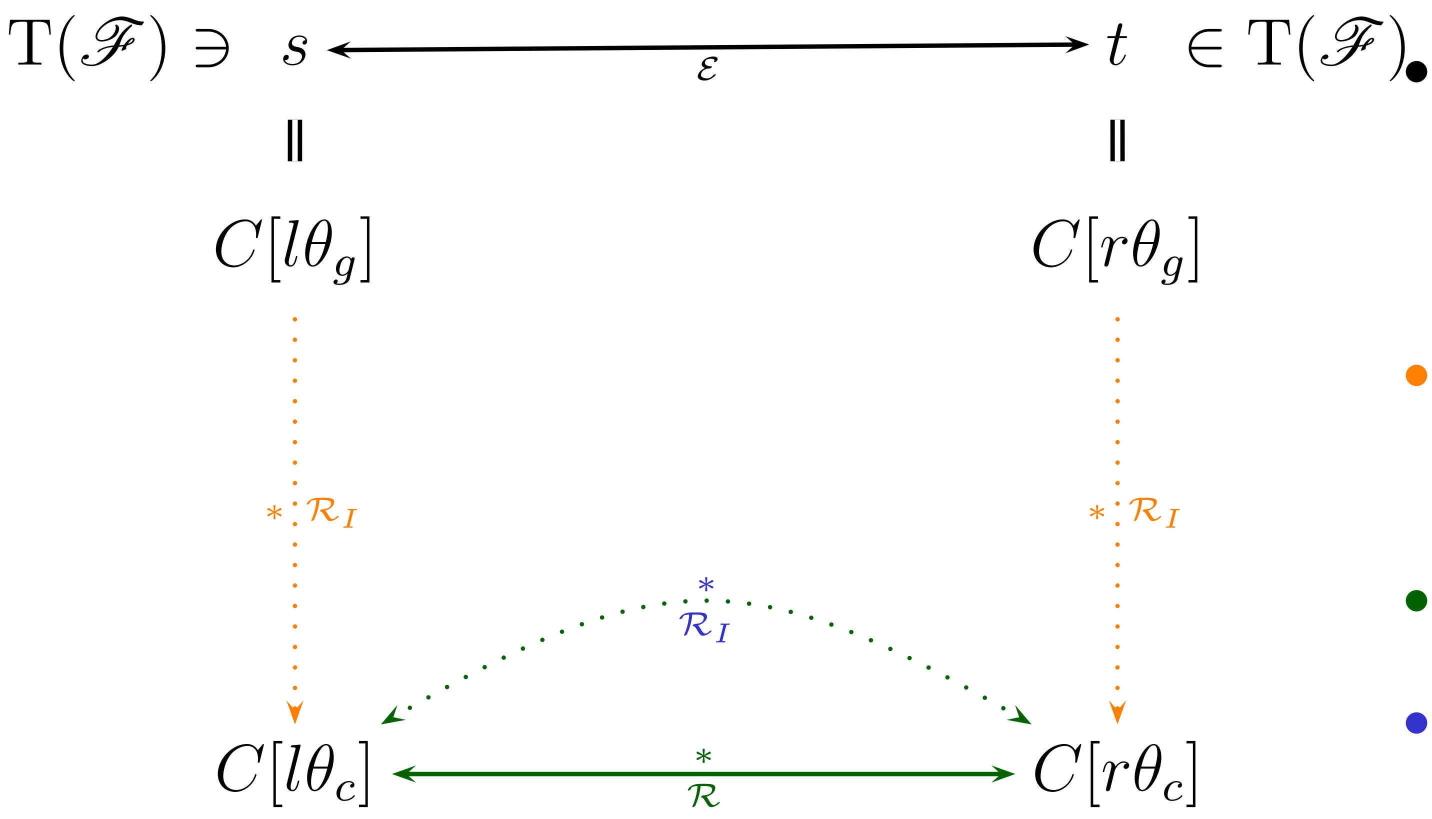
- The definition of reduction relation where
  - $\theta_g : \mathscr{V} \to \mathrm{T}(\mathscr{F})$
- From 1, where
  - $\theta_c : \mathscr{V} \to \mathrm{T}(\mathscr{G})$
- $\mathcal{R}, \mathscr{G} \vdash_{ind} \mathcal{E}$
- $\mathcal{R} \subseteq \mathcal{R}_I$

$\overset{*}{\leftrightarrow}_{\mathcal{R}_I} = \overset{*}{\leftrightarrow}_{\mathcal{R}_A}$ on $\mathrm{T}(\mathscr{F})$

$\mathcal{R} \underset{I}{\overset{*}{\Rightarrow}} \mathcal{R}_I \underset{A}{\overset{*}{\Rightarrow}} \mathcal{R}_A \underset{E}{\overset{*}{\Rightarrow}} \mathcal{R}'$

- Addition

  $\mathcal{R}_k \underset{A}{\Rightarrow} \mathcal{R}_k \cup \{l \to r\}$ $(= \mathcal{R}_{k+1})$
  - $l \overset{*}{\leftrightarrow}_{\mathcal{R}_k \cup \mathcal{E}} r$



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- $\mathcal{R}, \mathscr{G} \vdash_{ind} \mathcal{E}$
- $\mathcal{R} \subseteq \mathcal{R}_I$

$\overset{*}{\leftrightarrow}_{\mathcal{R}_I} = \overset{*}{\leftrightarrow}_{\mathcal{R}'}$ on $\mathrm{T}(\mathscr{G}')$

$\mathcal{R} \overset{*}{\underset{I}{\Rightarrow}} \mathcal{R}_I \overset{*}{\underset{A}{\Rightarrow}} \mathcal{R}_A \overset{*}{\underset{E}{\Rightarrow}} \mathcal{R}'$

- Elimination
  $\mathcal{R}_k \underset{E}{\Rightarrow} \mathcal{R}_k \setminus \{l \to r\}$

$$\mathrm{T}(\mathscr{G}') \ni s \overset{*}{\underset{\mathcal{R}_I}{\longleftrightarrow}} t \in \mathrm{T}(\mathscr{G}')$$

$\mathcal{R}_I$ *   * $\mathcal{R}'$ * $\mathcal{R}'$ $\mathcal{R}_I$ *

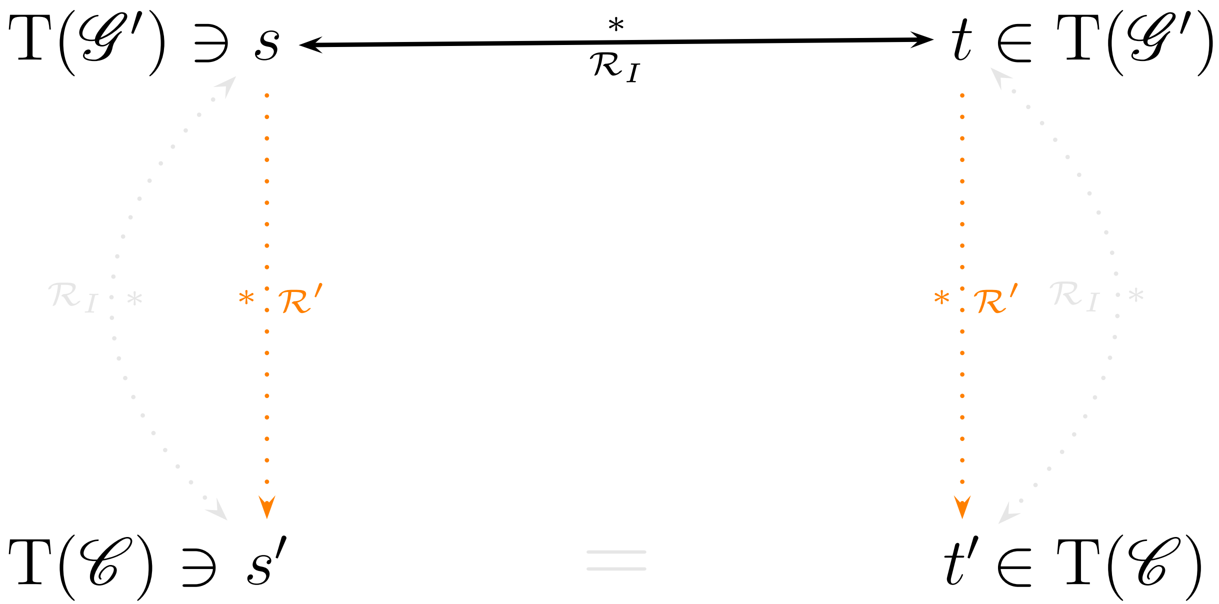
$$\mathrm{T}(\mathscr{C}) \ni s' \quad = \quad t' \in \mathrm{T}(\mathscr{C})$$

- $\mathrm{SC}(\mathcal{R}', \mathscr{G}')$,
- $\overset{*}{\leftrightarrow}_{\mathcal{R}'} \subseteq \overset{*}{\leftrightarrow}_{\mathcal{R}_A} = \overset{*}{\leftrightarrow}_{\mathcal{R}_I}$
- $\mathrm{CR}(\mathcal{R}_I)$,
  $\mathrm{T}(\mathscr{C}) \subseteq \mathrm{NF}(\mathcal{R}_I)$

$\overset{*}{\leftrightarrow}_{\mathcal{R}_I} = \overset{*}{\leftrightarrow}_{\mathcal{R}'}$ on $\mathrm{T}(\mathscr{G}')$

$\mathcal{R} \underset{I}{\overset{*}{\Rightarrow}} \mathcal{R}_I \underset{A}{\overset{*}{\Rightarrow}} \mathcal{R}_A \underset{E}{\overset{*}{\Rightarrow}} \mathcal{R}'$

- Elimination
  $\mathcal{R}_k \underset{E}{\Rightarrow} \mathcal{R}_k \setminus \{l \to r\}$

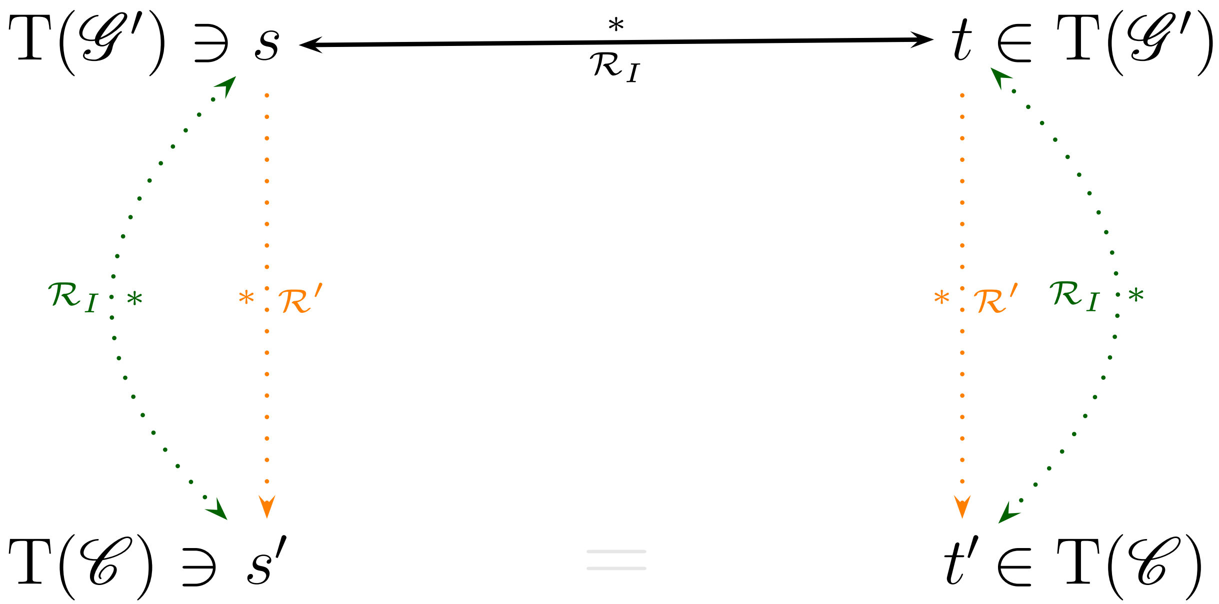


- $\mathrm{SC}(\mathcal{R}', \mathscr{G}')$,
- $\overset{*}{\leftrightarrow}_{\mathcal{R}'} \subseteq \overset{*}{\leftrightarrow}_{\mathcal{R}_A} = \overset{*}{\leftrightarrow}_{\mathcal{R}_I}$
- $\mathrm{CR}(\mathcal{R}_I)$,
  $\mathrm{T}(\mathscr{C}) \subseteq \mathrm{NF}(\mathcal{R}_I)$

$\overset{*}{\leftrightarrow}_{\mathcal{R}_I} = \overset{*}{\leftrightarrow}_{\mathcal{R}'}$ on $\mathrm{T}(\mathscr{G}')$

$\mathcal{R} \underset{I}{\overset{*}{\Rightarrow}} \mathcal{R}_I \underset{A}{\overset{*}{\Rightarrow}} \mathcal{R}_A \underset{E}{\overset{*}{\Rightarrow}} \mathcal{R}'$

- Elimination
  $\mathcal{R}_k \underset{E}{\Rightarrow} \mathcal{R}_k \setminus \{l \to r\}$



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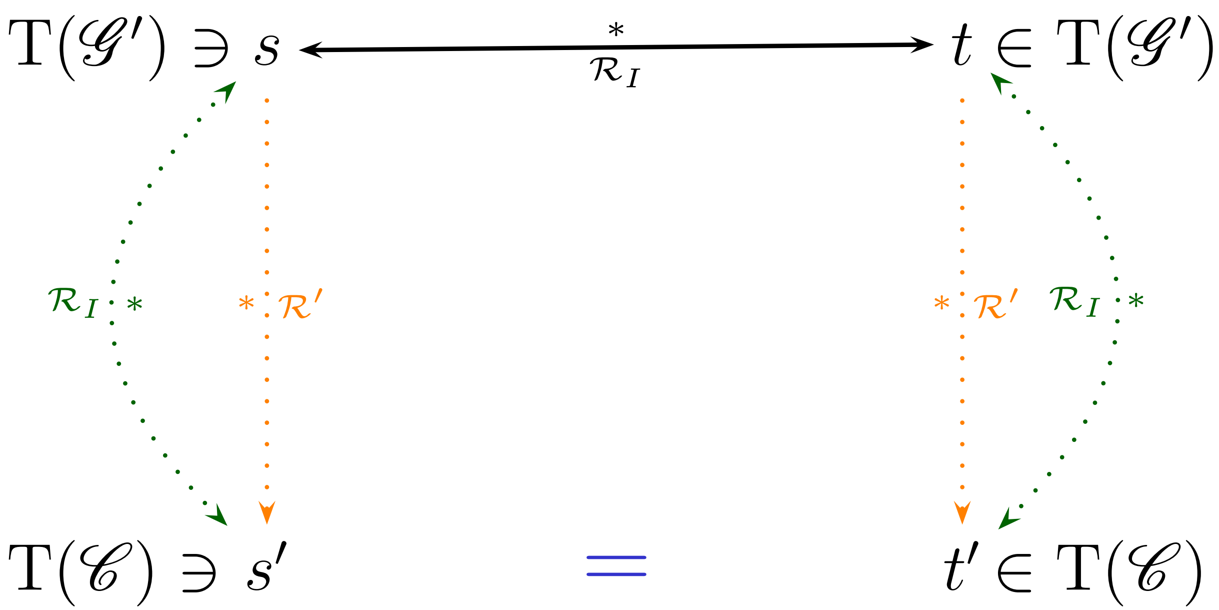
  $\mathrm{T}(\mathscr{C}) \subseteq \mathrm{NF}(\mathcal{R}_I)$

$\overset{*}{\leftrightarrow}_{\mathcal{R}_I} = \overset{*}{\leftrightarrow}_{\mathcal{R}'}$ on $\mathrm{T}(\mathscr{G}')$

$\mathcal{R} \underset{I}{\overset{*}{\Rightarrow}} \mathcal{R}_I \underset{A}{\overset{*}{\Rightarrow}} \mathcal{R}_A \underset{E}{\overset{*}{\Rightarrow}} \mathcal{R}'$

- Elimination

  $\mathcal{R}_k \underset{E}{\Rightarrow} \mathcal{R}_k \setminus \{l \to r\}$

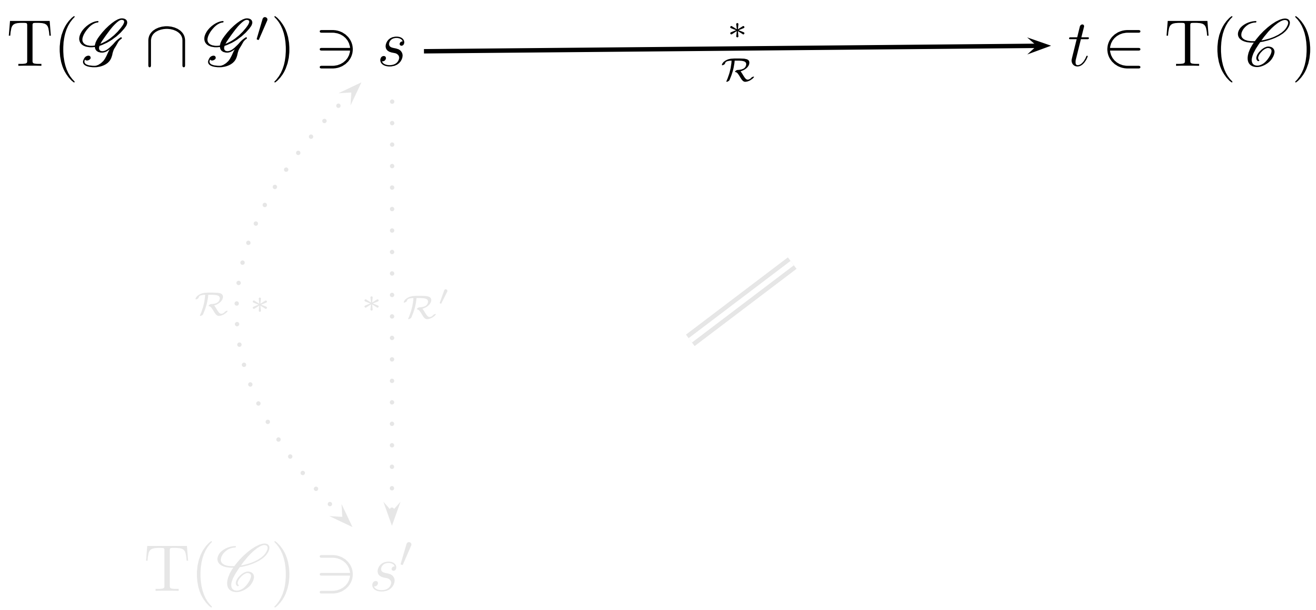


- $\mathrm{SC}(\mathcal{R}', \mathscr{G}')$,
- $\overset{*}{\leftrightarrow}_{\mathcal{R}'} \subseteq \overset{*}{\leftrightarrow}_{\mathcal{R}_A} = \overset{*}{\leftrightarrow}_{\mathcal{R}_I}$
- $\mathrm{CR}(\mathcal{R}_I)$,

  $\mathrm{T}(\mathscr{C}) \subseteq \mathrm{NF}(\mathcal{R}_I)$

$\mathcal{R} \simeq_{\mathscr{G}\cap\mathscr{G}'} \mathcal{R}'$ ($\subseteq$)

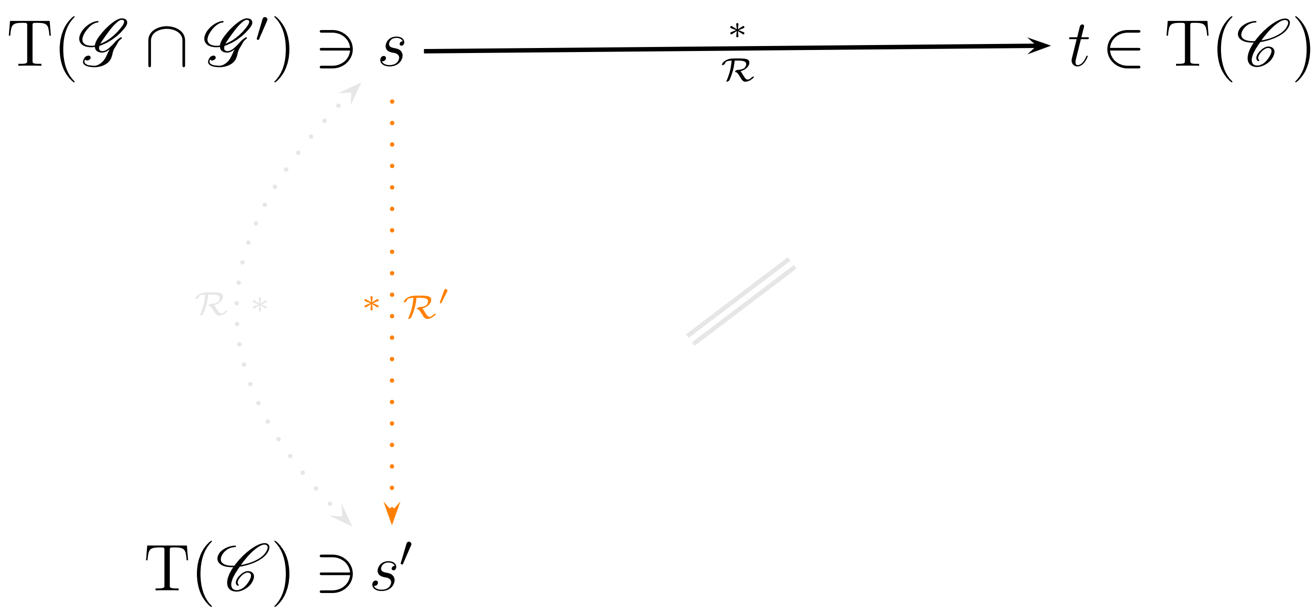
From 2 to 4, $\overset{*}{\leftrightarrow}_{\mathcal{R}} = \overset{*}{\leftrightarrow}_{\mathcal{R}'}$ on $\mathrm{T}(\mathscr{G}' \cap \mathscr{G}')$



- $\mathrm{SC}(\mathcal{R}', \mathscr{G})$,
- $\overset{*}{\leftarrow}_{\mathcal{R}'} \subseteq \overset{*}{\leftrightarrow}_{\mathcal{R}'} = \overset{*}{\leftrightarrow}_{\mathcal{R}}$
- $\mathrm{CR}(\mathcal{R})$, $\mathrm{T}(\mathscr{C}) \subseteq \mathrm{NF}(\mathcal{R})$

$\mathcal{R} \simeq_{\mathscr{G}\cap\mathscr{G}'} \mathcal{R}'$ ($\subseteq$)

From 2 to 4, $\overset{*}{\leftrightarrow}_{\mathcal{R}} = \overset{*}{\leftrightarrow}_{\mathcal{R}'}$ on $\mathrm{T}(\mathscr{G}' \cap \mathscr{G}')$



- $\mathrm{SC}(\mathcal{R}', \mathscr{G})$,
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$\mathcal{R} \simeq_{\mathscr{G} \cap \mathscr{G}'} \mathcal{R}'$ ($\subseteq$)

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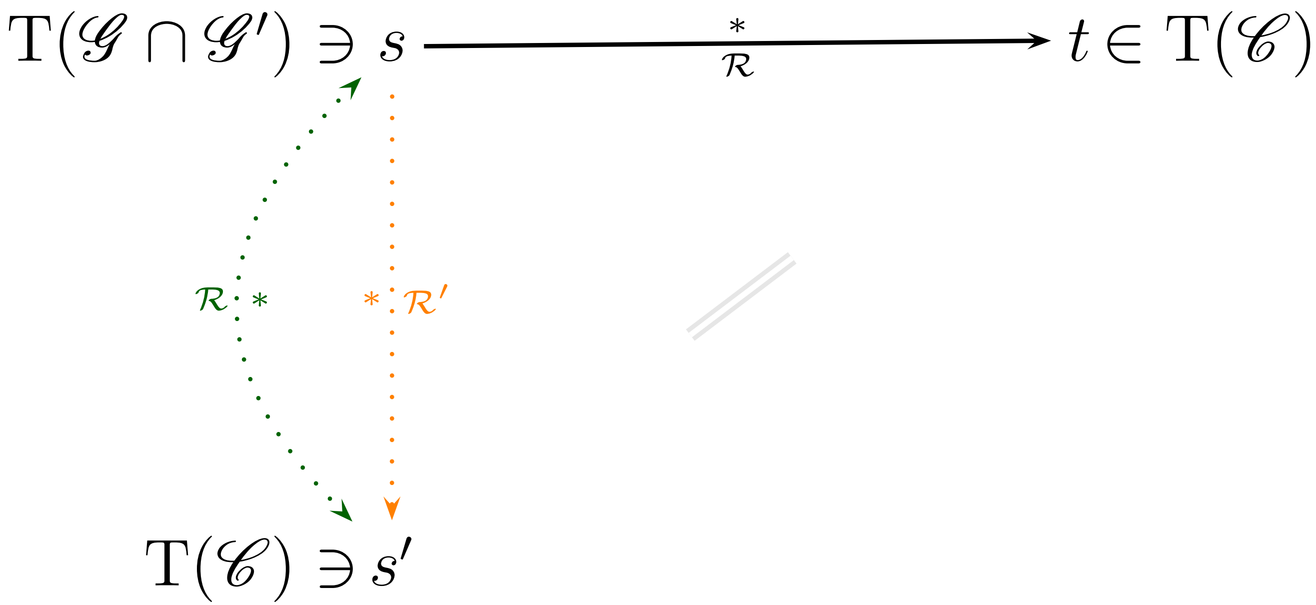

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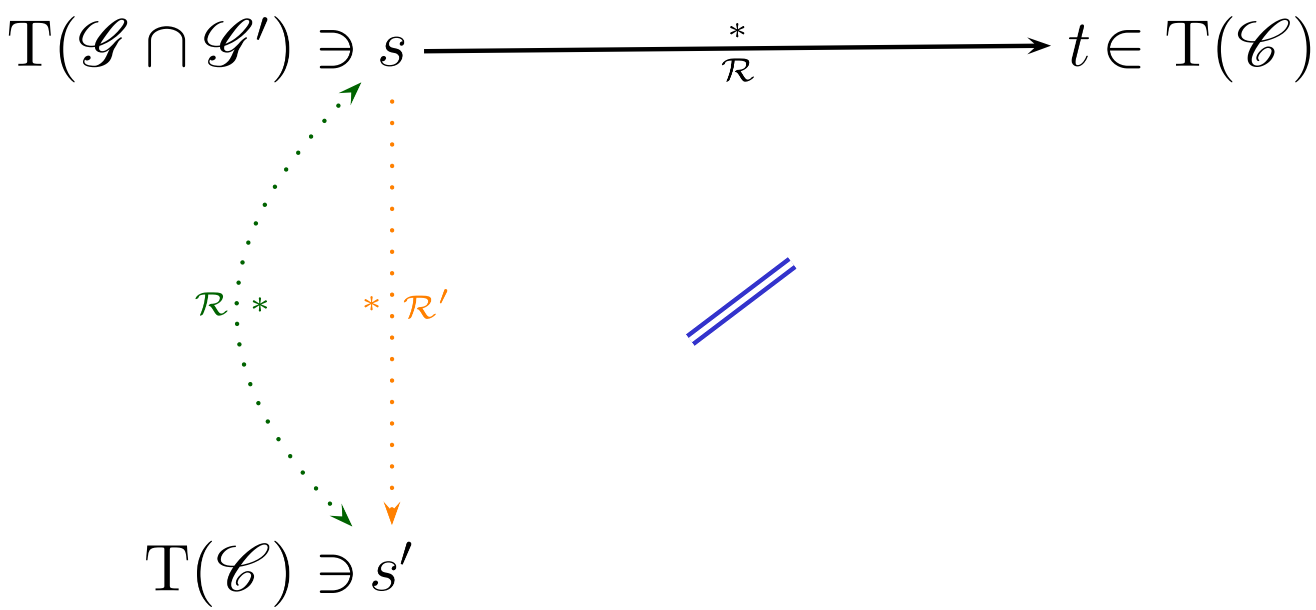

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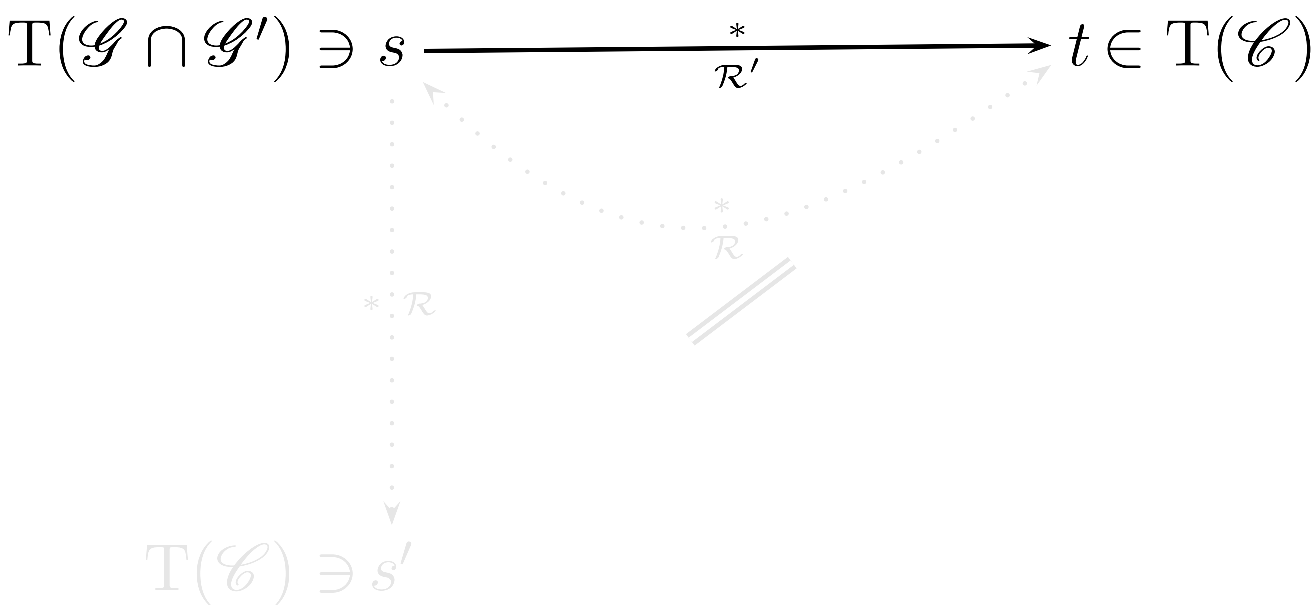
- $\mathrm{CR}(\mathcal{R})$,
  $\mathrm{T}(\mathscr{C}) \subseteq \mathrm{NF}(\mathcal{R})$

$\mathcal{R} \simeq_{\mathscr{G} \cap \mathscr{G}'} \mathcal{R}'$ $(\supseteq)$

From 2 to 4, $\overset{*}{\leftrightarrow}_{\mathcal{R}} = \overset{*}{\leftrightarrow}_{\mathcal{R}'}$ on $\mathrm{T}(\mathscr{G}' \cap \mathscr{G}')$



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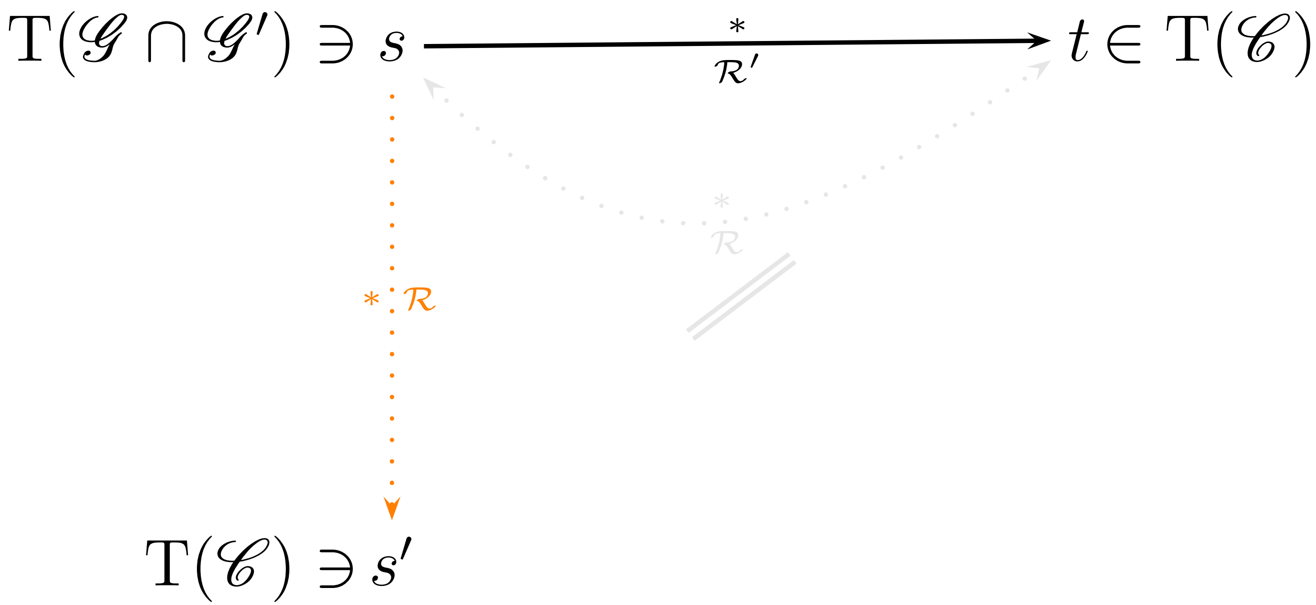
$\mathcal{R} \simeq_{\mathscr{G} \cap \mathscr{G}'} \mathcal{R}'$ ($\supseteq$)

From 2 to 4, $\overset{*}{\leftrightarrow}_{\mathcal{R}} = \overset{*}{\leftrightarrow}_{\mathcal{R}'}$ on $\mathrm{T}(\mathscr{G}' \cap \mathscr{G}')$



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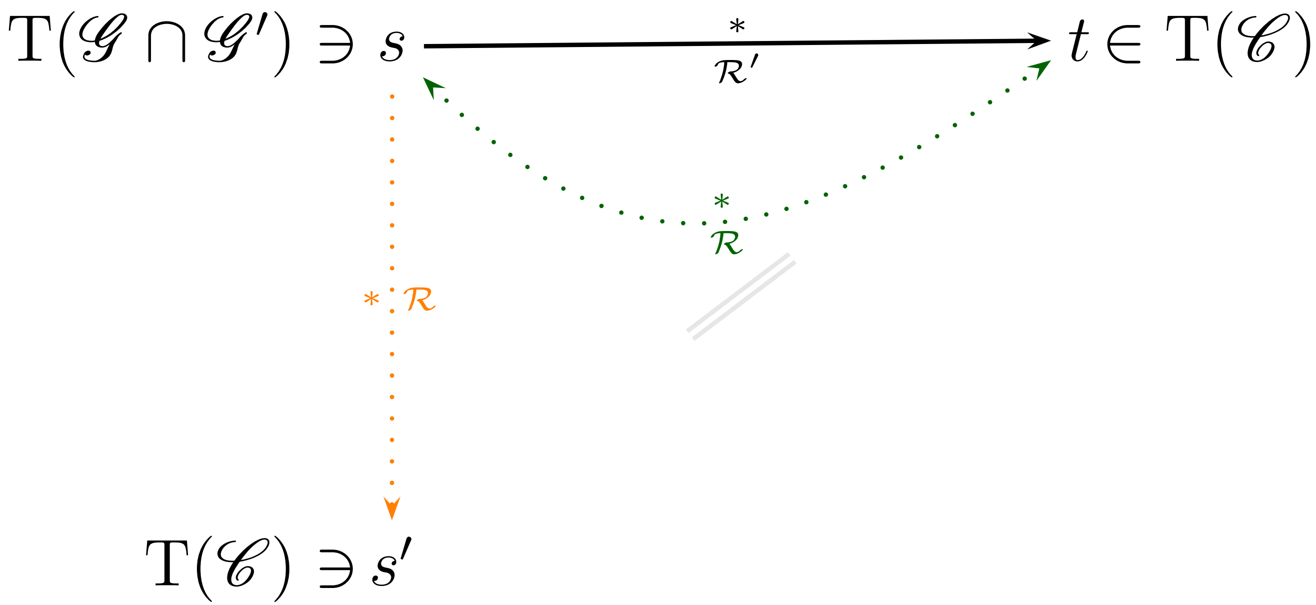
$\mathcal{R} \simeq_{\mathscr{G}\cap\mathscr{G}'} \mathcal{R}'$ ($\supseteq$)

From 2 to 4, $\overset{*}{\leftrightarrow}_{\mathcal{R}} = \overset{*}{\leftrightarrow}_{\mathcal{R}'}$ on $\mathrm{T}(\mathscr{G}' \cap \mathscr{G}')$



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$\mathcal{R} \simeq_{\mathscr{G}\cap\mathscr{G}'} \mathcal{R}'$ ($\supseteq$)

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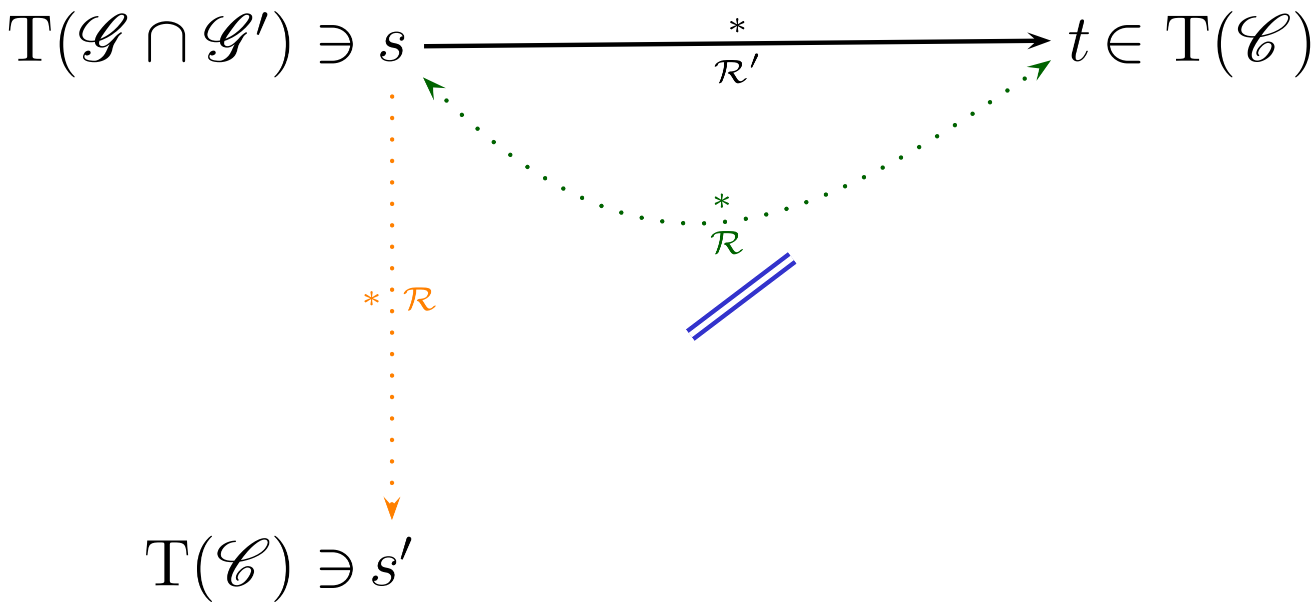

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- $\overset{*}{\leftarrow}_{\mathcal{R}'} \subseteq \overset{*}{\leftrightarrow}_{\mathcal{R}'} = \overset{*}{\leftrightarrow}_{\mathcal{R}}$
- $\mathrm{CR}(\mathcal{R})$,
  $\mathrm{T}(\mathscr{C}) \subseteq \mathrm{NF}(\mathcal{R})$

Proof

$$\mathcal{R} \overset{*}{\underset{I}{\Rightarrow}} \mathcal{R}_I \overset{*}{\underset{A}{\Rightarrow}} \mathcal{R}_A \overset{*}{\underset{E}{\Rightarrow}} \mathcal{R}'$$

1. $\mathrm{CR}(\mathcal{R}_I)$ and $\forall s \in \mathrm{T}(\mathscr{F}).\exists s' \in \mathrm{T}(\mathscr{G}).s \xrightarrow{*}_{\mathcal{R}_I} s'$
2. $\overset{*}{\leftrightarrow}_{\mathcal{R}} = \overset{*}{\leftrightarrow}_{\mathcal{R}_I}$ on $\mathrm{T}(\mathscr{G})$
3. $\overset{*}{\leftrightarrow}_{\mathcal{R}_I} = \overset{*}{\leftrightarrow}_{\mathcal{R}_A}$ on $\mathrm{T}(\mathscr{F})$
4. $\overset{*}{\leftrightarrow}_{\mathcal{R}_I} = \overset{*}{\leftrightarrow}_{\mathcal{R}'}$ on $\mathrm{T}(\mathscr{G}')$
5. $\mathcal{R} \simeq_{\mathscr{G} \cap \mathscr{G}'} \mathcal{R}'$

Proof

$$\mathcal{R} \underset{I}{\overset{*}{\Rightarrow}} \mathcal{R}_I \underset{A}{\overset{*}{\Rightarrow}} \mathcal{R}_A \underset{E}{\overset{*}{\Rightarrow}} \mathcal{R}'$$

1. $\mathrm{CR}(\mathcal{R}_I)$ and $\forall s \in \mathrm{T}(\mathscr{F}).\exists s' \in \mathrm{T}(\mathscr{G}).s \overset{*}{\to}_{\mathcal{R}_I} s'$
2. $\overset{*}{\leftrightarrow}_{\mathcal{R}} = \overset{*}{\leftrightarrow}_{\mathcal{R}_I}$ on $\mathrm{T}(\mathscr{G})$
3. $\overset{*}{\leftrightarrow}_{\mathcal{R}_I} = \overset{*}{\leftrightarrow}_{\mathcal{R}_A}$ on $\mathrm{T}(\mathscr{F})$
4. $\overset{*}{\leftrightarrow}_{\mathcal{R}_I} = \overset{*}{\leftrightarrow}_{\mathcal{R}'}$ on $n\mathrm{T}(\mathscr{G}')$
5. $\mathcal{R} \simeq_{\mathscr{G} \cap \mathscr{G}'} \mathcal{R}'$

Problems

- Higher-order functions

  `map`, `foldr`, `filter`, $\cdots$.

Problems

- Higher-order functions

  `map`, `foldr`, `filter`, $\cdots$.

  - Simply typed term rewriting system

Simply Typed Term Rewriting System (Yamada 2001)

$$
\left\{
\begin{array}{ll}
((\mathsf{foldr}\ f)\ e)\ [\,] & \to e \\
((\mathsf{foldr}\ f)\ e)\ (:\ x\ xs) & \to f\ x\ (((\mathsf{foldr}\ f)\ e)\ xs) \\
\mathsf{map}\ f & \to (\mathsf{foldr}\ (\mathsf{sub}\ f))\ [\,] \\
(\mathsf{sub}\ f)\ x\ xs & \to :\ (f\ x)\ xs \\
\mathsf{sum}\ xs & \to ((\mathsf{foldr}\ +)\ \mathsf{0})\ xs \\
+\ \mathsf{0}\ y & \to y \\
+\ (\mathsf{s}\ x)\ y & \to \mathsf{s}\ (+\ x\ y)
\end{array}
\right\}
$$

$\mathscr{F}^{\mathsf{Nat}} = \{\mathsf{0}\}$, $\mathscr{F}^{\mathsf{Nat}\to\mathsf{Nat}} = \{\mathsf{s}\}$, $\mathscr{F}^{\mathsf{Nat}\times\mathsf{Nat}\to\mathsf{Nat}} = \{+\}$, $\mathscr{F}^{\mathsf{List}} = \{[\,]\}$,
$\mathscr{F}^{\mathsf{Nat}\times\mathsf{List}\to\mathsf{List}} = \{:\}$, $\mathscr{F}^{(\mathsf{Nat}\times\mathsf{Nat}\to\mathsf{Nat})\to\mathsf{Nat}\to\mathsf{List}\to\mathsf{List}} = \{\mathsf{foldr}\}$,
$\mathscr{F}^{(\mathsf{Nat}\to\mathsf{Nat})\to\mathsf{List}\to\mathsf{List}} = \{\mathsf{map}\}$, $\mathscr{F}^{((\mathsf{Nat}\to\mathsf{Nat})\to\mathsf{Nat}\times\mathsf{List})\to\mathsf{List}} = \{\mathsf{sub}\}$,
$\mathscr{F}^{\mathsf{List}\to\mathsf{Nat}} = \{\mathsf{sum}\}$.

Equivalence of STTRSs

$\mathcal{R} \simeq_{\mathscr{G}} \mathcal{R}' \stackrel{\text{def}}{\Leftrightarrow} \forall s \in \mathrm{T}^{\mathrm{b}}(\mathscr{G}). \forall t \in \mathrm{T}^{\mathrm{b}}(\mathscr{C}).\ s \xrightarrow{*}_{\mathcal{R}} t \text{ iff } s \xrightarrow{*}_{\mathcal{R}'} t$

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$$\left\{ \begin{array}{ll} (\mathsf{map}\ f)\ [\,] & \to [\,] \\ (\mathsf{map}\ f)\ (:\ x\ xs) & \to :\ (f\ x)\ ((\mathsf{map}\ f)\ xs) \end{array} \right\}$$

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$$\simeq_{\{\mathsf{map},[\ ],:,\mathsf{s},\mathsf{0}\}}$$

$$\left\{ \begin{array}{ll} (\mathsf{map}\ f)\ [\,] & \to [\,] \\ (\mathsf{map}\ f)\ (:\ x\ xs) & \to :\ (f\ x)\ ((\mathsf{map}\ f)\ xs) \end{array} \right\}$$

Equivalent Transformation

$\mathcal{R}_0$: left-linear STTRS over $\mathscr{F}_0$, $\mathcal{E}$: set of equations over $\mathscr{F}_0$

- Introduction

$$\mathcal{R}_k \underset{I}{\Rightarrow} \mathcal{R}_k \cup \{l \to r\} \ (= \mathcal{R}_{k+1})$$
$$\mathscr{F}_{k+1} = \mathscr{F}_k \cup \{\mathrm{head}(l)\}$$

  - l is linear and basic
  - $\mathrm{head}(l) \notin \mathscr{F}_k$,
  - $\mathrm{args}(l) \subseteq \mathscr{V}$
  - $r \in T(\mathscr{F}_k, \mathscr{V})$, and
  - $\mathrm{WN}(\mathcal{R}_{k+1})$

- Addition

$$\mathcal{R}_k \underset{A}{\Rightarrow} \mathcal{R}_k \cup \{l \to r\} \ (= \mathcal{R}_{k+1})$$

  - $l \overset{*}{\leftrightarrow}_{\mathcal{R}_k \cup \mathcal{E}} r$

- Elimination

$$\mathcal{R}_k \underset{E}{\Rightarrow} \mathcal{R}_k \setminus \{l \to r\} \ (= \mathcal{R}_{k+1})$$

Problem

- Introduction

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$$\mathcal{R}_{\mathsf{sum}} \underset{I}{\Rightarrow} \mathcal{R}_{\mathsf{sum}} \cup \{(\mathsf{sum1}\ xs)\ y \to +\ (\mathsf{sum}\ xs)\ y\}.$$

Problem

- Introduction

$$\mathcal{R}_k \underset{I}{\Rightarrow} \mathcal{R}_k \cup \{l \to r\} \ (= \mathcal{R}_{k+1})$$
$$\mathscr{F}_{k+1} = \mathscr{F}_k \cup \{\text{head}(l)\}$$

- l is linear and basic
- $\text{head}(l) \notin \mathscr{F}_k$,
- $\text{args}(l) \subseteq \mathscr{V}$
- $r \in T(\mathscr{F}_k, \mathscr{V})$, and
- $\text{WN}(\mathcal{R}_{k+1})$

$$\mathcal{R}_{\textsf{sum}} \underset{I}{\Rightarrow} \mathcal{R}_{\textsf{sum}} \cup \{(\textsf{sum1}\ xs)\ y \to +\ (\textsf{sum}\ xs)\ y\}.$$

$$\textsf{map}\ (\textsf{sum1}\ [1,2,3])\ [4,5,6,7,8,9]$$

Higher-order sufficient completeness

(Aoto, Yamada and Toyama, 2004)

Definition

$$\mathrm{HSC}(\mathcal{R}, \mathscr{G}) \stackrel{\text{def}}{=} \forall s \in \mathrm{T}^{\mathrm{b}}(\mathscr{G}, \mathscr{V}^{\mathrm{h}}).\exists t \in \mathrm{T}^{\mathrm{b}}(\mathscr{G}, \mathscr{V}^{\mathrm{h}}) \text{ s.t.}$$

1. $t \in \mathrm{T}^{\mathrm{b}}(\mathscr{C})$, or
2. $t = C[F{\uparrow}\theta]$ for some C, θ, and $F \in \mathscr{V}^{\mathrm{h}}$.

$$\begin{aligned}
((\mathsf{foldr}\ f)\ e)\ [\,] &\to e \\
((\mathsf{foldr}\ f)\ e)\ (:\ x\ xs) &\to f\ x\ (((\mathsf{foldr}\ f)\ e)\ xs) \\
+\ \mathsf{0}\ y &\to y \\
+\ (\mathsf{s}\ x)\ y &\to \mathsf{s}\ (+\ x\ y)
\end{aligned}$$

$$((\mathsf{foldr}\ f)\ 0)\ [1,2,3] \to_{\mathcal{R}} f\ 1\ (((\mathsf{foldr}\ f)\ 0)\ [2,3])$$

Proof

Suppose $\mathcal{R}_0 \overset{*}{\underset{I}{\Rightarrow}} \mathcal{R}_k$, $f \in \mathscr{F}_k \setminus \mathscr{F}_0$ and $\mathrm{HSC}(\mathcal{R}_0)$

Lemma

If a basic ground term t contains f, then $t \notin \mathrm{NF}(\mathcal{R}_k)$

Lemma

For any basic ground term t s.t. $f \in \mathscr{F}(t)$, there exists a term $s \in \mathrm{T}^{\mathrm{b}}(\mathscr{F}_k)$.

- Introduction

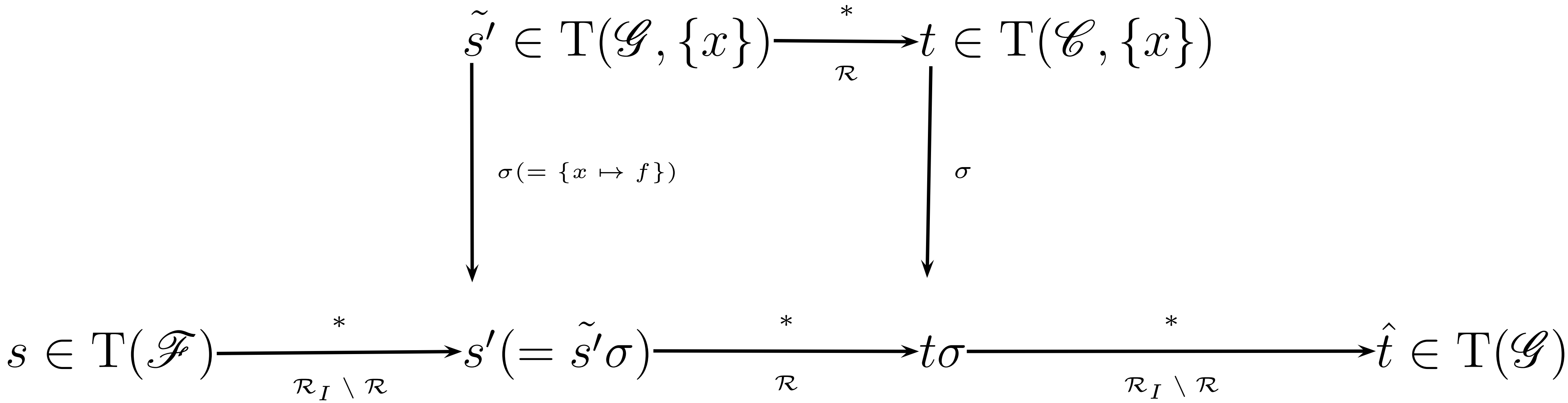
$$\mathcal{R}_k \underset{I}{\Rightarrow} \mathcal{R}_k \cup \{l \to r\} \ (= \mathcal{R}_{k+1})$$

$$\mathscr{F}_{k+1} = \mathscr{F}_k \cup \{\mathrm{head}(l)\}$$

- l is linear and basic
- $\mathrm{head}(l) \notin \mathscr{F}_k$,
- $\mathrm{args}(l) \subseteq \mathscr{V}$
- $r \in T(\mathscr{F}_k, \mathscr{V})$, and
- $\mathrm{WN}(\mathcal{R}_{k+1})$

Proof (sketch)

$\mathrm{HSC}(\mathcal{R})$ and $\mathrm{WN}(\mathcal{R})$



Verifying Equivalence of STTRSs

Theorem

- $\mathcal{R}$ is a left-linear STTRS over $\mathscr{G}$
- $\mathcal{R}'$ is a STTRS over $\mathscr{G}'$
- $\mathcal{E}$ is a set of equations over $\mathscr{G}$
- $\mathcal{R} \overset{*}{\underset{I}{\Rightarrow}} \cdot \overset{*}{\underset{A}{\Rightarrow}} \cdot \overset{*}{\underset{E}{\Rightarrow}} \mathcal{R}'$ under $\mathcal{E}$ $\quad\Longrightarrow\quad$ $\mathcal{R} \simeq_{\mathscr{G} \cap \mathscr{G}'} \mathcal{R}'$
- $\mathcal{R}, \mathscr{G} \vdash_{hind} \mathcal{E}$
- $CR(\mathcal{R})$, $SN(\mathcal{R})$, $HSC(\mathcal{R}, \mathscr{G})$
- $HSC(\mathcal{R}', \mathscr{G}')$

Conclusion

- Program transformation by templates
- Equivalence of rewriting systems
  - TRS
  - STTRS (higher order)
- A key part for extension

Future works

- Other rewriting systems
  - Order-sorted
  - Conditional
- Transformation templates