Tense operators on Łukasiewicz-Moisil algebras

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Tense operators on LM_n-algebras

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Tense operators on MV-Algebras and Łukasiewicz-Moisil Algebras

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Preliminaries - Łukasiewicz-Moisil algebras

Definition 1 Let $n \geq 2$. An **n-valued Łukasiewicz-Moisil algebra** (for short, **LM_n-algebra**) is a distributive lattice with a first and a last element, $\mathcal{L} = (L, \vee, \wedge, 0, 1)$, such that:

LM-1) There is a map $\neg : L \rightarrow L$ with the properties:

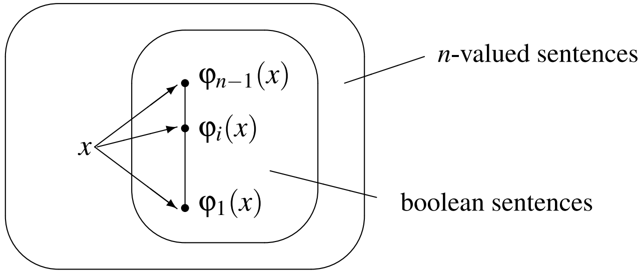
- $\neg(x \vee y) = \neg x \wedge \neg y$,
- $\neg(x \wedge y) = \neg x \vee \neg y$,
- $\neg(\neg x) = x$.

LM-2) There are $(n-1)$ maps $\varphi_i : L \rightarrow L$, $i = 1, \ldots, n-1$ (known as **chrysippian endomorphisms**) which have the properties:

1. $\varphi_i(0) = 0$, $\varphi_i(1) = 1$, for any i=1,...,n-1,
2. $\varphi_i(x \wedge y) = \varphi_i(x) \wedge \varphi_i(y)$ and $\varphi_i(x \vee y) = \varphi_i(x) \vee \varphi_i(y)$, for any $x, y \in L$ and i=1,...,n-1,
3. $\varphi_i(x) \vee \neg\varphi_i(x) = 1$ and $\varphi_i(x) \wedge \neg\varphi_i(x) = 0$, for any $x \in L$ and i=1,...,n-1,
4. $\varphi_h \circ \varphi_k = \varphi_k$, for any h,k=1,...,n-1,
5. $\varphi_1(x) \leq \varphi_2(x) \leq \ldots \leq \varphi_{n-1}(x)$, for any $x \in L$,
6. $\varphi_i(\neg x) = \neg\varphi_j(x)$, where $i + j = n$,
7. If $\varphi_i(x) = \varphi_i(y)$, for any i=1,...,n-1, then $x = y$, for any $x, y \in L$.

Axiom 7 is known as **Moisil's determination principle**.

Preliminaries - Łukasiewicz-Moisil algebras



Moisil's determination principle.

Example 1 Let $L_n = \{0, \frac{1}{n-1}, \frac{2}{n-1}, \ldots, \frac{n-2}{n-1}, 1\}$. We consider the following operations on L_n:

$$x \vee y = \max(x,y)\ ,\ x \wedge y = \min(x,y)\ ,\ \neg x = 1 - x$$

for any $x, y \in L_n$. We define the maps $\varphi_1, \ldots, \varphi_{n-1} : L_n \longrightarrow L_n$ by:

$$\varphi_i\left(\frac{j}{n-1}\right) = \begin{cases} 0, & if\ i+j < n \\ 1, & if\ i+j \geq n \end{cases},$$

L_n with the operations $\vee, \wedge, \neg, \varphi_1, \ldots, \varphi_{n-1}$ becomes an n-valued Łukasiewicz-Moisil algebra.

Theorem 1 (Moisil's representation theorem) For any LM_n-algebra $\mathcal{L}$, there exists a nonempty set I and an injective morphism of LM_n-algebras $d : L \longrightarrow L_n^I$.

Preliminaries - Tense Boolean algebras

Let $\mathcal{B} = (B, \vee, \wedge, \neg, 0, 1)$ be a Boolean algebra and two maps $G, H : B \to B$.
We define $F, P : B \to B$ by $F(x) = \neg G(\neg x)$ and $P(x) = \neg H(\neg x)$, for any $x \in B$.

Definition 2 $(\mathcal{B}, G, H)$ is a **tense Boolean algebra** if the following hold:

a) $G(1) = 1$, $H(1) = 1$,

b) $G(x \wedge y) = G(x) \wedge G(y)$ and $H(x \wedge y) = H(x) \wedge H(y)$, for any $x, y \in B$,

c) $x \leq GP(x)$ and $x \leq HF(x)$, for any $x \in B$.

Definition 3 A **frame** is a pair (X, R), where X is a nonempty set and R is a binary relation on X.

Given a frame (X, R), we define $G^*, H^* : L_2^X \to L_2^X$ by:

$$G^*(p)(x) = \bigwedge\{p(y) \mid y \in X,\, xRy\}, \quad H^*(p)(x) = \bigwedge\{p(y) \mid y \in X,\, yRx\},$$

for all $p \in L_2^X$ and $x \in X$.

Proposition 1 For any frame (X, R), (L_2^X, G^*, H^*) is a tense Boolean algebra.

Preliminaries - Tense Boolean algebras

The derivate tense operators F^*, P^* are given by:

$$F^*(p)(x) = \bigvee\{p(y) \mid y \in X,\ xRy\}, \quad P^*(p)(x) = \bigvee\{p(y) \mid y \in X,\ yRx\}$$

for all $p \in L_2^X$ and $x \in X$.

Definition 4 For any two tense Boolean algebras $(\mathcal{B}, G, H)$ and $(\mathcal{B}', G', H')$, a **morphism of tense Boolean algebras** $f : (\mathcal{B}, G, H) \to (\mathcal{B}', G', H')$ is a morphism of Boolean algebras which satisfies $f(G(b)) = G'(f(b))$ and $f(H(b)) = H'(f(b))$, for any $b \in B$.

Theorem 2 (**The representation theorem for tense Boolean algebras**) For any tense Boolean algebra $(\mathcal{B}, G, H)$, there exists a frame (X, R) and an injective morphism of tense Boolean algebras $d : (\mathcal{B}, G, H) \to (L_2^X, G^*, H^*)$.

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Tense operators on LM_n-algebras

Definition 5 Let $\mathcal{L} = (L, \vee, \wedge, \neg, \varphi_1, \ldots, \varphi_{n-1}, 0, 1)$ be an LM_n-algebra and $G, H : L \to L$ two unary operations on L. We say that $(\mathcal{L}, G, H)$ is a **tense LM_n-algebra** if the following axioms are satisfied:

(a) $G(1) = 1$ and $H(1) = 1$,

(b) $G(x \wedge y) = G(x) \wedge G(y)$ and $H(x \wedge y) = H(x) \wedge H(y)$, for any $x \in L$,

(c) $G \circ \varphi_i = \varphi_i \circ G$ and $H \circ \varphi_i = \varphi_i \circ H$, for any $i = 1, \ldots, n-1$,

(d) $x \leq GP(x)$ and $x \leq HF(x)$, for any $x \in L$,

where $F, P : L \to L$ are defined by $Fx = \neg G(\neg x)$, $Px = \neg H(\neg x)$.

Let (X, R) be a frame. We define $G^*, H^* : L_n^X \to L_n^X$ as follows:

$$G^*(p)(x) = \bigwedge\{p(y) \mid y \in X,\ xRy\}, \qquad H^*(p)(x) = \bigwedge\{p(y) \mid y \in X,\ yRx\},$$

for all $p \in L_n^X$ and $x \in X$.

Proposition 2 For any frame (X, R), (L_n^X, G^*, H^*) is a tense LM_n-algebra.

Tense operators on LM_n-algebras

Definition 6 If $(\mathcal{L}, G, H)$ and $(\mathcal{L}', G', H')$ are two tense LM_n-algebras, then a **morphism of tense LM_n-algebras** $f : (\mathcal{L}, G, H) \to (\mathcal{L}', G', H')$ is a morphism of LM_n-algebras such that $f(G(a)) = G'(f(a))$ and $f(H(a)) = H'(f(a))$, for any $a \in L$.

Proposition 3 Let $(\mathcal{L}, G, H)$ be a tense LM_n-algebra. Then there exists a frame (X, R) and an injective morphism of tense LM_n-algebras

$$\Phi : (\mathcal{L}, G, H) \to (L_n^X, G^*, H^*).$$

1. Preliminaries

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Tense Moisil logic

The alphabet of $\mathcal{TM}_n$ has the following primitive symbols:

(1) propositional variables $p, q, r, \ldots$ (the set V of propositional variables will be assumed infinite);

(2) logical connectives $\vee, \wedge, \Rightarrow, \neg, \varphi_1, \ldots, \varphi_{n-1}$;

(3) tense operators G, H;

(4) parentheses $(,)$.

The set E of sentences of $\mathcal{TM}_n$ is defined by the canonical induction.

We shall use the following abbreviations: $\alpha \Leftrightarrow \beta$ for $(\alpha \Rightarrow \beta) \wedge (\beta \Rightarrow \alpha)$, $F\alpha$ for $\neg G \neg \alpha$ and $P\alpha$ for $\neg H \neg \alpha$.

$\mathcal{TM}_n$ has the following axioms:

(M0) the axioms of the Moisil logic $\mathcal{M}_n$;

(M1) $G(\alpha \Rightarrow \beta) \Rightarrow (G\alpha \Rightarrow G\beta)$; $H(\alpha \Rightarrow \beta) \Rightarrow (H\alpha \Rightarrow H\beta)$;

(M2) $G\varphi_i(\alpha) \Leftrightarrow \varphi_i G(\alpha)$ and $H\varphi_i(\alpha) \Leftrightarrow \varphi_i H(\alpha)$, for any $i = 1, \ldots, n-1$;

(M3) $\alpha \Rightarrow GP\alpha$; $\alpha \Rightarrow HF\alpha$.

Tense Moisil logic

The notion of formal proof in $\mathcal{TM}_n$ is defined in terms of the above axioms and the following inference rules:

$$\frac{\alpha, \alpha \Rightarrow \beta}{\beta}, \quad \frac{\alpha}{\varphi_1 \alpha}, \quad \frac{\alpha}{G\alpha}, \quad \frac{\alpha}{H\alpha}.$$

We shall denote by $\vdash_{\mathcal{TM}_n} \alpha$ if α is **provable** in $\mathcal{TM}_n$.

Let us consider on E the following equivalence relation: $\alpha \sim \beta$ iff $\vdash_{\mathcal{TM}_n} \alpha \Leftrightarrow \beta$.

For any sentence $\alpha \in E$ denote by $[\alpha]$ the equivalence class of α.

We can define on the quotient set $E/_\sim$ the following operations:

$$[\alpha] \vee [\beta] = [\alpha \vee \beta], \quad [\alpha] \wedge [\beta] = [\alpha \wedge \beta], \quad \neg[\alpha] = [\neg\alpha] \quad G([\alpha]) = [G\alpha], \quad H([\alpha]) = [H\alpha],$$

$$\varphi_i([\alpha]) = [\varphi_i \alpha], \text{ for } i = 1, \ldots, n-1.$$

Lemma 1 $< E/_\sim, \vee, \wedge, \neg, \varphi_1, \ldots, \varphi_{n-1}, G, H, 0, 1 >$, the Lindenbaum-Tarski algebra of $\mathcal{TM}_n$, is a tense LM_n-algebra.

Tense Moisil logic

Let (X,R) be a frame. An **evaluation** of $\mathcal{TM}_n$ in (X,R) is a function $I : E \times X \longrightarrow L_n$ such that, for all $\alpha, \beta \in E$ and $x \in X$, the following equalities hold:

- $I(\alpha \Rightarrow \beta, x) = I(\alpha, x) \Rightarrow I(\beta, x)$,
- $I(\alpha \vee \beta, x) = I(\alpha, x) \vee I(\beta, x)$,
- $I(\alpha \wedge \beta, x) = I(\alpha, x) \wedge I(\beta, x)$,
- $I(\neg\alpha, x) = \neg I(\alpha, x)$,
- $I(\varphi_i \alpha, x) = \varphi_i I(\alpha, x)$, for any $i = 1, \ldots, n-1$,
- $I(G\alpha, x) = \bigwedge_{\text{xRy}} I(\alpha, y)$, $I(P\alpha, x) = \bigvee_{\text{yRx}} I(\alpha, y)$.

A sentence $\alpha \in E$ is **universally valid** in $\mathcal{TM}_n$ ($\models_{\mathcal{TM}_n} \alpha$) if for every frame (X,R) and for any evaluation $I : E \times X \longrightarrow L_n$ we have $I(\alpha, x) = 1$, for all $x \in X$.

Theorem 3 (Completeness Theorem)
A sentence α is provable in $\mathcal{TM}_n$ if and only if it is universally valid ($\vdash_{\mathcal{TM}_n} \alpha$ iff $\models_{\mathcal{TM}_n} \alpha$).