ASYMPTOTIC STABILITY FOR A CLASS OF MARKOV SEMIGROUPS

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ABSTRACT. Let $U \subset K$ be an open and dense subset of a compact metric space such that $\partial U \neq \emptyset$. Let Φ be a Markov operator with the strong Feller property acting on the space of bounded Borel measurable functions on U. Suppose that for each $x \in \partial U$ there exists a barrier $h \in C(K)$ at x such that $\Phi(h) \geq h$. Suppose moreover that every real-valued $g \in C(K)$ with $\Phi(g) \geq g$ and which attains its global maximum at a point inside U is constant. Then for each $f \in C(K)$ there exists the uniform limit $F = \lim_{n\to\infty} \Phi^n(f)$. Moreover $F \in C(K)$, agrees with f on ∂U and $\Phi(F) = F$.

1. Introduction

Let E be a locally compact Hausdorff space and let $M_b(E)$ be the space of all complex-valued, bounded and Borel measurable functions on E. Let also $C_b(E)$ be the set of all continuous functions in $M_b(E)$. A linear map

$$\Phi : M_b(E) \to M_b(E)$$

is called a Markov operator if for each $x \in E$ there exists a probability Borel measure μ_x on E such that

$$\Phi(f)(x) = \int f d\mu_x \quad \forall f \in M_b(E).$$

A Markov operator Φ is said to have the strong Feller property if $\Phi(f) \in C_b(E)$ for every $f \in M_b(E)$.

In [1] J. Arazy and M. Engliš studied the iterates of Markov operators acting on $M_b(U)$ where U is a bounded domain in $\mathbb{C}^d$. One of their main results is the following (see Theorem 1.4 in [1]):

Theorem 1.1. *Let $U \subset \mathbb{C}^d$ be a bounded domain and let $H^\infty(U)$ be the algebra of all bounded analytic function on U. Let $\partial_p(U) \subset \partial U$ be the set of all peak points for $H^\infty(U) \cap C(\overline{U})$. Let*

$$\Phi : M_b(U) \to M_b(U)$$

be a Markov operator with the strong Feller property such that $\Phi(h) = h$ for every $h \in H^\infty(U)$,

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Then for every continuous function F on the closure $\bar{U}$ the sequence $\{\Phi^n(F)\}$ converges uniformly on every subset of U whose closure is contained in $U \cup \partial_p(U)$ to a function $G \in C_b(U)$. Moreover, $\Phi(G) = G$ and

$$\lim_{y \to x} G(y) = F(x)$$

for every $x \in \partial_p(U)$.

The authors provide a large class of such operators, including Berezin transforms on strictly pseudoconvex domains and convolution operators on bounded symmetric domains.

In this paper we prove a result concerning the iterates of a class of Markov operators with the strong Feller property associated with dense open subsets of compact metric spaces. Under suitable boundary conditions and assuming a maximum principle for the interior points, we obtain a uniform convergence similar to that of Theorem 1.1. Theorem 2.1 below can be used to give an alternate proof for the main result in [2]. The methods used in [1] strongly motivated and inspired this research.

2. The main result

The main result of this paper is the following:

Theorem 2.1. *Let K be a compact metric space and let $U \subset K$ be a dense open subset such that $\partial U \neq \emptyset$. Let $C(K)$ be the space of all continuous complex-valued functions on K and let $C(\partial U)$ be the corresponding space for ∂U.*

Let $\Phi : M_b(U) \to M_b(U)$ be a Markov operator with the strong Feller property. Suppose that Φ satisfies the following conditions:

- (A) *For each point $x \in \partial U$ there exists $h \in C(K)$ such that $h(x) = 0$, $h(y) < 0$ for all $y \in K \setminus \{x\}$ and $\Phi(h_U) \geq h_U$ on U, where h_U is the restriction of h to U;*
- (B) *If $g \in C(K)$ is a real valued function with $\Phi(g_U) \geq g_U$ and if there exists $z \in U$ such that $g(z) = \max\{g(x) : x \in K\}$ then g is constant on K.*

Then, for each $f \in C(\partial U)$ there exists a unique function $G \in C(K)$ such that $\Phi(G_U) = G_U$ and $G(x) = f(x)$ for all $x \in \partial U$. Moreover, if $F \in C(K)$ is an arbitrary continuous extension of f to K then the sequence $\{\Phi^n(F_U)\}_n$ converges uniformly on U to G_U.

Proof. Let us assume that ∂U contains at least two points.

(1) First of all, it can be proved, exactly as in Proposition 1.3 from [1], that the boundary condition (A) implies the following. For each $f \in C(K)$ and for each $x \in \partial U$

$$\lim_{y \to x} \sup_{n \geq 1} |(\Phi^n(f_U))(y) - f(x)| = 0.$$

In particular, this shows that for each $f \in C(K)$ the function $\Phi(f_U)$ extends continuously up to ∂U and this extension agrees with f on ∂U. By abuse of notation, we shall continue to denote this extension by $\Phi(f)$.

(2) We show that if $f \in C(K)$ is a real-valued function such that $\Phi(f) \geq f$ then the sequence $\{\Phi^n(f)\}$ converges uniformly on K. Indeed, this sequence is monotone increasing and if g is its pointwise limit then g is Borel measurable and

$\Phi(g_U) = g_U$. Since Φ has the strong Feller property, it follows that g is continuous on U. Moreover (1) implies that

$$\lim_{y \to x} g(y) = f(x)$$

for every $x \in \partial U$. Since g agrees with f on ∂U and $f \in C(K)$ we see that $g \in C(K)$ and Dini's theorem shows that the convergence is uniform on K.

(3) Let

$$\mathcal{T}(\Phi) = \{h \in C(K) : \Phi(h) = h\}$$

and let $C^*(\mathcal{T}(\Phi))$ be the norm closed subalgebra of $C(K)$ generated by $\mathcal{T}(\Phi)$. Since $\mathcal{T}(\Phi)$ is a selfadjoint subspace of $C(K)$ it follows that $C^*(\mathcal{T}(\Phi))$ is a commutative C^*-algebra.

Let $C(\Phi)$ be the set of all $f \in C(K)$ for which the sequence $\{\Phi^n(f)\}$ is uniformly convergent on K and denote $\pi(f)$ its limit. It is clear that $\pi(f) \in \mathcal{T}(\Phi)$ for every $f \in C(\Phi)$. Let also

$$C(\Phi)_0 = \{f \in C(\Phi) : \pi(f) = 0\}.$$

We will show that $C^*(\mathcal{T}(\Phi)) \subset C(\Phi)$. It suffices to prove that for any finite set of k functions from $\mathcal{T}(\Phi)$ their product belongs to $C(\Phi)$.

Let $k = 2$. Let $h \in \mathcal{T}(\Phi)$. Since $\Phi(|f|^2) \geq |\Phi(f)|^2$ for every $f \in C(K)$ we see that $\Phi(|h|^2) \geq |h|^2$. In this case (2) shows that $|h|^2 \in C(\Phi)$ therefore $g = \pi(|h|^2) - |h|^2 \in C(\Phi)_0$. Since $g \geq 0$ we see that $gf \in C(\Phi)_0$ for every $f \in C(K)$. Indeed, it is easy to see that if $F \in C(\Phi)_0$ is non-negative on K then $Ff \in C(\Phi)_0$ for every $f \in C(K)$.

If $h_1, h_2 \in \mathcal{T}(\Phi)$ then

$$h_1 h_2 = (1/4) \sum_{m=0}^{3} i^m |g_m|^2$$

where $i = \sqrt{-1}$ and $g_m = (h_1 + i^m \bar{h_2})$. Since $g_m \in \mathcal{T}(\Phi)$ we see that $h_1 h_2 \in C(\Phi)$.

Let $k \geq 3$ and assume that every product of at most $k-1$ elements from $\mathcal{T}(\Phi)$ belongs to $C(\Phi)$. Let $h_1, \ldots, h_k$ in $\mathcal{T}(\Phi)$ and let $g = h_1 \cdot \cdots \cdot h_k$. Then

$$g = (h_1 h_2 - \pi(h_1 h_2)) \cdot h_3 \cdot \cdots \cdot h_k + \pi(h_1 h_2) \cdot h_3 \cdot \cdots \cdot h_k.$$

By what we have already proved the first summand belongs to $C(\Phi)_0$ and the second belongs, by the induction hypothesis, to $C(\Phi)$. This shows that $C^*(\mathcal{T}(\Phi)) \subset C(\Phi)$.

(4) Consider the map

$$\rho : C^*(\mathcal{T}(\Phi)) \to C(\partial U)$$

that takes any $f \in C^*(\mathcal{T}(\Phi))$ into its restriction to ∂U. It turns out that ρ is onto. To see this, we first observe that the boundary condition (A) together with (2) implies that for each $x \in \partial U$ there exists $g \in \mathcal{T}(\Phi)$ such that $g(x) = 0$ and $g(y) < 0$ for every $y \in \partial U \backslash \{x\}$. This shows that the range of ρ separates the points of ∂U and the Stone-Weierstrass theorem shows that this image is norm-dense in $C(\partial U)$. On the other hand, it is well-known that the image of a *-homomorphism between two C^*-algebras is norm-closed. The conclusion is that ρ is onto. Since $C^*(\mathcal{T}(\Phi)) \subset C(\Phi)$, and since for each $f \in C(\Phi)$ the function $\pi(f) \in \mathcal{T}(\Phi)$ and agrees with f on ∂U we see that for each $f \in C(\partial U)$ there exists a unique function $\theta(f) \in \mathcal{T}(\Phi)$ which agrees with f on ∂U. Uniqueness follows easily from (B). The map $f \mapsto \theta(f)$ is obviously linear and unit preserving.

(5) Let $L = \{g \in C(K) : g = \pi(|h|^2) - |h|^2$ for some $h \in \mathcal{T}(\Phi)\}$. We will show that $L \neq \{0\}$. Assume, on the contrary, that $L = \{0\}$ which means that $\pi(|h|^2) = |h|^2$ for every $h \in \mathcal{T}(\Phi)$. In this case, if h_1 and h_2 are functions in $\mathcal{T}(\Phi)$ then $\pi(h_1h_2) - h_1h_2$ can be written as a linear combination of elements from L (see step (3)). It then follows that $\pi(h_1h_2) = h_1h_2$. It follows that $\mathcal{T}(\Phi)$ is closed under multiplication hence it equals $C^*(\mathcal{T}(\Phi))$. This shows that the map $\theta : C(\partial U) \to C(K)$ defined in (4) is multiplicative on $C(\partial U)$. It follows that there exists a continuous map $\gamma : K \to \partial U$ such that

$$\theta(f) = f \circ \gamma \quad \forall f \in C(\partial U).$$

Let $z \in U$ and let $x = \gamma(z)$. Let $h \in C(\partial U)$ be a nonconstant real-valued function such that

$$h(x) = sup\{h(y); y \in \partial U\}.$$

Then

$$h(x) = (h \circ \gamma)(z) = (\theta(h))(z).$$

However, since $\theta(h) \in \mathcal{T}(\Phi)$ and is nonconstant, it attains its global maximum only on ∂U. We get a contradiction. The conclusion is that $L \neq \{0\}$.

(6) Let $h \in \mathcal{T}(\Phi)$ such that the function $g = \pi(|h|^2) - |h|^2$ is not identically zero on U. As pointed out in (3), $g \geq 0$ on K and $g \in C(\Phi)_0$. Moreover, one can see that $\Phi(g) \leq g$. We will show that $g > 0$ on U. Suppose, on the contrary, that there exists $z \in U$ such that $g(z) = 0$. Since $\Phi(g) \leq g$ and since $g \geq 0$ It follows from (B) that $g = 0$ on K. This contradiction shows that $g > 0$ on U. Since $g = 0$ on ∂U this shows that the closed ideal $I(g)$ of $C(K)$ generated by g equals

$$Z(\partial U) = \{f \in C(K) : f = 0 \text{ on } \partial U\}.$$

Since, as we pointed out in (3), $I(g) \subset C(\Phi)_0$ this shows that $f - \theta(f_{\partial U}) \in C(\Phi)_0$ for every $f \in C(K)$, where $f_{\partial U}$ is the restriction of f to ∂U. This means that the sequence $\{\Phi^n(f)\}$ converges uniformly on K to $\theta(f_{\partial U})$ for every $f \in C(K)$. This completes the proof for the case when ∂U contains at least two points.

(7) Suppose now that ∂U reduces to a singleton $\{x_0\}$. The first two steps hold true in this case as well. Moreover, the boundary condition (A) shows that there exists $g \in C(K)$ such that $g(x_0) = 0$, $g(y) < 0$ on U and $\Phi(g) \geq g$. It then follows from (2) that $g \in C(\Phi)$ and, since $g = 0$ on ∂U, we see that $\pi(g) = 0$ hence $g \in C(\Phi)_0$. It then follows, exactly as in (6), that $I(g) = Z(\partial U)$. This shows that $f - f(x_0) \in C(\Phi)_0$ fot every $f \in C(K)$ therefore

$$\lim_{n\to\infty} \Phi^n(f) = f(x_0)$$

uniformly on K, for all $f \in C(K)$. This completes the proof of this theorem.

□

We close with several remarks concerning Theorem 2.1.

(a) This proof works as well for Markov semigroups with continuous parameter.
(b) Instead of the strong Feller property, it suffices to assume that $\Phi(C_b(U)) \subset C_b(U)$ and that every $h \in M_b(U)$ with $\Phi(h) = h$ is continuous on U.
(c) For the existence of fixed points for Φ with prescribed boundary values, it suffices to assume only the boundary condition (A).
(d) The uniform convergence of the iterates holds true if, instead of (B), we assume that the fixed points of Φ separate the points of K.

(e) Condition (B) (the maximum principle) holds true, for instance, in the case when every point $y \in U$ is accessible for Φ. This means that for each $x \in U$ and each open neighborhood ω of y there exists $k \geq 1$ such that $\Phi^k(\chi_\omega)(x) > 0$, where $\chi_\omega = 1$ on ω and 0 elsewhere.

References

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