A SHORT PROOF OF A CHOQUET-DENY THEOREM FOR ABELIAN TOPOLOGICAL SEMIGROUPS

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Abstract. We give a short proof of an extended Choquet-Deny theorem for topological semigroups due to G. Szekely and W.B. Zeng.

In [5] G. Szekely and W. B. Zeng proved the following version of the well-known Choquet-Deny theorem [2]:

Theorem 1. *Let $(S, +)$ be an abelian topological (Hausdorff) semigroup and let μ be a regular Borel probability measure on S.*

Suppose that $h : S \to \mathbb{C}$ is a bounded Borel measurable function such that

$$h(x) = \int_S h(x+y)\mu(dy) \quad \forall x \in S.$$

Then, for each $x \in S$

$$h(x) = h(x+y)$$

for μ almost all $y \in S$.

The proof in [5] makes use of the martingale convergence theorem and the Hewitt-Savage zero-one law. A more elementary, still probabilistic proof of this theorem was given in [3].

In this note we give a non-probabilistic proof of Theorem 1. Our approach is in part inspired by that used in [4] for the case of measurable semigroups. In the proof we need to use the following topological Fubini theorem (see Theorem 7.6.5 in [1]):

Theorem 2. *Let X, Y be topological Hausdorff spaces and let μ and ν be Borel regular probability measures on X and Y respectively. Let $\phi : X \times Y \to \mathbb{C}$ be a bounded Borel measurable function (relative to the product topology). Then the functions*

$$x \mapsto \int_Y \phi(x, y)\nu(dy)$$

and

$$y \mapsto \int_X \phi(x, y)\mu(dx)$$

are Borel measurable, and

$$\int_X \int_Y \phi(x, y)\nu(dy)\mu(dx) = \int_Y \int_X \phi(x, y)\mu(dx)\nu(dy).$$

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We are now ready to give our proof of Theorem 1.

Proof. Let $M_b(S)$ denote the Banach algebra of all complex-valued, bounded and Borel measurable functions on S. For any $f \in M_b(S)$ define

$$\Phi(f)(x) = \int_S f(x+y)\mu(dy) \quad \forall x \in S.$$

It then follows from Theorem 2 that $\Phi(f) \in M_b(S)$ for every $f \in M_b(S)$. One obtains in this way a linear positive contractive and unit preserving mapping

$$\Phi : M_b(S) \to M_b(S).$$

Let $h \in M_b(S)$ such that $\Phi(h) = h$ and let $g \in M_b(S)$ be defined by

$$g(x) = \int_S |h(x) - h(x+y)|^2 \mu(dy) \quad \forall x \in S.$$

One easily sees that $g = \Phi(|h|^2) - |h|^2$. We will show that $\Phi(g) \geq g$. Indeed, using Theorem 2, the Schwarz inequality and the commutativity of S, we get:

$$\begin{aligned}
\Phi(g)(x) &= \int_S g(x+z)\mu(dz) \\
&= \int_S \int_S |h(x+z) - h(x+z+y)|^2 \mu(dy)\mu(dz) \\
&= \int_S \int_S |h(x+z) - h(x+z+y)|^2 \mu(dz)\mu(dy) \\
&\geq \int_S \left| \int_S (h(x+z) - h(x+z+y))\mu(dz) \right|^2 \mu(dy) \\
&= \int_S \left| \int_S (h(x+z) - h(x+y+z))\mu(dz) \right|^2 \mu(dy) \\
&= g(x)
\end{aligned}$$

for all $x \in S$. It then follows that

$$\Phi^{n+1}(|h|^2) - \Phi^n(|h|^2) \geq g \quad \forall n \geq 1.$$

Since the sequence $\{\Phi^n(|h|^2)\}$ is uniformly bounded these inequalities imply that $g(x) = 0$ for all $x \in S$. This completes the proof.

□

References

[1] V. Bogachev, *Measure Theory*, Springer-Verlag Berlin Heidelberg 2007.
[2] G. Choquet, J. Deny, *Sur l'equation de convolution $\mu = \mu * \sigma$*. C. R. Acad. Sci. Paris 250 (1960) 799-801.
[3] C. R. Rao, D. N. Shanbhag, *An elementary proof for an extended version of the Choquet-Deny theorem.* J. Multivariate Anal. 38 (1991) no. 1, 141-148.
[4] A. Raugi, *A general Choquet-Deny theorem for nilpotent groups.* Ann. Inst. H. Poincare, Probab. Statist. 40 (2004), no. 6, 677-683.
[5] G. Szekely, W. B. Zeng, *The Choquet-Deny convolution equation $\mu = \mu * \sigma$ for probability measures on abelian semigroups.* J. Theoret. Probab. 3 (1990), no. 2, 361-365.

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