ON THE RELATION BETWEEN COMPLEX AND REAL SECOND ORDER FREE INDEPENDENCE

JAMES A. MINGO[*] AND MIHAI POPA[*][**]

Abstract. We show that real second order freeness also appears in the study of Haar unitary and unitarily invariant random matrices when transposes are also considered. Moreover, it appears that the cases of complex second order freeness from the existing literature are in fact particular cases of real second order freeness.

1. Introduction

The notion of (complex) second order freeness was introduced in [MS], as a way to describe the asymptotic behavior of fluctuations for some important classes of random matrices. In particular, second order freeness is connected to the asymptotic behavior of Haar unitary and unitary invariant random matrices. A similar theory for Haar orthogonal and orthogonally invariant random matrices lead to a different type of relation, the *real* second order freeness (see [R], [MPR]).

The paper discusses aspects of the relation between the two types of second order freeness. More precisely we show that real second order freeness also appears in the study of Haar unitary and unitary invariant random matrices when their transposes are also considered. Also, as shown in [MPR], real second order freeness is defined for a certain class of random matrix ensembles, the ones with second order limit t-distribution (see Definition 1 below). In the last section, we prove that all unitary invariant random matrices with second order limit distribution (particularly the models studied in [MS] and [MSS]) have in fact second order limit t-distribution, hence the cases of complex second order freeness from the literature are in fact particular cases of real second order freeness.

[*] Research supported by a Discovery Grant from the Natural Sciences and Engineering Research Council of Canada.

[**] Research supported by the Natural Science Foundation of China Grant No. 11150110456, and the Romanian National Authority for Scientific Research, CNCS UEFISCDI, Project Number PN-II-ID-PCE-2011-3-0119.

2. Preliminaries

2.1. **Second order distributions.**

Definition 1. Suppose that for each N we have the set of $N \times N$ random matrices $\{A_{N,1}, A_{N,2}, \dots A_{N,s}\}$ and denote by $\mathcal{A}$ the ensemble $\coprod_{N\geq 1}\{A_{N,1}, A_{N,2}, \dots A_{N,s}\}$.

(1) We say that the ensemble $\mathcal{A}$ has a *second order bounded distribution*, (respectively *second order bounded t-distribution*) if for all polynomials $p_1, p_2, p_3, \dots$ in the non-commuting variables $\{x_1, x_2, \dots, x_s\}$ (respectively $\{x_1, x_2, \dots, x_s, y_1, y_2, \dots, y_s\}$) and for all $k = 1, 2, \dots,$

$$W_{N,k} = p_k(A_{N,1}, A_{N,2}, \dots A_{N,s})$$

(respectively $W_{N,k} = p_k(A_{N,1}, A_{N,2}, \dots A_{N,s}, A_{N,1}^t, A_{N,2}^t, \dots A_{N,s}^t)$) we have:

(i) $\sup_{N\to\infty} |E(\mathrm{tr}(W_1))| < \infty$

(ii) $\sup_{N\to\infty} |k_r(\mathrm{Tr}(W_1), \mathrm{Tr}(W_2), \dots, \mathrm{Tr}(W_r))| < \infty$ for all $r \geq 2$.

(2) We say that the ensemble $\mathcal{A}$ has a *second order limit distribution* (respectively *t-distribution*), if, in the definition above, the conditions (i)-(ii) are replaced by

(i)′ $\lim_{N\to\infty} |E(\mathrm{tr}(W_1))|$ exists and it is finite.

(ii)′ $\lim_{N\to\infty} k_2(\mathrm{Tr}(W_1), \mathrm{Tr}(W_2))$ exists and it is finite.

(iii)′ $\lim_{N\to\infty} k_r(\mathrm{Tr}(W_1), \dots, \mathrm{Tr}(W_r)) = 0$ for all $r \geq 3$.

Example 2. It is easy to construct ensembles of random matrices with second order limit distribution, but no second order limit t-distribution (see also [CS], Section 5). Consider $\mathcal{A} = \{A_N\}_{N\geq 1}$, with

$$A_N = [i^k \cdot \delta_{l-k,1}]_{k,l=1}^N,$$

i. e. all entries of A_N are zero, except for the $(k, k+1)$-type entries (on the diagonal above the main one), which are equal to i^k (here i is the complex number, $i^2 = -1$). Then $\mathrm{tr}(A_N^p) = 0$ for all positive integers p, but $\lim_{N\to\infty} \mathrm{tr}(A_N \cdot A_N^t)$ does not exist.

Throughout the rest of the paper, since all matrices appearing in the same relation will always have the same dimension, we will omit the first index when writing matrices from an ensemble, i. e. we will write A_1, A_2, A_2 for $A_{N,1}, A_{N,2}, A_{N,3}$.

If $\mathfrak{A}$ is a unital algebra and $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are unital subalgebras of $\mathfrak{A}$, we will say that a tuple $(a_1, \dots, a_n)$ of elements from $\mathfrak{A}_1 \coprod \mathfrak{A}_2$ is *alternating*, respectively *cyclically alternating*, if, for $i \leq n-1$, respectively for

all i, we have that $a_i \in \mathfrak{A}_{\epsilon(i)}$ with $\epsilon(i) \in \{1,2\}$ such that $\epsilon(i) \neq \epsilon(i+1)$. We count indices in a cyclic way mod n, i. e. $\epsilon(n) \neq \epsilon(1)$.

Definition 3. Let $\mathcal{A} = \{A_1, \ldots, A_s\}_{N\geq 1}$ and $\mathcal{B} = \{B_1, \ldots, B_p\}_{N\geq 1}$ be two ensemble of random matrices.

(i)We say that $\mathcal{A}$ and $\mathcal{B}$ are *asymptotically free* if $\mathcal{A} \cup \mathcal{B}$ satisfies the condition (i)$'$ from Definition 1 and for any positive integer n and any alternating n-tuple $D_1, D_2, \ldots, D_n$ from $\mathcal{A}$ and $\mathcal{B}$ such that $\mathrm{tr}(D_l) = 0$ for all l, we have that

$$\lim_{N\to\infty} \mathrm{tr}(D_1 D_2 \cdots D_n) = 0.$$

(ii)We say that $\mathcal{A}$ and $\mathcal{B}$ are *asymptotically complex second order free* if $\mathcal{A}$ and $\mathcal{B}$ are asymptotically free, $\mathcal{A} \cup \mathcal{B}$ has second order limit distribution and if, for all m, n and all cyclically alternating tuples $(C_1, \ldots, C_m)$, $(D_1, \ldots, D_n)$ from $\mathcal{A}$ and $\mathcal{B}$ such that $\mathrm{Tr}(C_j) = \mathrm{Tr}(D_l) = 0$ $(1 \leq j \leq m,\ 1 \leq l \leq n\ )$ we have that

$$\lim_{N\to\infty} \{\mathrm{cov}(\mathrm{Tr}(C_1 \cdots C_m), \mathrm{Tr}(D_1 \cdots D_n)) - \delta_{m,n} \cdot \sum_{k=1}^{n} \prod_{i=1}^{n} \mathrm{tr}(C_i \cdot D_{k-i})\} = 0$$

(iii)We say that $\mathcal{A}$ and $\mathcal{B}$ are *asymptotically real second order free* if if $\mathcal{A}$ and $\mathcal{B}$ are asymptotically free, $\mathcal{A} \cup \mathcal{B}$ has second order limit t-distribution and if, for all m, n and all cyclically alternating tuples $(C_1, \ldots, C_m)$, $(D_1, \ldots, D_n)$ from $\mathcal{A}$ and $\mathcal{B}$ such that $\mathrm{Tr}(C_j) = \mathrm{Tr}(D_l) = 0$ $(\ 1 \leq j \leq m,\ 1 \leq l \leq n\ )$ we have that

$$\begin{aligned}(1)\quad \lim_{N\to\infty} \{&\mathrm{cov}(\mathrm{Tr}(C_1 \cdots C_m), \mathrm{Tr}(D_1 \cdots D_n)) \\ &- \delta_{m,n} \cdot \sum_{k=1}^{n} [\prod_{i=1}^{n} \mathrm{tr}(C_i \cdot D_{k-i}) + \prod_{i=1}^{n} \mathrm{tr}(C_i \cdot D_{i-k}^t)]\} = 0.\end{aligned}$$

□

It was shown in [MPR], Proposition 29, that the relations above are associative.

Proposition 4. *Let $A_1, A_2, \ldots, A_m$, $B_1, \ldots, B_n$ be $N \times N$ random matrices with a second order bounded distribution such that $E(\mathrm{Tr}(B_l)) = 0$ for all $1 \leq l \leq n$. Then:*

(i) $E(\mathrm{Tr}(A_1) \cdots \mathrm{Tr}(A_m)) = E(\mathrm{Tr}(A_1)) \cdots E(\mathrm{Tr}(A_m)) + O(N^{m-1})$; in particular, $E(\mathrm{Tr}(A_1) \cdots \mathrm{Tr}(A_m)) = O(N^m)$.

(ii) $E(\mathrm{Tr}(A_1) \cdots \mathrm{Tr}(A_m)\, \mathrm{Tr}(B_1) \cdots \mathrm{Tr}(B_n))$
$= E(\mathrm{Tr}(A_1) \cdots \mathrm{Tr}(A_m)) \cdot E(\mathrm{Tr}(B_1) \cdots \mathrm{Tr}(B_n)) + O(N^{m-1})$
$= O(N^m)$.

Proof. For (i), using the moment-cumulant expansion from [NS], we have that

$$E(\mathrm{Tr}(A_1)\cdots\mathrm{Tr}(A_m)) = \sum_{\pi\in P(m)} k_\pi[\mathrm{Tr}(A_1),\ldots,\mathrm{Tr}(A_m)].$$

Since all cumulants of order higher than 2 are bounded, it follows that $k_\pi[\mathrm{Tr}(A_1),\ldots,\mathrm{Tr}(A_m) = O(N^{\#\text{of singletons of }\pi})$. But elements of $P(m)$ have at most m singletons, with equality for 0_m, hence the conclusion.

For (ii), note first that the moment-cumulant expansion gives

$$E(\mathrm{Tr}(A_1)\cdots\mathrm{Tr}(A_m)\,\mathrm{Tr}(B_1)\cdots\mathrm{Tr}(B_n)) = \sum_{\pi\in P(m+n)} k_\pi[\mathrm{Tr}(A_1),\ldots,\mathrm{Tr}(B_n)].$$

All cumulants of order higher than 1 are bounded; moreover, for all indices l, we have that $k_1(\mathrm{Tr}(B_l)) = 0$, since B_l's are centered, therefore, denoting $S_\pi^m = \{j : 1 \le j \le m, (j) \text{ is a singleton of } \pi\}$, we have that

$$k_\pi[\mathrm{Tr}(A_1),\ldots,\mathrm{Tr}(B_n)] = O(N^{\#S_\pi^m}).$$

But S_π^m has at most m elements, with equality when $(1),\ldots,(m)$ are blocks of π, particularly not when k_π contains mixed cumulants in A_j's and B_l's, hence the conclusion. □

Remark 5. Suppose that $\sigma \in S_m$ has the cycle decomposition $\sigma = c_1\cdot c_2\cdots c_r$ with each $c_k = (i_{k,1}, i_{k,2},\ldots,i_{k,s(k)})$. For A_j's and B_l's as in Proposition 4, denote

$$\mathrm{Tr}_\sigma(A_1,\ldots,A_m) = \prod_{k=1}^{r}\mathrm{Tr}(A_{i_{k,1}}\cdots A_{i_{k,s(k)}}).$$

With this notation, Proposition 4 implies that

$$E(\mathrm{Tr}_\sigma(A_1A_2\cdots A_m)) = O(N^r),$$

respectively

$$E(\mathrm{Tr}_\sigma(A_1A_2\cdots A_m)\,\mathrm{Tr}(B_1)\cdots\mathrm{Tr}(B_n)) \\ = E(\mathrm{Tr}_\sigma(A_1A_2\cdots A_m))\cdot E(\mathrm{Tr}(B_1)\cdots\mathrm{Tr}(B_n)).$$

2.2. **Combinatorial results.** For n a positive integer, let $[n] = \{1,2,\ldots,n\}$, $[-n] = \{-n,-n+1,\ldots,-1\}$ and $[\pm n] = [-n]\cup[n]$. As in [MPR], we will regard a permutation σ on $[n]$ also as a permutation on $[\pm n]$ where for $1 \le k \le n$ we let $\sigma(-k) = -k$.

The sets of permutations on $[n]$, respectively $[\pm n]$ such that each one of their cycles has exactly 2 elements will be denoted by $P_2(n)$, respectively $P_2(\pm n)$ (if n is odd, then by convention $P_2(n) = \emptyset$). It was shown in [MPR] that if $p, q \in P_2(\pm n)$, then pq has a cycle decomposition

of the type $(c_1, c_1', \ldots, c_l, c_l')$ with the property that for all k, if $c_k = (i_1, i_2, \ldots, i_l)$, then $c_k' = (q(i_l), q(i_{l-1}), \ldots, q(i_1))$.

We will denote by δ the element of $P_2(\pm n)$ given by $\delta(k) = -k$. If σ is a permutation on some finite set, the number of cycles of σ will be denoted by $\#(\sigma)$.

For $i : [\pm n] \to [N]$, a multi-index, and $p \in P_2(\pm n)$ we write $i = i \circ p$ to mean that for every $(r, s) \in p$ we have $i(r) = i(s)$. For a $N \times N$ matrix A let $A^{(-1)} = A^t$, the transpose of A, and $A^{(1)} = A$. For $\epsilon = (\epsilon_1, \epsilon_2, \ldots, \epsilon_n) \in \mathbb{Z}_2^n$ and $\pi \in S_n$, let $\mathrm{Tr}_{(\pi,\epsilon)}(A_1, \ldots, A_n) = \mathrm{Tr}_\pi(A_1^{(\epsilon_1)}, \ldots, A_n^{(\epsilon_n)})$

We saw that the cycle decomposition of $p\delta$ may be written $c_1 c_1{}' \cdots c_s c_s{}'$ where $c_i{}' = \delta c_i^{-1} \delta$. Since $c'' = c$, it is arbitrary which of the pair $\{c_i, c_i{}'\}$ is called c_i and which $c_i{}'$.

For each i, choose a representative of each pair $\{c_i, c_i{}'\}$, say $c_1, c_2, \ldots, c_s$. We shall construct a permutation $\pi(p) \in S_n$ and $\epsilon \in \mathbb{Z}_2^n$ which will depend on our choice of $\{c_1, \ldots, c_s\}$. For each i we construct a cycle $\tilde{c}_i$ as follows. Suppose $c_i = (l_1, \ldots, l_r)$. Let $\tilde{c}_i = (j_1, j_2, \ldots, j_r)$ where $j_k = |l_k|$ and $\epsilon_{j_k} = l_k/|l_k|$. Define $\pi(p) = \tilde{c}_1 \cdots \tilde{c}_s$ and $\epsilon = (\epsilon_1, \ldots, \epsilon_n)$.

Remark 6. In the above framework, if $p([n]) = [-n]$, then all $\epsilon_j = 1$ for all $j = 1, \ldots, n$.

The following result was proved in [MPR], Lemma 5:

Lemma 7. *With the above notations, we have that*

$$\sum_{i=i\circ p} a^{(1)}_{i_1 i_{-1}} a^{(2)}_{i_2 i_{-2}} \cdot \cdots \cdot a^{(n)}_{i_n i_{-n}} = \mathrm{Tr}_{\pi(p)}(A_1^{(\epsilon_1)}, \ldots, A_n^{(\epsilon_n)}).$$

2.3. **Results on asymptotic second order freeness.** In this paper we will utilize some results presented in [MPR] and [CS]. More precisely, Remark 39, Corollary 48 and Theorem 50 from [MPR] imply part (i) of the Theorem 8 and Remark 39 and Proposition 52 from [MPR] imply part (ii) of the Theorem 8 below.

Theorem 8. *Let $\mathcal{A} = \{A_1, A_2, \ldots, A_s\}_{N\geq 1}$ and $\mathcal{B} = \{B_1, B_2, \ldots, B_p\}_{N\geq 1}$ be two ensembles of square random matrices such that the entries of $\mathcal{A}$ and the entries of $\mathcal{B}$ form two independent families of random variables. Let $\mathcal{O} = \{O\}_{N\geq 1}$ be a family of $N \times N$ Haar orthogonal random matrices with entries independent from the entries of $\mathcal{A}$ and $\mathcal{B}$. Then:*

(i) *If the ensemble $\mathcal{A}$ has real second order bounded t-distribution, then $\mathcal{A}$ and $\mathcal{O}$ satisfy equation 1 and the ensemble $\mathcal{O} \cup \mathcal{A}$ has second order bounded t-distribution; if $\mathcal{A}$ has second order limit t-distribution, then $\mathcal{A}$ and $\mathcal{O}$ real asymptotically real second order free.*

(ii) *If the ensembles $\mathcal{A}$ and $\mathcal{B}$ have second order bounded t-distribution, then the ensembles $\widetilde{A} = \{OAO^t : A \in \mathcal{A}, O \in \mathcal{O}\}$ and $\mathcal{B}$ satisfy equation* (1) *and $\widetilde{A} \cup \mathcal{B}$ have second order bounded t-distribution; if $\mathcal{A}$ and $\mathcal{B}$ have second order limit t-distribution, then $\widetilde{A}$ and $\mathcal{B}$ are asymptotically real second order free.*

Also, Corollaries 2.4 and 2.7 from [CS] can be formulated as follows.

Theorem 9. *We will denote by $\mathcal{U}(N)$ the group of $N \times N$ unitary matrices with complex entries; we will use the notation $\int_{\mathcal{U}(N)} \cdot \, d\mathcal{U}$ for the integral on $\mathcal{U}(N)$ with respect to the Haar measure, and the notation $u_{i,j}$ for the function $u_{i,j} : \mathcal{U}(N) \longrightarrow \mathbb{C}$ that gives the value of the (i,j)-entry.*

Let n, m be two positive integers and let $i = (i_1, i_2, \ldots, i_n)$ and $i' = (i'_1, i'_2, \ldots, i'_m)$, respectively $j = (j_1, j_2, \ldots, j_n)$ and $j' = (j'_1, j'_2, \ldots, j'_m)$ be n-, respectively m-tuples from the set $\{1, 2, \ldots, N\}$. Then

$$\int_{\mathcal{U}(N)} u_{i_1,j_1} \cdots u_{i_n,j_n} \overline{u_{i'_1,j'_1}} \cdots \overline{u_{i'_n,j'_n}} d\mathcal{U} =$$

$$\delta_{m,n} \cdot \sum_{\sigma,\tau \in S_n} \delta_{i_1,i'_{\sigma(1)}} \cdots \delta_{i_n,i'_{\sigma(n)}} \cdot \delta_{i_1,j'_{\tau(1)}} \cdots \delta_{j_n,j'_{\tau(n)}} \cdot Wg_{\mathcal{U},d}(\tau\sigma^{-1}).$$

where $Wg_{\mathcal{U},N}(\tau\sigma^{-1})$ is the value of the Weingarten function on $\mathcal{U}(N)$ at $\tau\sigma^{-1}$. Moreover

$$Wg_{\mathcal{U},N}(\sigma) = N^{-2n+\#(\sigma)} Moeb(\sigma) + O(N^{-2n+\#(\sigma)-2}).$$

where $Moeb : S_n \longrightarrow \mathbb{R}$ is the Moebius function (see [B]*), given explicitely, for $\sigma \in S_n$ with cycle decomposition $C_1 \cdots C_m$, by*

$$Moeb(\sigma) = \prod_{1 \leq k \leq m} (-1)^{|C_k|-1} \frac{[2(|C_k|-1)]!}{(|C_k|)!(|C_k|-1)!}.$$

An immediate consequence of the above Theorem 9 is the following

Corollary 10. $\lim_{N\to\infty} \int_{\mathcal{U}(N)} \operatorname{tr}\left(U^t U^*\right) d\mathcal{U} = 0.$

Proof. We have that

$$\lim_{N\to\infty} \int_{\mathcal{U}(N)} \operatorname{tr}\left(U^t U^*\right) d\mathcal{U} = \lim_{N\to\infty} \frac{1}{N} \int_{\mathcal{U}(N)} \sum_{i,j=1}^{N} u_{i,j} \overline{u_{j,i}} d\mathcal{U}.$$

But, applying Theorem 9, $\int_{\mathcal{U}(N)} u_{i,j} \overline{u_{j,i}} d\mathcal{U} = 0$ if $i \neq j$, hence

$$\lim_{N\to\infty} \int_{\mathcal{U}(N)} u_{i,i} \overline{u_{i,i}} d\mathcal{U} = \lim_{N\to\infty} (N^{-1} + O(N^{-2})) = 0.$$

□

3. Haar Unitaries

3.1. We will use a slight reformulation of Theorem 9 in the spirit of [MPR]. First we define the map $S_n \ni \sigma \mapsto \widehat{\sigma} \in P_2(2n)$ given by $\widehat{\sigma}(i) = 2n - \sigma(i)$ for $1 \leq i \leq n$. Note that $\widehat{\sigma}(2n-i) = \sigma^{-1}(i)$ and that $\sigma \mapsto \widehat{\sigma}$ is a bijection from S_n to $P_2^0(2n) = \{\pi \in P_2(2n) : \pi([n]) = [2n] \setminus [n]\}$.

Proposition 11. *Let σ, τ be permutations from S_n and $(c_1, c_2, \ldots, c_p)$ be the cycle decomposition of $\tau\sigma^{-1}$. Then $\widehat{\tau}\widehat{\sigma}$ has cycle decomposition $(d_1, d_1', d_2, d_2', \ldots, d_p, d_p')$ where if $c_k = (i_1, i_2, \ldots, i_l)$ then $d_k = (2n - i_1, 2n - i_2, \ldots, 2n - i_l)$ and $d_k' = (\sigma^{-1}(i_l), \sigma^{-1}(i_{l-1}), \ldots, \sigma^{-1}(i_1))$.*

Proof. Suppose $(i_1, \ldots, i_l)$ is a cycle of $\tau\sigma^{-1}$ and denote $j_k = \sigma^{-1}(i_k)$, $1 \leq k \leq l$. To simplify the writing, we will consider that $i_{l+1} = i_1$ and $j_{l+1} = j_1$. Then $\widehat{\tau}\widehat{\sigma}(2n - i_k) = \widehat{\tau}(\sigma^{-1}(i_k)) = \widehat{\tau}(j_k) = 2n - i_{k+1}$, while

$$\widehat{\tau}\widehat{\sigma}(\sigma^{-1}(i_k)) = \widehat{\tau}\widehat{\sigma}\widehat{\sigma}(2n - i_k) = \widehat{\tau}(2n - i_k) = j_{k-1} = \sigma^{-1}(i_{k-1}).$$

□

We introduce now the map $\Phi_N : P_2(\pm n) \times P_2(\pm n) \longrightarrow \mathbb{C}$ as follows. If $p, q \in P_2(\pm n)$ then pq has the cycle decomposition $(c_1, c_1', \ldots, c_l, c_l')$ such that c_k and c_k' have the same number of elements. Considering $\sigma \in S_n$ with cycle decomposition $(d_1, d_2, \ldots, d_l)$ such that each d_k has the same number of elements as c_k, we define $\Phi_N(p, q) = \mathrm{Wg}_{\mathcal{U},N}(\sigma)$. As shown in [CS], the value $\mathrm{Wg}_{\mathcal{U},N}(\sigma)$ depends only on the cycle decomposition of σ, so Φ_N is well-defined. Remark that Theorem 9 and the fact that the Moebius function is multiplicative and also depends only on the cycle decomposition give

$$\Phi_N(p, q) = N^{-2n + \frac{\#(pq)}{2}} \cdot \mathrm{Moeb}(\sigma) + O(N^{-2n + \frac{\#(pq)}{2} - 2}). \tag{2}$$

Finally, let $u_{i,j} = u_{i,j}^{(1)}$ and $\overline{u_{i,j}} = u_{i,j}^{(2)}$. With the above notations, Theorem 9 can be reformulated as

Proposition 1. Consider the multi-indices $i : [\pm n] \longrightarrow [N]$ and $\alpha : [n] \longrightarrow \{(1), (2)\}$ and define

$$P_2^{\alpha}(n) = \{p \in P_2(n) : \text{ if } p(k) = l \text{ then } \alpha_k \neq \alpha_l\}.$$

Then

$$\int_{\mathcal{U}(N)} u_{i_1, i_{-1}}^{\alpha_1} \cdots u_{i_n, i_{-n}}^{\alpha_n} d\mathcal{U} = \sum_{p,q \in P_2^{\alpha}(n)} \delta^i_{i \circ p \delta q \delta} \cdot \Phi_N(p, q).$$

Proof. If $\#\{s:\ \alpha_s=(1)\}\neq\#\{s:\ \alpha_s=(2)\}$ then the left-hand side of the equation above vanishes from Theorem 9 while the right-hand side also vanishes from the definition of the Kronecker's symbol δ. Hence we can suppose that we have equality in the last relation, particularly $n=2m$. Consider now some α satisfying this assumption and let

$$A=\{p\leq m:\alpha(p)=(2)\}$$
$$B=\{p>m:\alpha(p)=(1)\}.$$

Then A and B have the same number of elements, denoted by $w(\alpha)$ and we can write $A=\{s_1,s_2,\ldots,s_{w(\alpha)}\}$ and $B=\{s_{-1},s_{-2},\ldots,s_{-w(\alpha)}\}$. Define $\pi\in S_{2m}$ as follows

$$\pi(k)=\begin{cases} k & \text{if } k\in[2m]\setminus(A\cup B)\\ s_{-p} & \text{if } k=s_p\in A\cup B.\end{cases}$$

Then $\pi^2=1$ and let $\widetilde{\alpha}$ be defined by

$$\widetilde{\alpha}(k)=\alpha\circ\pi(k)=\begin{cases}(1) & \text{if } k\leq m\\ (2) & \text{if } k>m.\end{cases}$$

Remark that if $p\in P_2^{\alpha}(2m)$, then $\widetilde{p}=\pi\circ p\circ\pi\in P_2^{\widetilde{\alpha}}(2m)$. Also $i=i\circ p$ is equivalent to $i\circ\pi=(i\circ\pi)\circ\widetilde{p}$ and $\Phi_N(p,q)=\Phi_N(\widetilde{p},\widetilde{q})$. Therefore, without loss of generality, we can suppose that $\alpha_1,\ldots,\alpha_m=(1)$ and $\alpha_{m+1},\ldots,\alpha_{2m}=(2)$. With this re-arrangement, the map $\sigma\mapsto\widehat{\sigma}$ is a bijection from S_m to $P_2^{\alpha}(2m)$.

For $1\leq k\leq m$, with the notations $i'_k=i_{-k}$, $j_k=i_{k+m}$ and $j'_k=i_{-k-m}$, Theorem 9 gives

$$\begin{aligned}\int_{\mathcal{U}(N)}&u^{\alpha_1}_{i_1,i_{-1}}\cdots u^{\alpha_n}_{i_n,i_{-n}}d\mathcal{U}\\ &=\sum_{\sigma,\tau\in S_m}\delta_{i_1,i'_{\sigma(1)}}\cdots\delta_{i_n,i'_{\sigma(m)}}\cdot\delta_{j_1,j'_{\tau(1)}}\cdots\delta_{j_n,j'_{\tau(m)}}\cdot\mathrm{Wg}_{\mathcal{U},N}(\tau\sigma^{-1})\\ &=\sum_{\sigma,\tau\in S_m}\delta^i_{i\circ\widehat{\sigma}\delta\widehat{\tau}\delta}\cdot\Phi_N(\widehat{\sigma},\widehat{\tau})=\sum_{p,q\in P_2^{\alpha}(2m)}\delta^i_{i\circ p\delta q\delta}\cdot\Phi_N(p,q)\end{aligned}$$

□

3.2. For the rest of this section, U_N will denote $N\times N$ unitary random matrix, and $\mathfrak{U}$ will denote the ensemble $\{U_N,U_N^t,U_N^*,(U_N^*)^t\}_N$. As before, we will omit the index N under the assumption that we will only multiply matrices of the same size. Also, to simplify the writing, we will use the conventions $U^{(1)}=U$, $U^{(2)}=U^t$, $U^{(3)}=U^*$, $U^{(4)}=(U^*)^t$, and if $u_{i,j}$ is the (i,j)-entry of U, then $u^{(1)}_{i,j}=u_{i,j}$, $u^{(2)}_{i,j}=\overline{u_{i,j}}$.

Definition 12. An ensemble $\mathcal{A}$ of random matrices is said to be *orthogonally invariant*, respectively *unitarily invariant*, if for any positive integer N, any $N \times N$ orthogonal, respectively unitary matrices O, respectively U, and any $N \times N$ matrices $A_1, \ldots, A_s$ from $\mathcal{A}$ and any $1 \leq i_k \leq N$, $1 \leq |k| \leq s$, we have that

$$E(a^{(1)}_{i_1,i_{-1}} a^{(2)}_{i_2,i_{-2}} \cdots a^{(s)}_{i_s,i_{-s}}) = E(b^{(1)}_{i_1,i_{-1}} b^{(2)}_{i_2,i_{-2}} \cdots b^{(s)}_{i_s,i_{-s}})$$

where $a^{(k)}_{ij}$ is the (i,j)-entry of A_k and $b^{(k)}_{ij}$ is the (i,j)-entry of OA_kO^t, respectively of UA_kU^*.

Lemma 13. *The ensemble $\mathfrak{U}$ is orthogonally invariant, but not unitarily invariant.*

Proof. To simplify the writing, for A a square random matrix, let us denote by $(A)_{i,j}$ the (i,j)-th entry of A and by $A^{(1)}, A^{(2)}, A^{(3)}$, respectively $A^{(4)}$ the random matrices $A, A^t, A^*, (A^t)^*$ and by $e_{i,j}$ the $N \times N$ matrix with all entries 0 except for the (i,j)-entry which is 1.

Let s be a positive integer and $k_p \in \{1,2,3,4\}$, $i_l \in [N]$ for $p \in [s]$ and $l \in [\pm s]$, and consider the function $f : \mathcal{U}(N) \to \mathbb{C}$,

$$f(U) = \mathrm{Tr}(e_{1,i_1} U^{(k_1)} e_{i_{-1},i_2} U^{(k_2)} e_{i_{-2},i_3} \cdots U^{(k_s)} e_{i_{-s},1}).$$

Then $\displaystyle\int_{\mathcal{U}(N)} f(U) d\mathcal{U} = E\left((U^{(k_1)})_{i_1,i_{-1}} \cdots (U^{(k_s)})_{i_s,i_{-s}}\right)$, so, according to Theorem 9, $f \in L^\infty(\mathcal{U}(N), d\mathcal{U})$.

Let now O be a $N \times N$ orthogonal matrix. From the translation invariance of Haar measure, $\displaystyle\int_{\mathcal{U}(N)} f(U) d\mathcal{U} = \int_{\mathcal{U}(N)} f(OUO^t) d\mathcal{U}$.

Since, with the notations above, $(OUO^t)^{(k)} = OU^{(k)}O^t$ for $1 \leq k \leq 4$, we obtain

$$\begin{aligned}
\int_{\mathcal{U}(N)} f(U) d\mathcal{U} &= \int_{\mathcal{U}(N)} f(OUO^t) d\mathcal{U} \\
&= \int_{\mathcal{U}(N)} \mathrm{Tr}(e_{1,i_1} (OUO^t)^{(k_1)} e_{i_{-1},i_2} \cdots (OUO^t)^{(k_s)} e_{i_{-s},1}) d\mathcal{U} \\
&= \mathrm{Tr}(e_{1,i_1} OU^{(k_1)} O^t e_{i_{-1},i_2} \cdots OU^{(k_s)} O^t e_{i_{-s},1}) d\mathcal{U} \\
&= E\left((OU^{(k_1)}O^t)_{i_1,i_{-1}} \cdots (OU^{(k_1)}O^t)_{i_s,i_{-s}}\right)
\end{aligned}$$

therefore $\mathfrak{U}$ is orthogonally invariant.

For the second part, let V be the unitary matrix with the $(1,1)$-entry equal to i (where $i^2 = -1$), all other entries on the main diagonal 1

and all entries off the main diagonal 0. Theorem 9 gives

$$E\left((U^{(1)})_{1,2}\cdot(U^{(4)})_{1,2}\right)=\int_{\mathcal{U}(N)}u_{1,2}\overline{u_{1,2}}d\mathcal{U}=\frac{1}{N},$$

while

$$\begin{aligned}E\left((VUV^*)^{(1)})_{1,2}\cdot(VUV^*)^{(4)})_{1,2}\right)&=E\left(VUV^*)_{1,2}\cdot(V^*U^{(4)}V^t)_{1,2}\right)\\&=\int_{\mathcal{U}(N)}i\cdot u_{1,2}\cdot i\cdot\overline{u_{1,2}}d\mathcal{U}=-\frac{1}{N}\end{aligned}$$

□

Theorem 14. *If $\mathcal{A}$ is an ensemble of random matrices independent from $\mathfrak{U}$ and having second order bounded t-distribution, then the ensemble $\mathfrak{U}\cup\mathcal{A}$ has second order bounded t-distribution.*

Proof. To simplify the writing, we will suppose that $\mathcal{A}$ is closed under multiplication and centering (i.e. if $A_1,A_2\in M_N(\mathbb{C})\cap\mathcal{A}$, then $A_1\cdot A_2\in\mathcal{A}$ and $A_1-\mathrm{tr}(A_1)\cdot I\in\mathcal{A}$).

Let r,R be positive integers such that $r\leq R$, let $l_1,l_2,\ldots,l_r$ also be positive integers and put $M_0=0$, $M_s=M_{s-1}+l_s$ for $1\leq s\leq r$ and let $M=M_r$.

Remark first that it suffices to prove that

$$\sup_{N\to\infty}|E(\mathrm{Tr}(W_1)\,\mathrm{Tr}(W_2)\cdots\mathrm{Tr}(W_R)|<\infty$$

for all $A_i\in\mathcal{A}\cup\{I\}$ and all $\epsilon_1,\ldots,\epsilon_M\in\{(1),(2),(3),(4)\}$, where (with the notations from Section 3.2) $W_1,\ldots W_R$ are as follows:

(i) if $0<s\leq r$, then

$$W_s=U^{\epsilon_{M_{s-1}+1}}A_{M_{s-1}+1}U^{\epsilon_{M_{s-1}+2}}A_{M_{s-1}+2}\cdots U^{\epsilon_{M_s}}A_{M_s}$$

is a nontrivial reduced word in $U,U^t,U^*,(U^*)^t$ and centered elements from $\mathcal{A}$;

(ii) if $0<s\leq R-r$, then $W_{r+s}=A'_s$ with A'_s a centered element from $\mathcal{A}$.

Indeed, writing each $A\in\mathcal{A}$ as $(A-E(\mathrm{tr}(A))\cdot I)+E(\mathrm{tr}(A))\cdot I$, a general monomial with factors from $\mathcal{A}$ and $\mathfrak{U}$ becomes a finite sum of monomials W_s as above and some multiple of the identity matrix I and conditions from Definition 1(1) are trivial for cumulants having one scalar entry.

Note that the requirement that W_s be reduced amounts to the requirement that if $A_i=I$, then the set $\{\epsilon_i,\epsilon_{i+1}\}$ cannot be $\{(1),(3)\}$ or $\{(2),(4)\}$.

Denote by $a_{i,j}^{(k)}$ the (i,j)-entry of A_k. For $1 \leq s \leq r$, we have that

$$\operatorname{Tr}(W_s) = \sum_i u^{\beta(M_{s-1}+1)}_{i_{M_{s-1}+1}, i_{-M_{s-1}-1}} \cdot a^{(M_{s-1}+1)}_{i_{\varphi(M_{s-1}+1)}, i_{\varphi(-M_{s-1}-1)}} \cdot u^{\beta(M_{s-1}+2)}_{i_{M_{s-1}+2}, i_{-M_{s-1}-2}} \cdot \cdots u^{\beta(M_s)}_{i_{M_s}, i_{-M_s}} \cdot a^{(M_s)}_{i_{\varphi(M_s)}, i_{\varphi(-M_s)}},$$

where $\beta(l) = \begin{cases} (1) & \text{if } \epsilon(l) \in \{(1),(2)\} \\ (2) & \text{if } \epsilon(l) \in \{(3),(4)\} \end{cases}$, and the map φ is given by the following relations

- if $M_{s-1}+1 \leq l \leq M_s$, then
$$\varphi(l) = \begin{cases} l & \text{if } \epsilon(l) \in \{(2),(3)\} \\ -l & \text{if } \epsilon(l) \in \{(1),(4)\}; \end{cases}$$
- if $-M_s \leq l \leq -M_{s-1}+1$, then
$$\varphi(l) = \begin{cases} l-1 & \text{if } \epsilon(-l+1) \in \{(2),(3)\} \\ -l+1 & \text{if } \epsilon(-l+1) \in \{(1),(4)\}; \end{cases}$$
- $\varphi(-M_s) = \begin{cases} M_{s-1}+1 & \text{if } \epsilon(M_{s-1}+1) \in \{(2),(3)\} \\ -M_{s-1}-1 & \text{if } \epsilon(M_{s-1}+1) \in \{(1),(4)\}. \end{cases}$

Finally, denote $\mathfrak{a} = \operatorname{Tr}(A'_1) \cdots \operatorname{Tr}(A'_{R-r})$. With the above notations and taking into account Proposition 1, we have that

$$\begin{aligned} &E(\operatorname{Tr}(W_1)\operatorname{Tr}(W_2)\cdots\operatorname{Tr}(W_R)) \\ &= \sum_i E(u^{\beta(1)}_{i_1,i_{-1}} a^{(1)}_{i_{\varphi(1)},i_{\varphi(-1)}} \cdots u^{\beta(M)}_{i_M,i_{-M}} a^{(M)}_{i_{\varphi(M)},i_{\varphi(-M)}} \cdot \operatorname{Tr}(A'_1)\cdots\operatorname{Tr}(A'_{R-r})) \\ &= \sum_i \sum_{\substack{p,q \in P_2^\beta(M) \\ i = i\circ p\delta q\delta}} E(u^{\beta(1)}_{i_1,i_{-1}} \cdots u^{\beta(M)}_{i_M,i_{-M}}) E([a^{(1)}_{i_{\varphi(1)},i_{\varphi(-1)}} \cdots a^{(M)}_{i_{\varphi(M)},i_{\varphi(-M)}}] \cdot \mathfrak{a}) \\ &= \sum_{p,q \in P_2^\beta(M)} \Phi_N(p,q) \sum_{i = i\circ p\delta q\delta} E([a^{(1)}_{i_{\varphi(1)},i_{\varphi(-1)}} \cdots a^{(M)}_{i_{\varphi(M)},i_{\varphi(-M)}}] \cdot \mathfrak{a}). \end{aligned}$$

Note that, for $\widetilde{p} \in P_2(\pm M)$ given by $\widetilde{p} = \varphi^{-1} p\delta q\delta\varphi$ we have that

$$\sum_{i = i\circ p\delta q\delta} a^{(1)}_{i_{\varphi(1)},i_{\varphi(-1)}} \cdots a^{(M)}_{i_{\varphi(M)},i_{\varphi(-M)}} = \sum_{j = j\circ\widetilde{p}} a^{(1)}_{j_1,j_{-1}} \cdots a^{(M)}_{j_M,j_{-M}}.$$

From Remark 5 and Lemma 7, there exist some maps $\pi : P_2(M)^2 \to S_M$ and $\eta : P_2(M)^2 \to \mathbb{Z}_2^M$ such that, with the notation $A^{(1)} = A$, $A^{(-1)} = A^t$, we have that

$$\begin{aligned} &\sum_{j = j\circ\widetilde{p}} E([a^{(1)}_{j_1,j_{-1}} \cdots a^{(M)}_{j_M,j_{-M}}] \cdot \mathfrak{a}) \\ &\qquad = E(\operatorname{Tr}_{\pi(p,q)}(A_1^{(\eta_1)} \cdots A_M^{(\eta_M)}) \cdot \operatorname{Tr}(A'_1)\cdots\operatorname{Tr}(A'_{R-r})) = O(N^{\#\pi}). \end{aligned}$$

Let now $\mathbf{A}(W_1,\ldots,W_R) = \{p \in P_2^\beta(M) : \pi(p,p) \text{ is also a pairing}\}$, and, for $p \in \mathbf{A}(W_1,\ldots,W_R)$, define

$$\mathcal{E}(p) = N^{-\frac{M}{2}} \cdot \sum_{i=i\circ p\delta p\delta} E(a^{(1)}_{i_{\varphi(1)},i_{\varphi(-1)}} \cdots a^{(M)}_{i_{\varphi(M)},i_{\varphi(-M)}}) \cdot E(\mathfrak{a}).$$

We will finish the proof by showing that

$$E(\mathrm{Tr}(W_1)\cdots\mathrm{Tr}(W_R)) = \sum_{p\in\mathbf{A}(W_1,\ldots,W_R)} \mathcal{E}(p) + o(1). \tag{3}$$

To see that (3) implies the conclusion, it suffice to note that, according to Remark 5

$$\mathcal{E}(p) = N^{-\frac{M}{2}} \cdot O(N^{\#\pi(p,p)}) = N^{-\frac{M}{2}} \cdot O(N^{\frac{M}{2}}) = O(1). \tag{4}$$

In order to prove (3), note first that, according to Proposition 4(ii),

$$\mathcal{E}(p) = N^{-\frac{M}{2}} \cdot \sum_{i=i\circ p\delta p\delta} E(a^{(1)}_{i_{\varphi(1)},i_{\varphi(-1)}} \cdots a^{(M)}_{i_{\varphi(M)},i_{\varphi(-M)}} \cdot \mathfrak{a}) + o(1) \tag{5}$$

From Theorem 9, $\Phi_N(p,q) = O(N^{-\frac{M}{2}-1})$ if $p \neq q$ and $\Phi_N(p,p) = N^{-\frac{M}{2}}$. Hence the proof will be complete if we show that either $\#\pi(p,q) \leq M/2$ or $E(\mathrm{Tr}_{\pi(p,q)}(A_1^{\eta_1}\cdots A_M^{\eta_M})) = 0$ whenever $p \neq q$ or $p \notin \mathbf{A}(W_1,\ldots,W_R)$. For the lattert statement it suffices to show that either $\pi(p,q)$ does not have any singletons or $E(\mathrm{Tr}_{\pi(p,q)}(A_1^{(\eta_1)}\cdots A_M^{(\eta_M)})) = 0$

Suppose that (k) is a singleton of $\pi(p,q)$. If $A_k \neq I$, then A_k is centered, so $E(\mathrm{Tr}(A_k^{(\eta_k)})) = 0$. If $A_k = I$, then Lemma 7 gives that $(k,-k)$ is a cycle of $\widetilde{p}$, that is $(\varphi(k),\varphi(-k))$ is a cycle of $p\delta q\delta$. Since $p\delta q\delta$ is sign preserving, it follows that $\varphi(k)$ and $\varphi(-k)$ have the same sign. If $\varphi(k),\varphi(-k) > 0$, then $\epsilon(k) \in \{(2),(3)\}$ and $\epsilon(k+1) \in \{(1),(4)\}$, which contradicts either $p,q \in P_2^\beta(M)$ or the fact that $W_1,\ldots,W_r$ are reduced words. The case $\varphi(k),\varphi(-k) < 0$ is similar. □

Corollary 15. *The ensembles $\{U_N, U_N^*\}_N$ and $\{U_N^t, (U_N^t)^*\}_N$ are asymptotically free.*

Proof. Since both $\{U_N, U_N^*\}_N$ and $\{U_N^t, (U_N^t)^*\}_N$ verify condition (i)′ from Definition 1, it suffices to show that for any non-zero integers $p_1, p_2, \ldots, p_{2M}$, we have

$$\lim_{N\to\infty} E\left(\mathrm{tr}\left(U^{p_1}\cdot(U^t)^{p_2}\cdots U^{p_{2M-1}}(U^t)^{p_{2M}}\right)\right) = 0.$$

But taking $\mathcal{A}$ to be the ensemble given by the identity matrices, Theorem 14 implies that

$$\sup_{N\to\infty} E\left(\mathrm{Tr}\left(U^{p_1}\cdot(U^t)^{p_2}\cdots U^{p_{2M-1}}(U^t)^{p_{2M}}\right)\right) < \infty,$$

hence the conclusion. □

Also, an immediate consequence of Lemma 13, Theorem 14 and Theorem 8 is the following

Corollary 16. *If $\mathcal{A}$ is an ensemble of random matrices with second order bounded t-distribution and independent from $\mathfrak{U}$, then $\mathfrak{U} \cup \mathcal{A}$ satisfies equation* (1) *from Definition 3.*

Theorem 17. *If $\mathcal{A}$ has second order limit t-distribution, then $\mathfrak{U}$ and $\mathcal{A}$ are asymptotically real second order free.*

Proof. We only need to show that $\mathfrak{U} \cup \mathcal{A}$ has second order limit t-distribution, and the conclusion will follow from Corollary 16. For this, applying (3), it suffices to show that if $W_1, \ldots, W_R$ satisfy conditions (i) and (ii) from Theorem 14, then

$$\sum_{\mathcal{U} \in P_{1,2}(R)} k_{\mathcal{U}}(\mathrm{Tr}(W_1), \ldots, \mathrm{Tr}(W_R)) = \sum_{p \in \mathbf{A}(W_1, \ldots, W_R)} \mathcal{E}(p) + o(1). \tag{6}$$

where $P_{1,2}(R)$ is the set of all partitions on $[R]$ whose blocks have no more than 2 elements.

From Theorem 17, $k_2(\mathrm{Tr}(W_i), \mathrm{Tr}(W_j)) = o(1)$ whenever $i \leq r$ and $r < j \leq R$. Thus

$$\sum_{\mathcal{U} \in P_{1,2}(R)} k_{\mathcal{U}}(\mathrm{Tr}(W_1), \ldots, \mathrm{Tr}(W_R)) = \sum_{\mathcal{U} \in P_{1,2}(r)} k_{\mathcal{U}}(\mathrm{Tr}(W_1), \ldots, \mathrm{Tr}(W_r)) \cdot \sum_{\mathcal{V} \in P_{1,2}(R-r)} k_{\mathcal{V}}(\mathrm{Tr}(W_{r+1}), \ldots, \mathrm{Tr}(W_R)) + o(1).$$

Since $A'_1, \ldots, A'_{R-r}$ are centered and have second order limited t-distribution, their first order cumulants are 0, second order cumulants are $O(1)$ and cumulants of higher order are $o(1)$, hence we have that

$$E(\mathrm{Tr}(W_{r+1}) \cdots \mathrm{Tr}(W_R)) = \sum_{\mathcal{V} \in P_{1,2}(R-r)} k_{\mathcal{V}}(\mathrm{Tr}(W_{r+1}), \ldots, \mathrm{Tr}(W_R)) + o(1)$$

so it suffices to prove (6) for $r = R$, i.e. we have to show that

$$\sum_{\mathcal{U} \in P_{1,2}(R)} k_{\mathcal{U}}(\mathrm{Tr}(W_1), \ldots, \mathrm{Tr}(W_r)) = \sum_{p \in \mathbf{A}(W_1, \ldots, W_r)} \mathcal{E}(p) + o(1) \tag{7}$$

where for each $s \in [r]$,

$$W_s = U^{\epsilon_{M_{s-1}+1}} A_{M_{s-1}+1} U^{\epsilon_{M_{s-1}+2}} A_{M_{s-1}+2} \cdots U^{\epsilon_{M_s}} A_{M_s}$$

is a nontrivial reduced word in $U = U^{(1)}$, $U^t = U^{(2)}$, $U^* = U^{(3)}$, $(U^*)^t = U^{(4)}$ and centered elements from $\mathcal{A}$, $M_0 = 0$ and $M_r = M$.

Let γ be the partition on $[\pm M]$ with blocks $B_k = [\pm M_k] \setminus [\pm M_{k-1}]$; for $l, s \in [r]$, define $\gamma_{l,s} = \gamma_{|B_l \cup B_s}$.

Let $p \in \mathbf{A}(W_1, \ldots, W_r)$ and suppose that $p\delta p\delta \vee \gamma$ has the blocks $C_1, \ldots, C_t$. Each C_j is a union of blocks of γ, hence p determines a partition $\rho(p) \in P(r)$, with blocks $D_1, \ldots, D_t$ given by $l \in D_j$ if $B_l \subset C_j$.

We will first show that

$$\mathcal{E}(p) = \mathcal{E}(p_{|C_1}) \cdots \mathcal{E}(p_{|C_t}). \tag{8}$$

With the notations from the proof of Theorem 14, for (8) it suffices to prove that the blocks C_j are invariant for the partition $\pi(p, p)$. But, as described in Section 2.2, the blocks of $\pi(p, p)$ are obtaining by taking absolute value of the elements of the cycles of $\delta\widetilde{p}$, for $\widetilde{p}$ defined as in the proof of Theorem 14, i. e. $\widetilde{p} = \varphi^{-1}p\delta p\delta\varphi$. But, by construction, $\varphi(B_l) = B_l$ and $\delta(B_l) = B_l$ for all B_l, and $p(C_j) = C_j$, therefore $\delta\widetilde{p}(C_j) = C_j$ for all C_j.

Also, note that each C_j has at most two blocks of γ, therefore $\rho(p) \in P_{1,2}(r)$. Indeed, suppose that there exist B_u, B_v, B_w three blocks of γ such that $B_v \neq B_w$ and for some $j, k \in B_u$, with $j < k$, we have that $p(j) \in B_v$ and $p(k) \in B_w$. Since B_v is symmetric, $p(j) \in B_v$ implies that $\delta p(j) = \delta p\delta(-j)$ is also in B_v. But $\varphi(B_v) = B_v$ and $\varphi(j) \in \{j, -j\}$, therefore $\widetilde{p}(j) = \varphi^{-1}p\delta p\delta\varphi(j) \in B_v$; using again that b_v is symmetric, we get $\delta\widetilde{p}(j) \in B_v$. Since $\pi(p, p)$ is a pairing, $\delta\widetilde{p}$ is also a pairing, hence B_v contains $(\delta\widetilde{p})^{-1}(j) = \widetilde{p}\delta(j)$.

On the other side, $\widetilde{p}\delta(j) = \widetilde{p}(-j) = \varphi^{-}p\delta p\delta\varphi(-j)$, therefore

$$p\delta p\delta(\varphi(-j)) \in \varphi(B_v) = B_v.$$

Since $\varphi(-j) \in j + 1, -j - 1$, it follows that $p(j + 1) \in B_v$. Iterating the argument, we obtain $p(k) \in B_v$, contradicting $p(k) \in B_w$.

The proof of Theorem 17 is finished if we show that

$$\sum_{\substack{p \in \mathbf{A}(W_1, \ldots, W_r) \\ \rho(p) = \mathcal{U}}} \mathcal{E}(p) = k_{\mathcal{U}}(\mathrm{Tr}(W_1), \ldots, \mathrm{Tr}(W_r)) + o(1).$$

Since $\mathcal{E}(p)$ factorizes over the blocks of $p\delta p\delta \vee \gamma$, it suffices to show that if (j) , respectively (k, l) are blocks of $\rho(p)$, then

$$k_1(\mathrm{Tr}(W_j)) = \sum_{p \in \mathbf{A}(W_j)} \mathcal{E}(p) + o(1),$$

respectively

$$k_2(\mathrm{Tr}(W_l), \mathrm{Tr}(W_k)) = \sum_{\substack{p \in \mathbf{A}(W_l, W_k) \\ p\delta p\delta \vee \gamma_{l,k} = \mathbf{1}}} \mathcal{E}(p) + o(1).$$

The first relation is the definition of $\mathcal{E}(p)$ for $r = 1$. For the second relation, we have that, according to (8),

$$\begin{aligned} k_1(\mathrm{Tr}(W_l))k_1(\mathrm{Tr}(W_k)) &= \sum_{p\in\mathbf{A}(W_l)} \mathcal{E}(p) \cdot \sum_{p\in\mathbf{A}(W_k)} \mathcal{E}(p) + o(1) \\ &= \sum_{\substack{p\in\mathbf{A}(W_l,W_k) \\ p\delta p\delta\vee\gamma_{l,k}=(B_l,B_k)}} \mathcal{E}(p) + o(1). \end{aligned}$$

The set $\mathbf{A}(W_l, W_k)$ is the disjoint union of the sets $\{p \in \mathbf{A}(W_l, W_k) : p\delta p\delta \vee \gamma_{l,k} = \mathbf{1}\}$ and $\{p \in \mathbf{A}(W_l, W_k) : p\delta p\delta \vee \gamma_{l,k} = (B_l, B_k)\}$, therefore

$$\begin{aligned} k_2(\mathrm{Tr}(W_l), \mathrm{Tr}(W_k)) &= E(\mathrm{Tr}(W_l)\,\mathrm{Tr}(W_k)) - k_1(\mathrm{Tr}(W_l))k_1(\mathrm{Tr}(W_k)) \\ &= \sum_{p\in\mathbf{A}(W_l,W_k)} \mathcal{E}(p) - \sum_{\substack{p\in\mathbf{A}(W_l,W_k) \\ p\delta p\delta\vee\gamma_{l,k}=(B_l,B_k)}} \mathcal{E}(p) + o(1) \\ &= \sum_{\substack{p\in\mathbf{A}(W_l,W_k) \\ p\delta p\delta\vee\gamma_{l,k}=\mathbf{1}}} \mathcal{E}(p) + o(1). \end{aligned}$$

□

Since the ensemble generated by the identity matrices has second order limit t-distribution, Theorem 17 implies:

Corollary 18. *The Haar unitary U has second order limit t-distribution.*

An immediate consequence of Theorem 8, Theorem 17 from above and Corollary 21 from [MSS] is the following

Corollary 19. *Suppose that $\mathcal{A}$, an ensemble of random matrices with second order limit t-distribution, that $\mathcal{O}$ is an ensemble of Haar orthogonal random matrices, that $\mathcal{U}$ is an ensemble of Haar unitary random matrices,and suppose that $\mathcal{A}$, $\mathcal{O}$ and $\mathcal{U}$ are independent. Then $\mathcal{O}$ and $\mathcal{U}$ are both real and complex second order free, $\mathcal{U}$ and $\mathcal{A}$ are complex second order free, $\mathcal{O}$ and $\mathcal{A}$ are real second order free.*

There is no contradiction in $\mathcal{O}$ and $\mathcal{U}$ being both asymptotically complex and real second order free, since, from Corollary 10, all traces from the second summation of Equation (1) cancel asymptotically for Haar unitaries. In the next section we will show that a similar situation takes place for a larger class of random matrices.

4. Unitarily-Invariant Random Matrices

Theorem 20. *Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be two ensembles of random matrices such that:*

(i) *$\mathcal{A}_1$ is unitarily invariant and has second order bounded distribution;*
(ii) *$\mathcal{A}_2$ has second order bounded t-distribution;*
(iii) *$\mathcal{A}_1$ and $\mathcal{A}_2$ are independent.*

Then $\mathcal{A}_1 \cup \mathcal{A}_2$ has second order bounded t-distribution. Moreover, if $\mathcal{A}_1$ has second order limit distribution and $\mathcal{A}_2$ has second order limit t-distribution,then $\mathcal{A}_1 \cup \mathcal{A}_2$ has second order limit t-distribution.

Proof. To simplify the writing we will suppose that $\mathcal{A}_1$ and $\mathcal{A}_2$ are closed to centering and taking products, that $\mathcal{A}_2$ is closed to taking transposes. Also, for $A \in \mathcal{A}_1$, we introduce the notation $A^{(1)} = A$, $A^{(2)} = A^t$.

Let $W_1, W_2, \ldots, W_r$ be reduced products in centered elements from $\mathcal{A}_1$, transposes of centered elements from $\mathcal{A}_1$ and centered elements from $\mathcal{A}_2$. Henceforth we can suppose that there exist $l_1, l_2, \ldots, l_r$ positive integers such that for $M_0 = 0$, $M_s = M_{s-1} + l_s$, $M_r = M$ for each $s = 1, \ldots, r$ we have

$$W_s = A_{M_{s-1}+1}^{\epsilon(M_{s-1}+1)} B_{M_{s-1}+1} A_{M_{s-1}+2}^{\epsilon(M_{s-1}+2)} \cdots A_{M_s}^{\epsilon(M_s)} B_{M_s}$$

where A_i's are centered elements from $\mathcal{A}_1$ and B_i's are either centered elements from $\mathcal{A}_2$ or identity matrices. Remark that the condition that W_s is a reduced product implies that B_i can be an identity matrix only if $\epsilon(i) \neq \epsilon(i+1)$, if $i < M_s$, respectively if $\epsilon(M_{s-1}+1) \neq \epsilon(M_s)$, if $i = M_s$.

Note that, for $A = [a_{i,j}]_{i,j=1}^N \in \mathcal{A}_1$, the (i_1, i_2)-entries of UAU^*, respectively of $(UAU^*)^t$ are respectively

$$[UAU^*]_{i_1,i_2} = \sum_{i_{-1},i_{-2}=1}^{N} u_{i_1,i_{-1}} a_{i_{-1},i_{-2}} \overline{u_{i_2,i_{-2}}}$$

$$\left[(UAU^*)^t\right]_{i_1,i_2} = \sum_{i_{-1},i_{-2}=1}^{N} u_{i_2,i_{-1}} a_{i_{-1},i_{-2}} \overline{u_{i_1,i_{-2}}} = [UAU^*]_{i_2,i_1} .$$

It follows that

$$\begin{aligned}\mathrm{Tr}(W_1) &= \mathrm{Tr}((UA_1U^*)^{\epsilon(1)}B_1\cdots(UA_{M_1}U^*)^{\epsilon(M_1)}B_{M_1})\\ &= \sum_i u_{i_1,i_{-1}}a^{(1)}_{i_{-1},i_{-2}}\overline{u_{i_2,i_{-2}}}\cdot b^{(1)}_{i_{\varphi(1)},i_{\varphi(-1)}}\cdots u_{i_{2M_1-1},i_{-2M_1+1}}\\ &\qquad\cdot a^{(M_1)}_{i_{-2M_1+1},i_{-2M_1}}\overline{u_{i_{2M_1},i_{-2M_1}}}\cdot b^{(M_1)}_{i_{\varphi(M_1)},i_{\varphi(-M_1)}}\end{aligned}$$

and the analogues for $W_2,\ldots,W_r$, where the map φ is given by

- if $1\leq k\leq M_1$, $\varphi(k)=\begin{cases}2k-1, & \text{if } \epsilon(k)=(1)\\ 2k, & \text{if } \epsilon(k)=(-1)\end{cases}$
- if $1\leq k\leq M_1-1$, $\varphi(-k)=\begin{cases}2k+1, & \text{if } \epsilon(k+1)=(1)\\ 2k+2, & \text{if } \epsilon(k+1)=(-1)\end{cases}$
- $\varphi(-M_1)=\begin{cases}1, & \text{if } \epsilon(1)=(1)\\ 2, & \text{if } \epsilon(1)=(-1).\end{cases}$

Let $P_2'(2M)=\{p\in P_2(2M): l+p(l)=1(\mathrm{mod}\ 2) \text{ for all } \ l\in[2M]\}$. Proposition 1 gives

$$\begin{aligned}&E(\mathrm{Tr}(W_1)\,\mathrm{Tr}(W_2)\cdots\mathrm{Tr}(W_r))\\ &= \sum_i E(u_{i_1,i_{-1}}a^{(1)}_{i_{-1},i_{-2}}\overline{u_{i_2,i_{-2}}}\cdot b^{(1)}_{i_{\varphi(1)},i_{\varphi(-1)}}\cdots u_{i_{2M-1},i_{-2M+1}}\\ &\qquad\cdot a^{(M)}_{i_{-2M+1},i_{-2M}}\overline{u_{i_{2M},i_{-2M}}}\cdot b^{(M)}_{i_{\varphi(M)},i_{\varphi(-M)}})\\ &= \sum_{p,q\in P_2'(2M)}\ \sum_{i=i\circ p\delta q\delta} E(u_{i_1,i_{-1}}\overline{u_{i_2,i_{-2}}}\cdots u_{i_{2M-1},i_{-2M+1}}\overline{u_{i_{2M},i_{-2M}}})\\ &\qquad\cdot E([a^{(1)}_{i_{-1},i_{-2}}\cdots a^{(M)}_{i_{-2M+1},i_{-2M}}]\cdot[b^{(1)}_{i_{\varphi(1)},i_{\varphi(-1)}}\cdots b^{(M)}_{i_{\varphi(M)},i_{\varphi(-M)}}])\\ &= \sum_{p,q\in P_2'(2M)}\Phi_N(p,q)\cdot\sum_{i=i\circ p\delta q\delta}E([a^{(1)}_{i_{-1},i_{-2}}\cdots a^{(M)}_{i_{-2M+1},i_{-2M}}]\\ &\qquad\cdot[b^{(1)}_{i_{\varphi(1)},i_{\varphi(-1)}}\cdots b^{(M)}_{i_{\varphi(M)},i_{\varphi(-M)}}])\end{aligned}$$

For p,q as above, remark that p acts only on $[2M]$, while $\delta q\delta$ acts only on $[-2M]$, hence the condition $i=i\circ p\delta q\delta$ is equivalent to $i=i\circ p$ and $i=i\circ\delta q\delta$.

Moreover, denoting $l_k=i_{\varphi(k)}$, the condition $i=i\circ p$ is equivalent to $l=l\circ\varphi^{-1}p\varphi$.

Consider the map $\psi:[\pm M]\longrightarrow[2M]$, given by

$$\psi(k)=\begin{cases}2k, & \text{if } k<0\\ -2k+1, & \text{if } k>0\end{cases}$$

Denoting now $j_k = i_{-\psi(k)}$, the condition $i = i \circ \delta q \delta$ is equivalent to $j = j \circ \psi^{-1} q \psi$ and we obtain

$$\begin{aligned} &E(\mathrm{Tr}(W_1)\,\mathrm{Tr}(W_2)\cdots\mathrm{Tr}(W_r)) \\ &\quad = \sum_{p,q\in P_2'(2M)} \Phi_N(p,q)\cdot E([\sum_{j=j\circ\psi^{-1}q\psi} a^{(1)}_{j_1,j_{-1}}\cdots a^{(M)}_{j_M,j_{-M}}] \\ &\qquad\qquad \cdot [\sum_{l=l\circ\varphi^{-1}p\varphi} b^{(1)}_{l_1,l_{-1}}\cdots b^{(M)}_{l_M,l_{-M}}]) \end{aligned}$$

From Lemma 7, there exist some $\sigma_1 \in S_M$ and some $(\eta_1,\ldots,\eta_M)\in\mathbb{Z}_2^M$ such that

$$\sum_{l=l\circ\varphi^{-1}p\varphi} b^{(1)}_{l_1,l_{-1}}\cdots b^{(M)}_{l_M,l_{-M}} = \mathrm{Tr}_{\sigma_1}(B_1^{(\eta_1)}\cdots B_M^{(\eta_M)}).$$

Next, observe that $\psi^{-1}q\psi([M]) = [-M]$. Indeed, if $k>0$, then we have $\psi(k) = 1(\mathrm{mod}\ 2)$, hence $q\psi(k) = 0(\mathrm{mod}\ 2)$ and $\psi^{-1}q\psi(k) < 0$. Therefore, according to Remark 6, there exists some $\sigma_2 \in S_M$ such that

$$\sum_{j=j\circ\psi^{-1}q\psi} a^{(1)}_{j_1,j_{-1}}\cdots a^{(M)}_{j_M,j_{-M}} = \mathrm{Tr}_{\sigma_2}(A_1\cdots A_M).$$

Therefore in the development of $E(\mathrm{Tr}(W_1)\cdot\mathrm{Tr}(W_2)\cdots\mathrm{Tr}(W_r))$ there are no mixed products in elements from $\mathcal{A}_1$ and $\mathcal{A}_2$ and all matrices from $\mathcal{A}_1$ do not appear as transposes, hence the conclusion.

□

Remark 21. The condition that $\mathcal{A}_1$ and $\mathcal{A}_2$ are independent can be dropped, if we replace the condition that that $\mathcal{A}_1$ is unitary invariant is replaced by the condition that for any unitary U and any monomial P in matrices from $\mathcal{A}_1$, $\mathcal{A}_2$ and their transposes, the value $E(\mathrm{Tr}(P))$ is unchanged when all $A_1, A_2 \in \mathcal{A}_1$ such that A_1 and A_2^t are factors in P are replaced by UA_1U^*, respectively $(UA_2U^*)^t$. □

A similar argument to the proof of Corollary 15 gives

Corollary 22. *If $\mathcal{A}$ is a ensemble of unitarily invariant random matrices with second order limit distribution, then $\mathcal{A}$ and $\mathcal{A}^t = \{A^t :\ A\in\mathcal{A}\}$ are asymptotically free.*

Proof. Let n be a positive integer and, for $1 \le l \le n$, let A_l, B_l be centered $N\times N$ matrices from $\mathcal{A}$. Also, let U be a $n\times n$ Haar unitary random matrix with entries independent from the entris of matrices from $\mathcal{A}$. It suffices to show that

$$\lim_{N\longrightarrow\infty} E(\mathrm{tr}(\prod_{l=1}^{N} A_l B_l^t)) = 0. \tag{9}$$

Since $\mathcal{A}$ is unitarily invariant, it follows that

$$E(\mathrm{tr}(\prod_{l=1}^{N} A_l B_l^t)) = E(\mathrm{tr}(\prod_{l=1}^{N} U A_l U^* U^t B_l (U^t)^*)).$$

From Theorem 17, respectively Corollary 15, the ensembles $\mathcal{A} \cup \mathcal{A}^t$ and $\{U, U^*, U^t, (U^t)^*\}_N$, respectively the ensembles $\{U, U^*\}_N$ and $\{U^t, (U^*)^t\}_N$ are asymptotically free. The associativity property of free independence (see, for example [VDN]), it follows that the $\mathcal{A} \cup \mathcal{A}^t$, $\{U, U^*\}_N$ and $\{U^t, (U^*)^t\}_N$ are asymptotically free, and (9) follows from the definition of free independence.

□

Note that if $\mathcal{B} \subset \mathcal{A}$ is an ensemble of symmetric random matrices, then $\mathcal{B}$ is also contained in $\mathcal{A}^t$. This situation is discussed in Corollary 24 below.

Lemma 23. *Suppose that $\{B_N\}_N$ is an ensemble of random matrices with limit distribution and $\beta = \lim_{N \longrightarrow \infty} E(\mathrm{tr}(B_N))$.*

(i) *If $\{B_N\}_n$ and $\{B_N^*\}_n$ are asymptotically free, then $\{B_N - \beta I_N\}_N$ converges to 0 in $L^2(E \circ \mathrm{tr})$.*
(ii) *If $\{B_N\}_N$ is asymptotically free from itself, then $\{B_N - \beta I_N\}_N$ converges to 0 in distribution with respect to $E \circ \mathrm{tr}$, but not necessarily in $L^2(E \circ \mathrm{tr})$.*

Proof. Suppose that $\{B_N\}_N$ and $\{B_N^*\}_N$ are asymptotically free. The definition implies gives then

$$\lim_{N \longrightarrow \infty} E(\mathrm{tr}((B_N - \beta I_N)(B_N - \beta I_N)^*))) = \lim_{N \longrightarrow \infty} E(\mathrm{tr}((B_N - \beta I_N)) \cdot \lim_{N \longrightarrow \infty} E(\mathrm{tr}((B_N - \beta I_N)^*) = 0.$$

Suppose that $\{B_N\}_N$ is asymptotically free from itself. Then, for any positive integer m, we have that

$$\lim_{N \longrightarrow \infty} (E \circ \mathrm{tr})((B_N - \beta I_N)^{m+1}) = \lim_{N \longrightarrow \infty} (E \circ \mathrm{tr})((B_N - \beta I_N)^m) \cdot \lim_{N \longrightarrow \infty} (E \circ \mathrm{tr})((B_N - \beta I_N)) = 0$$

For the last part, take $\{B_N\}_N = \{U + U^t\}_N$. Then $\{B_N\}_N$ is an ensemble of unitarily invariant random matrices and, according to Corollary 22, it is asymptotically free from $\{B_N^t\}_N = \{B_N\}_N$. Corollary 15 gives

$$\lim_{N \longrightarrow \infty} (E \circ \mathrm{tr})((U + U^t)(U + U^t)^*) = \lim_{N \longrightarrow \infty} (E \circ \mathrm{tr})(UU^* + U(U^t)^* + (U^*U)^t + U^t U^*) = 2 \neq 0.$$

□

Corollary 24. *Suppose that $\mathcal{A}$ and $\{B_N\}_N$ are ensembles of unitarily invariant random matrices having second order limit distribution such that $\{B_N\}_N \subset \mathcal{A} \cup \mathcal{A}^t$ and let $\beta = \lim_{N\longrightarrow\infty} E(\mathrm{tr}(B_N))$.*

(i) *If $\mathcal{A}$ is closed under taking adjoints (i.e. $A \in \mathcal{A}$ implies $A^* \in \mathcal{A}$), then $\{B_N - \beta I_N\}_N$ converges to 0 in $L^2(E \circ \mathrm{tr})$.*

(ii) *If $\mathcal{A}$ is not closed under taking adjoints, then $\{B_N - \beta I_N\}_N$ converges to 0 in distribution with respect to $E \circ \mathrm{tr}$, but not necessarily in $L^2(E \circ \mathrm{tr})$.*

Proof. For part (i), since taking transposes and taking adjoint commute, it follows that if $\mathcal{A}$ is closed under taking adjoints, then so is $\mathcal{A}^t$ hence $\{B_N^*\}_N \subset \mathcal{A}^t$ is asymptocally free (according to Corollary 22) from $\{B_N\}_N \subset \mathcal{A}$, and the conclusion follows from Lemma 23(i).

For part (ii), it suffice to note that the example from the proof of Lemma 23(ii) is symmetric and unitarily invariant.

□

Since identity matrices have second order limit t-distribution, Theorem 20 implies the following Corollary.

Corollary 25. *If $\mathcal{A}$ is an ensemble of unitary invariant random matrices that has second order limit distribution, then it has second order limit t-distribution.*

Remark 26. Let $\mathcal{A}$ be an ensemble with second order limit distribution. The ensemble

$$\widetilde{\mathcal{A}} = \{\int_{\mathcal{U}(N} UAU^* d\mathcal{U}_N(U) : A \in \mathcal{A}\}$$

is unitary invariant (from the properties of the Haar measure) and has second order limit distribution (since $\mathcal{A}$ has second order limit distribution). Hence, Corollary 25 implies that $\widetilde{\mathcal{A}}$ has second order limit t-distribution.

Since orthogonal matrices are unitary, any unitary-invariant ensemble is also orthogonal invariant. From Corollary 25, a unitary invariant ensemble with second order limit distribution also has second order limit t-distribution, hence Theorem 8 and Corollary 25 implies the following

Corollary 27. *If $\mathcal{A}_1$ and $\mathcal{A}_2$ are two independent ensembles of random matrices such that $\mathcal{A}_1$ is unitary invariant and has second order limit distribution and $\mathcal{A}_2$ has second order limit t-distribution, then $\mathcal{A}_1$ and $\mathcal{A}_2$ are asymptotically real second order free.*

Remark 28. Some relevant classes of unitary invariant ensembles of random matrices (namely complex Gaussian and Wishart) are described in [M], [MS], [MSS]. Moreover, in [MSS] it is shown that a unitary invariant ensemble of random matrices with second order limit distribution is asymptotically *complex* second order free from any independent ensemble of random matrices with second order limit distribution. This result, together with Corollary 21 imply that unitary invariat ensembles of random matrices with second order limit distribution are *both real and complex* second order free from any independent ensemble of random matrices with second order limit distribution (see also [MPR] Section10). Particularly, the following relation (under a weaker form also mentioned in [MPR], Section 10) holds true:

If $\{A_N, B_N\}_N$ is an ensemble of unitary invariant random matrices with second order limit distribution, then $\lim_{N\to\infty} \mathrm{tr}(AB^t) = 0$. □

Proposition 29. *Suppose that $\mathcal{A}$ is a unitarily-invariant ensemble of random matrices that have a second order limit t-distribution. Then $\mathcal{A}$ and $\mathcal{A}^t = \{A^t : A \in \mathcal{A}\}$ are asymptotically real second order free.*

In particular, if U_N is a $N \times N$ Haar unitary random matrix, then $\{U_N, U_N^*\}_N$ and $\{U_N^t, (U_N^t)^*\}$ are asymptotically second order free.

Proof. The asymptotic free independence of the ensembles $\mathcal{A}$ and $\mathcal{A}^t$ is shown in Corollary 22.

Let $\{U_N\}_N$ be an ensemble of Haar unitary random matrices independent from $\mathcal{A}$ (hence from $\mathcal{A}^t$) and denote by $\mathfrak{U}$ the ensemble $\{U_N, U_N^t, U_N^*, (U_N^t)^*\}_N$.

Since $\mathcal{A}$ is unitarily invariant, any cumulant of traces of products of elements from $\mathcal{A}$ and $\mathcal{A}^t$ equals a cumulant of same order of traces of products of elemtns from $\mathcal{A}$, $\mathcal{A}^t$ and $\mathfrak{U}$, which vanishes asymptotically from Theorem 17 (i.e. from the asymptotic real second order freeness of $\mathfrak{U}$ and $\mathcal{A} \cup \mathcal{A}^t$).

It remains to prove that $\mathcal{A}$ and $\mathcal{A}^t$ satisfy equation (1). Let $A_1, \ldots, A_n$ and $B_1, \ldots, B_m$ be centered random matrices such that if k is odd then $A_k, B_k \in \mathcal{A}$ and if k is even then $A_k, B_k \in \mathcal{A}^t$. Then

$$\begin{aligned}\mathrm{Tr}(A_1 A_2 \cdots A_n) &= \mathrm{Tr}(U A_1 U^* (U^*)^t A_2 U^t U A_3 \cdots) \\ &= \mathrm{Tr}(A_1 (U^t U)^* A_2 U^t U A_3 \cdots)\end{aligned}$$

where the last fractors of the product above are $(U^t U)^* A_n$ if n is odd, respectively $A_n U^t U$ if n is even.

Since $E(\mathrm{tr}(U^tU)) = 0$ and $E(\mathrm{tr}(U^tU(U^tU)^*)) = 1$, applying again Theorem 17 we obtain

$$\lim_{N\to\infty} \mathbf{cov}(\mathrm{Tr}(A_1A_2\cdots A_n), \mathrm{Tr}(B_1, B_2\cdots B_m)) - \\ \delta_{m,n}\cdot\sum_{k=1}^{n}[\prod_{i=1}^{n}\mathrm{tr}(A_i\cdot B_{k-i}) + \prod_{i=1}^{n}\mathrm{tr}(A_i\cdot B_{i-k}^t)]\} = 0$$

hence the conclusion.

Moreover, from Remark 28, $\lim_{N\to\infty}\mathrm{tr}(A_l\cdot B_k) = 0$ for all l and k, therefore $\mathcal{A}$ and $\mathcal{A}^t$ are *not complex second order free.* □

It is an open question at this moment if there exist $\mathcal{A}$, $\mathcal{B}$ two ensembles of random matrices with second order limit t-distribution such that $\mathcal{A}$ and $\mathcal{B}$ are asymptotically complex second order free, but not real second order free.

Remark 30. We can show a stronger version of the result from Remark 28, more precisely

Proposition 31. *If $\mathcal{A} = \{A_N, B_N\}_N$ is an ensemble of random matrices such that*

$$\sup_{N\to\infty}|E(\,Tr(AB))| < \infty$$
$$\sup_{N\to\infty}|E(\,Tr(A)\,Tr(B))| < \infty$$

and $\{U_N\}_N$ is an ensemble of Haar unitary matrices independent from $\mathcal{A}$, then $\lim_{N\to\infty} E\left(tr(UAU^*(UBU^*)^t)\right) = 0$.

Proof. Denote $A = [a_{i,j}]_{i,j=1}^N$ and $B = [b_{i,j}]_{i,j=1}^N$. Then

$$E(\mathrm{tr}(UAU^*(UBU^*)^t)) = \frac{1}{N}\sum_{\substack{1\le i_k\le N\\ 1\le|k|\le 4}} E(u_{i_1,i_{-1}}a_{i_{-1},i_{-2}}\overline{u_{i_2,i_{-2}}}u_{i_3,i_{-3}}b_{i_{-3},i_{-4}}\overline{u_{i_4,i_{-4}}})$$

$$= \frac{1}{N}\sum_{\substack{1\le i_k\le N\\ 1\le|k|\le 4}} E(a_{i_{-1},i_{-2}}b_{i_{-3},i_{-4}})\cdot E(u_{i_1,i_{-1}}\cdot u_{i_3,i_{-3}}\cdot\overline{u_{i_2,i_{-2}}}\cdot\overline{u_{i_4,i_{-4}}}).$$

From Theorem 9, the right-hand side summand cancels unless $(i_{-1}, i_{-3}) \in \{(i_{-2}, i_{-4}), (i_{-4}, i_{-2})\}$. If $(i_{-1}, i_{-3}) = (i_{-2}, i_{-4})$, then from Remark 5 and Theorem 9, the right-hand side of the above equation equals

$$\frac{1}{N}E\left(\mathrm{Tr}(A)\mathrm{Tr}(B)\right)\sum_{\substack{1\le k\le N\\ 1\le k\le 4}} E(u_{i_1,j}\cdot u_{i_3,k}\overline{u_{i_2,j}}\cdot\overline{u_{i_4,k}}) = O(N^{-1}).$$

Similarly, if $(i_{-1}, i_{-3}) = (i_{-4}, i_{-2})$, then the summation equals

$$\frac{1}{N} E\left(\mathrm{Tr}(AB)\right) \sum_{\substack{1\leq k\leq N \\ 1\leq k\leq 4}} E(u_{i_1,j} \cdot u_{i_3,k}\overline{u_{i_2,j}} \cdot \overline{u_{i_4,k}}) = O(N^{-1}).$$

□

References

[B] Ph. Biane, Representations of symmetric groups and free prbability, *Adv. Math.*, 138(1) (1998), 126-181

[CS] B. Collins and P. Śniady, Integration with Respect to the Haar Measure on Unitary, Orthogonal, and Symplectic Group, *Comm. Math. Phy.*, **264** (2006), 773-795.

[M] M. L. Mehta , Random Matrices, Academic Press, 1991

[MPR] J. Mingo, M.Popa, E. Redelmeier, Second Order Freeness and Haar Orthogonal Matrices, preprint

[MSS] J. Mingo, R. Speicher, P.Sniady, Second Order Freeness and Fluctuations of Random Matrices: II. Unitary Random Matrices

[MS] J. Mingo, R. Speicher, Second Order Freeness and Fluctuations of Random Matrices: I. Gaussian and Wishart matrices and Cyclic Fock spaces

[NS] A. Nica and R. Speicher, *Lectures on the Combinatorics of Free Probability*, Cambridge Univ. Press, 2006.

[R] C. E. I. Redelmeier, *Real second-order freeness and the asymptotic real second-order freeness of several real matrix ensembles*, arXiv:1101.0422

[VOI] D.-V. Voiculescu, *Limit laws for random matrices and free products*. Invent. Math. 104 (1991), no. 1, 201220

[VDN] D.-V. Voiculescu, K. J. Dykema, A. Nica. *Free random variables. A noncommutative probability approach to free products with applications to random matrices, operator algebras and harmonic analysis on free groups*. CRM Monograph Series, 1. American Mathematical Society, Providence, RI, 1992

Department of Mathematics and Statistics, Queen's University, Jeffery Hall, Kingston, Ontario, K7L 3N6, Canada

E-mail address: mingo@mast.queensu.ca

Department of Mathematics and Statistics, Queen's University, Jeffery Hall, Kingston, Ontario, K7L 3N6, Canada, and

Institute of Mathematics "Simion Stoilow" of the Romanian Academy, P.O. Box 1-764, Bucharest, RO-70700, Romania

E-mail address: popa@mast.queensu.ca