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SUMMARY OF Ph.D. THESIS

On the spectrum of geometric differential operators on Riemannian manifolds

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Abstract

The first aim of this thesis is to study the behavior of the spectrum of the Dirac operator on degenerating families of Riemannian surfaces, when the length of a simple closed geodesic shrinks to zero, under the hypothesis that the spin structure along the pinched geodesic is non-trivial. It is well-known that the spectrum of an elliptic differential operator on a compact manifold varies continuously under smooth perturbations of the metric. The difficulty of our problem arises from the non-compactness of the limit surface, which is of finite area with two cusps. The main tool for this investigation is to construct an adapted pseudodifferential calculus (in the spirit of the celebrated b-algebra of Melrose) which includes both the family of Dirac operators on the family of compact surfaces and the Dirac operator on the limit (non-compact) surface, together with their resolvents.

The second aim of this work is to investigate heat kernel asymptotics for real powers of Laplacians. Let us fix a generalised Laplacian Δ (satisfying certain conditions) acting on smooth functions over a closed oriented manifold M. We first study the small-time heat kernel asymptotics of Δ^r, $r \in (0,1)$, along the diagonal of $M \times M$, and in a compact set away from it. Furthermore, we prove the non-triviality of the coefficients and the non-locality of some of the coefficients. In the special case $r = 1/2$, we give an *uniform description* of the transition between the on- and off-diagonal behavior by proving that the heat kernel of $\Delta^{1/2}$ is a polyhomogeneous conormal function on a certain heat blow-up space.

Contents

Chapter 1

Introduction

The aim of this thesis is to study the behavior of certain elliptic operators and their spectra in degeneracy problems, and furthermore, to investigate heat kernel asymptotics for real powers of Laplacians. The common ground of these two subjects is the analysis of functions having asymptotic expansions containing logarithmic terms, and the set-up of blow-ups of manifolds with corners.

A philosophical motivation to study such problems arises from physics. On one hand, a particle in classical mechanics is regarded as a point in $T\mathbb{R}^3$ (the tangent space of $\mathbb{R}^3$): a point in $\mathbb{R}^3$ and a vector representing its velocity. The trajectories of such particles are geodesics of the space. On the other hand, in quantum mechanics, a particle is an L^2 function. Its velocity is encoded in the Fourier transform and it evolves under the Schrödinger equation:

$$\frac{\partial f}{\partial t} = i\Delta_x f.$$

The periodic states of such particles are given by the eigenfunctions of the Laplacian, and the frequencies are related to the eigenvalues.

Following L. Schwartz [59], one can regard differential operators as *integral* operators given by distributional kernels. The theory of pseudo-differential operators on compact Riemannian manifolds, due to Hörmander [36] among others, provides a framework for the study of geometric elliptic differential operators (such as the Laplace operator or the Dirac operator) and their solutions. The algebra of pseudo-differential operators contains parametrices and the resolvent family of such operators. As a consequence, the normalised eigenfunctions of the Laplace operator form a Hilbert basis for the space of L^2 functions on a compact Riemannian manifold (see for instance [17]).

The classic pseudodifferential calculus is intimately related to index theory results such as the Atiyah-Patodi-Singer Theorem (see e.g. [5]-[7]). This formula computes the index of an elliptic differential operator on a compact, oriented manifold with boundary in terms of certain topological data of the manifold and spectral data of the boundary, namely the so-called eta invariant of the Dirac operator induced on the boundary.

There are ways to generalise the theory of pseudodifferential operators for open manifolds. Melrose's b-calculus [45] is a framework for dealing with differential operators on compact manifolds *with boundary*. A beautiful consequence of the b-calculus is that it provides a proof for a generalization of the Atiyah-Patodi-Singer Index Theorem in the context of compact manifolds with boundary with metrics having complete asymptotically cylindrical ends (exact b-metrics).

One of the central ideas in Melrose's machinery is to describe the distributional kernels of pseudodifferential operators as *polyhomogeneous conormal distributions* on manifolds with corners obtained through iterations of blow-ups. This fruitful idea led to the development of many other calculi: the ϕ-calculus [49], the zero-calculus [50], the edge-calculus [41], the scattering calculus (see e.g. [35]), the b-surgery calculus [42] and the φ-surgery calculus [2]. All these pseudodifferential calculi have proven to be powerful tools in solving a wide range of analytic problems on non-compact manifolds.

The present thesis fits into the framework of microlocal analysis on manifolds with corners and uses a pseudodifferential calculus constructed via the Melrose machinery.

The spectrum of the Dirac operator on degenerating surfaces

Context and Motivation

Let X be a smooth compact oriented surface of genus $g \geq 2$ and denote by $\mathcal{M}_{-1}(X)$ the set of hyperbolic metrics on S. By the classical Poincaré-Koebe Uniformization Theorem of Riemann surfaces (see e.g. [4] for a self-contained proof), $\mathcal{M}_{-1}(X)$ is in one-to-one correspondence with the set of complex structures on X. The Teichmüller space $\mathcal{T}_g$ is defined by factoring the set $\mathcal{M}_{-1}(X)$ by the connected component of id_M in the group of diffeomorphisms acting on X. It is known that $\mathcal{T}_g$ is a complex manifold of dimension $3g-3$ endowed with a non-complete Riemannian metric, the Weil-Peterson metric, and furthermore it is a Kähler manifold. We describe below a process to consider a path in $\mathcal{T}_g$ escaping towards infinity.

Let $\gamma \subset X$ be a simple closed geodesic in X. We consider a smooth family of metrics $(g_t)_{t\geq 0}$ on $X \setminus \gamma$ such that locally near γ

$$g_t = \frac{dx^2}{x^2+t^2} + \left(x^2+t^2\right) dy^2.$$

We remark that g_t does not need to be hyperbolic everywhere, but only locally, near the pinched geodesic γ. For $t > 0$, (X, g_t) is a compact surface, while $(X \setminus \gamma, g_0)$ is a non-compact surface with two cusps. We call this a *pinching process* along the geodesic γ.

In fact, this construction is valid when we pinch simultaneously several (up to $3g-3$) disjoint simple geodesics, but for simplicity we will treat below the case where we only pinch one geodesic.

If we choose the family of metrics $(g_t)_{t\geq 0}$ in the set $\mathcal{M}_{-1}(X)$, the pinching process described above corresponds to a path towards the boundary of the Teichmüller space. Such degeneration phenomena were studied by Ji [39], Bär [8], Schulze [58], and Stan [61].

It is well-known that the spectrum of a geometric elliptic differential operator like the Laplacian or the Dirac operator varies continuously under smooth perturbations of the metric. Our aim is to study the continuity of the spectrum of the Dirac operator during a pinching process. The difficulty of the problem arises from the non-compactness of the limit surface.

Key point

There are phenomena when the Laplacian has discrete spectrum at the limit of degeneracy process, for example the Laplacian acting on forms [28], and the magnetic Laplacian [27]. While it is true that the spectrum of the scalar Laplacian becomes continuous at the limit of a pinching process, Bär [8] proved that under some *invertibility condition* on each cusp, the spectrum of the Dirac operator remains discrete at the limit. More precisely, if we glue a circle at the "end" of each cusp, we require the spin structure to be non-trivial along this circle, which is equivalent to the invertibility of the Dirac operator on the circle defined with respect to the induced spin structure. Later on, Moroianu [54] generalised this result on manifolds of higher dimension and for a larger class of metrics. He also deduced a Weyl's law for the open manifold at the limit.

Idea of the solution

The main tool for investigating the continuity of the spectrum of the Dirac operator during a pinching process is to construct an adapted pseudodifferential calculus (in the spirit of the celebrated b-algebra of Melrose) which includes both the family of Dirac operators $\not{D}_t$ on the family of compact surfaces $(X, g_t)_{t>0}$ and the Dirac operator $\not{D}_0$ on the limit (non-compact) surface $(X \setminus \gamma, g_0)$. This adapted pseudodifferential calculus is closely related to the cusp calculus [51] (a particular case of the φ-calculus [49]). More precisely, it is the cusp-calculus with a time parameter, and we denote it by $\Psi_{\mathrm{cp}}^{*,*,*}(X)$.

Albin, Rochon, Sher [2] introduced the fibered cusp calculus with a parameter in order to study the spectrum of the Hodge Laplacian having coefficients in a flat bundle on a compact manifold which degenerates to a manifold with fibered cusps. Our cusp-surgery calculus is a particular case of their φ-surgery calculus, since they treat the more complicated case when the boundary fibrates over another closed manifold. We believe that it is worth including in Chapter 3 all the details of the construction, since the focus of our investigation regards fully-elliptic differential operators, leading to different and more straightforward proofs.

Intuitively, we will describe the cusp-surgery pseudodifferential operators in $\Psi_{\mathrm{cp}}^{*,*,*}(X)$ as distributions on a certain blown-up space $\mathrm{X}^2_{\mathrm{cp}}$, conormal to the closure of the $(0,\infty) \times \mathrm{Diag}$, where Diag is the diagonal inside $X \times X$ (see Fig. 3.6). Furthermore, we will impose certain polyhomogeneous behavior of the distributions towards the boundary faces of $\mathrm{X}^2_{\mathrm{cp}}$.

A challenging but crucial result in constructing this calculus is the so-called Composition Theorem, which establishes that the composition of two operators in the calculus also belongs to it. As customary, this theorem is proved through the use of a triple space, along with multiplication of conormal distributions, and the Pull-back and Push-forward Theorems for conormal distributions (see for instance [44], [31]), however we stress that the geometric structure of the triple space and of the companion b-fibrations are by no means trivial.

To each cusp-surgery pseudodifferential operator $A \in \Psi_{\mathrm{cp}}^{*,*,*}(X)$, we will associate three leading symbols: $\sigma_{\mathrm{cp}}(A)$, $\mathcal{N}(A)$, and $\mathcal{T}(A)$. More precisely, the cusp-parameter symbol $\sigma_{\mathrm{cp}}(A)$ is the leading term in the principal symbol of the conormal distribution k_A. The normal operator $\mathcal{N}(A)$ is a normalization of the restriction of A to the cusp front face ff_{c}. Finally, the temporal operator $\mathcal{T}(A)$ is the normalized restriction of A to tb, the lift of the temporal boundary $\{t = 0\} \times X \times X \subset [0,\infty) \times X \times X$ to the double space $\mathrm{X}^2_{\mathrm{cp}}$ (see Fig. 3.3).

An algebraic result ensures that if an operator $A \in \Psi_{\mathrm{cp}}^{*,*,*}(X)$ has all the three symbols invertible, one can construct a parametrix modulo residual operators, i.e., operators belonging to $\Psi_{\mathrm{cp}}^{-\infty,-\infty,-\infty}(X)$.

Main results

We manage to merge the family of Dirac operators $\left(\not{D}_t\right)_{t>0}$ on the Riemannian surfaces (X, g_t), together with the Dirac operator $\not{D}_0$ on the limit (non-compact) surface $(X \setminus$

$\gamma, g_0)$, to obtain a cusp-surgery differential operator

$$\not{D} \in \Psi_{\mathrm{cp}}^{1,1,0}(X).$$

A crucial result in Chapter 3 is to prove that the normal operator $\mathcal{N}(\not{D})$ of $\not{D}$ is invertible. Here we rely on the hypothesis of non-triviallity of the spin structure along the geodesic γ. We are able to prove that if λ is not an eigenvalue of $\not{D}_0$, then the resolvent family $(\not{D} - \lambda)^{-1}$ for small time t belongs to the calculus.

Theorem 1.1. *Let X be a compact oriented surface and let $\gamma \subset X$ be a simple closed geodesic. Consider a smooth family of metrics $(g_t)_{t\geq 0}$ on $X \setminus \gamma$ which near γ are given by:*

$$g_t = \frac{dx^2}{(x^2+t^2)^2} + \left(x^2+t^2\right) dy^2.$$

Furthermore, consider $(\not{D}_t)_{t\geq 0}$ the family of Dirac operators corresponding to the family of metrics $(g_t)_{t\geq 0}$ and to a fixed non-trivial spin structure $\mathrm{P}_{\mathrm{Spin}(2)} X$ relatively to γ. If $\lambda \in \mathbb{R} \setminus \operatorname{spec} \not{D}_0$, then there exists $t_0(\lambda) > 0$ such that the operator $(\not{D}_t - \lambda)$ is invertible for every $t \leq t_0(\lambda)$, and the resolvent is a cusp-surgery operator:

$$(\not{D} - \lambda)^{-1} \in \Psi_{\mathrm{cp}}^{-1,-1,0}(X).$$

We also prove the convergence of the spectral projectors under degeneration.

Theorem 1.2. *Suppose that we are under the hypothesis of Theorem 1.1, and let λ_0 be an eigenvalue for the limit operator $\not{D}_0$. Consider $\epsilon > 0$ such that*

$$[\lambda_0 - \epsilon, \lambda_0 + \epsilon] \cap \operatorname{spec} \not{D}_0 = \{\lambda_0\}.$$

Then the spectral projector $\mathrm{P}_{[\lambda_0-\epsilon,\lambda_0+\epsilon]}$ belongs to $\Psi_{\mathrm{cp}}^{-\infty,-\infty,0}(X)$. More precisely, its Schwartz kernel is smooth on $[0,\infty) \times X \times X$ and moreover, it vanishes rapidly at $\{t=0\} \times \gamma$.

If $A \in \Psi_{\mathrm{cp}}^{m,\alpha,\beta}(X)$ is a trace-class cusp-surgery operator, the cusp-trace is a function ${}^{\mathrm{cp}}\operatorname{Tr}(A) : [0,\infty) \longrightarrow \mathbb{C}$ which associates to each time t the integral over the t-time slice in the diagonal plane $\Delta \subset \mathrm{X}_{\mathrm{cp}}^2$ (see Fig. 3.6). In fact, for $t > 0$, the cusp trace associates to A exactly the L^2-trace of the operator A at time t acting on (X, g_t).

Theorem 1.3. *Let $A \in \Psi_{\mathrm{cp}}^{m,\alpha,\beta}(X)$ be a trace-class cusp-surgery pseudodifferential operator, meaning that the orders satisfy the following inequalities:*

$$m < -2, \quad \alpha < -1, \quad \beta \leq 0. \tag{1.1}$$

i) If $\alpha - \beta \notin \mathbb{Z}$*, then*

$$^{\mathrm{cp}}\operatorname{Tr} A \in t^{-\alpha}\mathcal{C}^\infty[0,\infty) + t^{-\beta}\mathcal{C}^\infty[0,\infty).$$

ii) If $\alpha - \beta \in \mathbb{Z}$*, then*

$$^{\mathrm{cp}}\operatorname{Tr} A \in t^{\min(-\alpha,-\beta)}\mathcal{C}^\infty[0,\infty) + t^{\max(-\alpha,-\beta)}\log t \cdot \mathcal{C}^\infty[0,\infty).$$

Notice that for $t > 0$, the cusp-surgery trace exists whenever for $\alpha > -1$, and is clearly a $\mathcal{C}^\infty$ function. The relevance of the result above is that it describes the behavior of the cusp surgery trace towards $\{t = 0\}$. More precisely, it is of class $\mathcal{C}^1$, and in particular, it is continuous. As a corollary, we study the cusp trace of the resolvent of the Dirac operator.

Theorem 1.4. *In the hypothesis of Theorem 1.1, consider an integer* $k \geq 3$*, and let* $\lambda \in \mathbb{R}$ *be in the complement of the spectrum of* $\not{D}_0$*. Denote by*

$$\mathrm{R}(\lambda) = (\not{D} - \lambda)^{-1}$$

the resolvent of the Dirac operator. Then the k^{th} *power of the resolvent is trace-class, and its cusp-surgery trace* $^{\mathrm{cp}}\operatorname{Tr}\left(\mathrm{R}(\lambda)^k\right)$ *is of Hölder class* $\mathcal{C}^{k-1,\alpha}$*, for any* $\alpha \in (0,1)$*.*

Remark that $^{\mathrm{cp}}\operatorname{Tr}\left(\mathrm{R}(\lambda)^k\right)$ is smooth for $t > 0$ so, as above, the content of this theorem lies in the polyhomogeneous behavior of the cusp-surgery trace as a function of t towards $t \to 0$.

Theorem 1.5. *Let* $\lambda, \lambda_0 \in \mathbb{R}$ *such that the cusp differential operators* $\not{D}_0^2 - \lambda$ *and* $\not{D}_0^2 - \lambda_0$ *are invertible. Denote the resolvents of the squared Dirac operator by*

$$\widetilde{\mathrm{R}}(\lambda) := \left(\not{D}^2 - \lambda\right)^{-1}, \qquad \widetilde{\mathrm{R}}(\lambda_0) := \left(\not{D}^2 - \lambda_0\right)^{-1}.$$

Then the relative resolvent $\widetilde{\mathrm{R}}(\lambda) - \widetilde{\mathrm{R}}(\lambda_0)$ *is trace-class and its cusp-surgery trace behaves as a function of t as* $t \searrow 0$ *is as follows:*

$$^{\mathrm{cp}}\operatorname{Tr}\left(\mathrm{R}(\lambda) - \mathrm{R}(\lambda_0)\right) \in \mathcal{C}^\infty[0,\infty) + t^2\log t\, \mathcal{C}^\infty[0,\infty).$$

We hope to apply this theorem to improve the result of Stan [61] for the asymptotic behavior of the Dirac Selberg zeta function on degenerating hyperbolic surfaces.

Heat kernel asymptotics for real powers of Laplacians

Let $\mathcal{E}$ be a hermitic vector bundle over a closed, oriented Riemannian manifold M of dimension n. Consider a non-negative self-adjoint generalized Laplacian Δ acting on the sections of the bundle $\mathcal{E}$. For example, if $\not{D}$ is a Dirac operator corresponding to a spin closed Riemannian manifold, then $\not{D}^*\not{D}$ is such a generalized Laplacian acting on the sections of the spinor bundle.

Background

A classical result due to Minakshisundaram-Pleijel [43] tells us that the heat kernel p_t of Δ (i.e. the Schwartz kernel of the operator $e^{-t\Delta}$) has a small-time asymptotic expansion near the diagonal:

$$p_t(x,y) \overset{t\searrow 0}{\sim} t^{-n/2} e^{-\frac{d(x,y)^2}{4t}} \sum_{j=0}^{\infty} t^j a_j(x,y), \tag{1.2}$$

where $d(x,y)$ is the geodesic distance between x and y. Moreover, the a_j's are recursively defined as solutions of certain ODE's along geodesics (see for instance [12], [14]). Using this result, one can prove the Weyl's law on counting the eigenvalues of a Laplacian (see e.g. [12]). Furthermore, if $\not{D}$ is a twisted Dirac operator, then the asymptotic expansion (1.2) applied to the generalized Laplacian $\Delta = \not{D}^*\not{D}$ plays a leading role in proving the Atiyah-Singer index theorem (see for instance [13], [15], [23]).

Small time heat asymptotic for real powers of Δ

The central object of study in Chapter 4 (already published as a paper in [3]) is the Schwartz kernel h_t of the operator $e^{-t\Delta^r}$, where $r \in (0,1)$. More precisely, we first investigate separately the short-time asymptotic expansion of h_t in $[0,\infty) \times \mathrm{Diag}$, and towards $[0,\infty) \times K$, where $K \subset M \times M$ is a compact set disjoint from the diagonal. The main idea used in this investigation is to relate the heat operator $e^{-t\Delta}$ and the operator $e^{-t\Delta^r}$ to the family of pseudodifferential operators Δ^s, $s \in \mathbb{C}$. One can obtain these connections by the Mellin and Inverse Mellin formulæ.

The Schwartz kernel h_t of the operator $e^{-t\Delta^r}$ is $\mathcal{C}^\infty$ on $[0,\infty) \times (M \times M \setminus \mathrm{Diag})$ and it vanishes at least at order 1 at $t = 0$. We give a precise description of the Taylor coefficients of h_t as $t \searrow 0$ in terms of the Schwartz kernels q_s of the complex powers Δ^s, $s \in \mathbb{C}$.

The asymptotic along the diagonal of h_t is more complicated than that of the classical heat kernel p_t in (1.2). It depends on the parity of n (like in [9]) and on the rationality of r. The most interesting case occurs when logarithmic terms appear. This happens only if n is odd, $r = \frac{\alpha}{\beta}$ is rational, and the denominator β is even.

Theorem 1.6. *Let Δ be a non-negative self-adjoint generalized Laplacian Δ acting on the sections of a hermitic vector bundle $\mathcal{E}$ over a closed manifold M of dimension n. Let $a_j(x,x)$ be the coefficients along the diagonal of the heat kernel p_t of Δ in (1.2). If n is odd, $r = \frac{\alpha}{\beta}$ is rational and its denominator β is even, then the asymptotic expansion of the Schwartz kernel h_t of the operator $e^{-t\Delta^r}$, $r \in (0,1)$ along the diagonal when $t \searrow 0$ is the following:*

$$\begin{aligned} h_{t|_{\mathrm{Diag}}} \overset{t\searrow 0}{\sim} & \sum_{j=0}^{(n-1)/2} t^{-\frac{n-2j}{2r}} \cdot A_{-\frac{n-2j}{2r}} + \sum_{\substack{j=1 \\ \alpha \nmid 2j+1}}^{\infty} t^{\frac{2j+1}{2r}} \cdot A_{\frac{2j+1}{2r}} + \sum_{\substack{j=1 \\ \frac{\beta}{2} \nmid j}}^{\infty} t^{j} \cdot A_j \\ & + \sum_{\substack{l=1 \\ l \text{ odd}}}^{\infty} t^{l\frac{\beta}{2}} \cdot A_{l\frac{\beta}{2}} + \sum_{\substack{l=1 \\ l \text{ odd}}}^{\infty} t^{l\frac{\beta}{2}} \log t \cdot B_{l\frac{\beta}{2}}. \end{aligned} \tag{1.3}$$

Moreover,

$$A_j(x) = \frac{(-1)^j}{j!} \cdot q_{rj}(x,x).$$

We prove similar expansions in all the other cases and we give explicit formulæ for all the coefficients appearing in (1.3).

Motivation to study real powers of Laplacians

If P is a scalar positive elliptic self-adjoint pseudodifferential operator of integer order, then the asymptotic expansion of the heat kernel of e^{-tP} was studied by Duistermaat and Guillemin [20]. Their result was generalized by Grubb [33, Theorem 4.2.2] in the context of fiber bundles when the order of P is positive, not necessary an integer. Later on, Bär and Moroianu [9] studied the short-time asymptotic behavior of the heat kernel of $\Delta^{1/m}$, $m \in \mathbb{N}^*$, for a strictly positive self-adjoint generalized Laplacian Δ. In Theorem 1.6, we obtain the vanishing of some terms appearing in [33, Corollary 4.2.7] in our particular case when $P = \Delta^r$ is a real power of a self-adjoint non-negative generalized Laplacian Δ, $r \in (0,1)$. Furthermore, we prove that in general the remaining terms do not vanish (see Theorem 4.1).

Non-locality of some coefficients

It is well-known that the heat coefficients in (1.2) of a generalized Laplacian are locally computable in terms of the curvature of the connection on $\mathcal{E}$, the Riemannian metric of M and their derivatives (see e.g. [12]). This is no longer the case for the coefficients of positive integer powers of t from Theorem 1.6:

Theorem 1.7. *If r is irrational, then the heat coefficients A_j for $j \in \mathbb{N}$, $j \geq 1$ from Theorem 1.6 are not locally computable. If $r = \frac{\alpha}{\beta}$ is rational, then A_j are not locally computable for $j \in \mathbb{N} \setminus \{l\beta : l \in \mathbb{N}\}$. Moreover, all the other coefficients can be written in terms of the heat coefficients of $e^{-t\Delta}$, thus they are locally computable.*

The heat kernel as a conormal section

The classic heat kernel p_t can be regarded as a polyhomogeneous conormal function on a certain blow-up space. More precisely, Melrose introduced the heat space M_H^2 by performing a parabolic blow-up of the diagonal in $M \times M$ at time $t = 0$. The new space M_H^2 is a manifold with corners with boundary hypersurfaces given by the boundary defining functions ρ and ω_0. Then the classic heat kernel p_t belongs to the space $\rho^{-n}\mathcal{C}^\infty(M_H^2)$, and moreover, it vanishes rapidly at the boundary face $\{\omega_0 = 0\}$ (see [45, Theorem 7.12]).

In the special case $r = 1/2$, we gave a *uniform* description of the transition between the on- and off-diagonal behavior of h_t as $t \searrow 0$. In this case, we can understand better the heat operator $e^{-t\Delta^{1/2}}$ on a *homogeneous* (rather than parabolic) *blow-up* heat space $\mathrm{M}_{\mathrm{heat}}$, obtained by the standard blow-up of $\{0\} \times \mathrm{Diag}$ in $[0, \infty) \times M \times M$.

Theorem 1.8. *If n is even, then the Schwartz kernel h_t of the operator $e^{-t\Delta^{1/2}}$ belongs to $\rho^{-n}\omega_0 \cdot \mathcal{C}^\infty(\mathrm{M}_{\mathrm{heat}})$. If n is odd, then $h_t \in \rho^{-n}\omega_0 \cdot \mathcal{C}^\infty(\mathrm{M}_{\mathrm{heat}}) + \rho \log \rho \cdot \omega_0 \cdot \mathcal{C}^\infty(\mathrm{M}_{\mathrm{heat}})$.*

Chapter 5

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