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STOCHASTIC INTEGRATION ON RIEMANNIAN MANIFOLDS FROM THE POINT OF VIEW OF FUNCTIONAL ANALYSIS

SUMMARY OF Ph.D. THESIS

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CONTENTS

INTRODUCTION AND MOTIVATIONS

The aim of the present text is to answer the question: can we integrate a smooth 1-form along a continuos curve? When the curve is differentiable, the answer is in the affirmative and given by the concept of line integral. Whe the curve is no longer differentiable, there are multiple possible answers: Young integral, integration in the sense of the ”rough paths” theory, stochastic integration; the latter will be the subject of interest of this thesis.

Stochastic integration is not new: after several attempts by various mathematicians in the area of Wiener processes, Kiyoshi Itô obtained in 1944 the first coherent construction in $\mathbb{R}^n$ of a new (back then) type of ”line” integral, later named ”Itô integral”. An alternative to the Itô integral, that was initially given only a lukewarm reception by the probabilistic community, was developed by Ruslan Stratonovich in the USSR, during the ’60s, and was later named ”Stratonovich integral”. The history of the gradual emergence and clarification of these concept is, of course, much richer, but it does not form the subject of this thesis; a detailed chronological presentation thereof can be fond in [JP04].

If stochastic integration is not new, the approach that we put forward in this text is. Traditionally, the concept is presented in $\mathbb{R}^n$, as a chapter of stochastic calculus - therefore using the language and techniques of probability theory. The few texts that venture on Riemannian manifolds do not stray away from this language. Furthermore, even though the problem, as stated in the first paragraph, is a geometrically intrinsic one, the construction if stochastic integration on Riemannian manifolds is often performed by resorting to extrinsic methods, usually by using Whitney’s theorem to embed the manifold in some Euclidean space where the usual properties of the Brownian motion may be used. The present text attempts a completely new approach: on the one hand, stochastic integration will be obtained using functional analysis and Riemannian geometry techniques; on the other hand, all the proofs will be intrinsic, this presentation of the concept under study having the advantage of exposing its geometrical underpinnings, which has the tendency of remaining obscured and unexplored in the purely probabilistoc approaches. Moreover, the construction put forward will show that the Itô and Stratonovich integrals are just two of an infinite family of possible stochastic integrals, this family being though very explicitly described, and any two integrals of which being connected by a very explicit and simple relationship (thus allowing us to say that there exists essentially a single stochastic integral, all the others being simple variations thereof). In particular, we shall see that the Stratonovich integral is the stochastic integral with the most convenient geometrical properties (of all the stochastic integrals it having the most properties in common with the usual line integral), while the Itô integral has the most convenient analytical properties. In order to ease the navigation through this thesis, we shall give in the following a brief presentation of its chapters.

Chapter 1 is dedicated to an in-depth understanding of the Laplace operator on a Riemannian manifold (M, g) (in this text M will always be connected) and contains classical, basic results: the Laplace-Beltrami operator Δ is defined and, in order to be able to use functional analysis results, the Friedrichs extension L of $-\Delta$ is constructed and its spectrum and a necessary condition for essential slef-djointness are studied. An essential object presented next is the heat kernel, understood as the minimal positive fundamental solution of the heat equation: its construction on arbitrary Riemannian manifolds being known, it is only sketched here (details can be found in the cited references), only the construction on compact Riemannian manifolds being carried out in detail. As the heat kernel on manifolds does not, in general, have an explicit formula, the chapter concludes with the statement and proof of some (upper and lower) Gaussian estimates of it.

Chapter 2 is entirely devoted to the construction of the Wiener measure on curve spaces in Riemannian manifolds. As in the case of stochastic integrals, this is constructed in the literature using probabilistic methods. Until 2011, purely geometric, intrinsic constructions were not known; in 2011, however, Christian Bär and Frank Pfäffle presented such an approach, which we adopt in this text. Thus, we will quickly obtain the Wiener measure on the space of (arbitrary, not necessarily continuous) trajectories in the manifold using the Kolmogorov extension theorem, all the effort then being to show that this measure is concentrated on continuous (in fact, even Hölder continuous) trajectories. This result is guaranteed if the the Kolmogorov continuity condition is satisfied. This, however, is an integral condition which, unfortunately, is not verified on arbitrary Riemannian manifolds, so we shall be forced to resort to an indirect approach: first we shall verify it on compact manifolds, then on relatively compact domains with smooth boundary (using the concept of "Riemannian double" to reduce the problem to the previous case); this will allow us to construct the Wiener measure intrinsically associated with the space of trajectories contained in such domains. Finally, we shall consider an exhaustion of M with such domains and define the Wiener measure associated with the entire manifold as the limit of the Wiener measures associated with the domains in the exhaustion. The chapter concludes with a classical application, namely the Feynman-Kac formula, which presents the solutions of the heat equation with given initial conditions as integrals with respect to the Wiener measure.

Chapter 3 presents the first original result of the thesis. Since L is a self-adjoint operator in $L^2(M)$, it will generate a semigroup of operators $(\mathrm{e}^{-tL})_{t\geq 0}$ called the "heat semigroup"; it can be shown that the heat kernel constructed in chapter 1 is the integral kernel corresponding to the heat semigroup. It is reasonable then to expand the context in which we talk about the heat kernel in the following way: we consider a Hermitian vector bundle E over a Riemannian manifold M, endowed with a Hermitian connection ∇ from which the operator $\nabla^*\nabla$ is immediately obtained, the Friedrichs extension of which generates in turn a "heat" semigroup in E. We ask ourselves the question: does this semigroup admit, in turn, an integral kernel in the fiberwise sense, with good integrability properties? When M is compact the result is known and the construction is analogous to that in chapter 1. For arbitrary manifolds, however, a purely analytic-functional construction of this kernel was given only recently, in 2015, by Batu Güneysu, under very general assumptions. This chapter offers an alternative construction, completely different from that of Güneysu, under slightly more restrictive assumptions (namely, we accept the existence of the heat kernel constructed in chapter 1) but having the advantage of being more intuitive and also presenting a series of concrete approximations of this kernel. If Güneysu's central tool is Lebesgue's differentiability theorem, the central tool in our construction proposed in this chapter is Chernoff's approximation theorem of semigroups of operators in Banach spaces in the strong operator topology. The chapter concludes with the study of some regularity properties of the heat kernel in bundles thus constructed.

The 4th chapter presents the second original result of this thesis, the main one in fact, namely the construction of stochastic integrals using only functional analysis techniques and Riemannian geometry, and the classification of these integrals. Using the existence of the heat kernel in vector bundles obtained in the previous chapter, we start by obtaining an essentially bounded function on the space of continuous trajectories in M, which we shall later show to have absolute value 1 by some technical reasoning, based on the same Chernoff theorem used above. Moreover, we shall show that this function is even a 1-parameter unitary continuous group which, using Stone's theorem, we shall deduce to have a self-adjoint generator, which will next be shown to be a measurable function. This function will later be shown to be the Stratonovich integral. On the other hand, starting from Borel regular probabilities P on the interval $[0,1]$, we shall construct some sums similar to the Riemann sums associated with a curvilinear integral, and we shall show that they have a limit in measure. If P is the Dirac measure δ_0, then the limit obtained will be the Itô integral itself, and if P is the Lebesgue measure on $[0,1]$ then the corresponding limit will be the Stratonovich integral. This motivates us to call "stochastic integral" any limit of such a Riemann sum associated with such a probability P. Even though it looks like there are as many stochastic integrals as there are Borel regular probabilities P, we shall see that in reality any two such probabilities with the same moment of order 1 produce the same stochastic integral, so the stochastic integrals understood in the above sense above are classified by the moments of order 1 of the regular Borel probabilities on $[0,1]$. Moreover, any two such stochastic integrals differ by a simple term that we shall be able to write explicitly, so the apparently infinite abundance of stochastic integrals essentially reduces to only one of them (any one), all others being translations of it by a very precise term. We shall obtain, in addition, some classical results of stochastic analysis, for example Itô's lemma (which in this theoretical framework will have an extremely short and simple proof), and a generalization of the Feynman-Kac formula corresponding to the situation where the heat propagation occurs in the presence of a magnetic field (mathematically encoded by a smooth 1-form, the Stratonovich integral of which will emerge naturally).

Chapter 5 improves some of the results from the previous one, under the slightly more restrictive assumption that the 1-forms we stochastically integrate have compact support. With this additional condition we shall show that the stochastic integrals are square integrable with respect to the Wiener measure. We shall also obtain an analogue of Itô's isometry, which will allow us to extend stochastic integrals from the space of smooth 1-forms with compact support to much larger spaces of 1-forms.

The last chapter, the 6th, follows a conceptual and technical structure similar to that of the 4th one. Its purpose is to show how the stochastic parallel transport in Hermitian bundles over Riemannian manifolds can be understood from a purely functional-analytic and geometric point of view, without resorting to concepts belonging to probability theory. We shall show that the stochastic parallel transport can naturally be understood as an square-integrable (in fact, even essentially bounded) section in a certain pull-back bundle, and that it is the limit of a series of concrete approximations in the topology of the Hilbert norm on this space of sections. The Feynman-Kac formula can be formulated in this context, too (and the formulation we shall give seems to be significantly more general than those currently in the literature), and it will make the stochastic parallel transport emerge naturally. If the considered bundle is trivial of rank 1, we recover the results from chapter 4 related to the stochastic Stratonovich integral.

1. THE LAPLACIAN, THE HEAT KERNEL AND THE DISTANCE FUNCTION ON RIEMANNIAN MANIFOLDS

This chapter aims to present, among the many properties of the distance function on a Riemann manifold, some of those that manifest themselves in relation to the Laplacian operator and the heat equation. The Minakshisundaram-Pleijel construction of the heat kernel on a compact Riemann manifold is also presented in detail.

1.1 The Laplacian on Riemannian manifolds

Let (M, g) be a connected (hence separable) Riemannian manifold with $\dim M = n$. Let $C(M)$ and $C_0(M)$ be the algebras of continuous and continuous functions with compact support, respectively, on M. Let Δ be the Laplacian acting on smooth functions, given in coordinates by

$$\Delta = \frac{1}{\sqrt{\det g}} \sum_{i,j} \partial_i (\sqrt{\det g}\, g^{ij} \partial_j) \ .$$

The volume form is given in coordinates by $\mathrm{vol} = \sqrt{\det g}\, \mathrm{d}x_1 \wedge \ldots \wedge \mathrm{d}x_n$. Let μ_M be the measure associated with the volume form, obtained by applying the Riesz theorem to the positive functional $C_0(M) \ni f \mapsto \int_M f \,\mathrm{vol} \in \mathbb{R}$. Let $d(x, y) = \inf_c \int_a^b \|c'(t)\| \,\mathrm{d}t$ be the distance associated to the Riemannian structure, where c sweeps through the set of smooth curves with $c(a) = x$ and $c(b) = y$. For $x \in M$ and $r > 0$ we shall write $B(x, r) = \{y \in M \mid d(x, y) < r\}$. If f is a continuous function we shall write $f_+ = \max(f, 0)$.

We shall consider the Laplace-Beltrami operator Δ to be densely-defined on $L^2(M)$ (in general, when considering some differential operator as acting on $L^2(M)$ we shall consider it defined on the dense subspace $C_0^\infty(M)$). This operator is not self-adjoint. Under these conditions, since many of the functional analysis theorems that we shall use require self-adjoint operators, we shall have to replace Δ with some self-adjoint extension in subsequent considerations; however, it is not clear if such extensions exist, and if there are several of them it is not clear how to choose a specific one among them. Fortunately, the following result shows that such extensions exist and, moreover, there is a maximal (and therefore unique) one among them in a sense that we shall specify immediately.

Teorema 1.1. *Every symmetric, lower-bounded, densely-defined operator $A : \mathrm{Dom}\, A \subseteq \mathcal{H} \to \mathcal{H}$ admits a self-adjoint extension $\hat{A}$ (called the Friedrichs extension) with the same lower bound. If A' is another symmetric extension of A with $\mathrm{Dom}\, A' \subseteq \widehat{\mathrm{Dom}\, A}$, then $\hat{A}|_{\mathrm{Dom}\, A'} = A'$.*

1.2 The metric completeness and the essential self-adjointness of the Laplacian

Teorema 1.2. *If M is metrically complete, then the operator $-\Delta : C_0^\infty(M) \subset L^2(M) \to L^2(M)$ is essentially self-adjoint in $L^2(M)$.*

1.3 The distance function and the spectrum of the Laplacian

The purpose of this section is to estimate the infimum $\lambda(M)$ of the spectrum of the positive-defined Laplacian using the distance function.

Teorema 1.3. *If there exists some Lipschitz function $f : M \to \mathbb{R}$ of Lipschitz constant 1, and some constant $c \geq 0$ such that $\Delta f \geq c$ in the distributional sense, then $\lambda(M) \geq \frac{c^2}{4}$.*

Teorema 1.4. *If there exists some Lipschitz function $f : M \to \mathbb{R}$ of Lipschitz constant 1, and some constant $c > 0$, such that $\lim_{x\to\infty} f(x) = \infty$ and $\mathrm{e}^{-cf} \in L^1(M)$, then $\lambda(M) \le \frac{c^2}{4}$.*

1.4 The heat kernel on Riemannian manifolds

The aim of this section is the construction of the heat kernel on Riemannian manifolds (in particular, we shall present its detailed construction on compact manifolds) and the obtention of lower and upper Gaussian bounds for it.

Definiția 1.5. *The **heat kernel** on M is the unique smooth positive solution $h : (0, \infty) \times M \times M \to [0, \infty)$ of the heat equation $(\partial_t - \Delta)u = 0$, that is minimal with respect to function inequality, subjected to the initial condition $\lim_{t\to 0+} \int_M h(t, x, y) f(x)\, \mathrm{d}x = f(y)$ for all $f \in C_b(M)$ and all $y \in M$.*

1.4.1 The construction of the heat kernel on compact Riemannian manifolds

In this section one constructs, with full details, the heat kernel on compact Riemannian manifolds, following the method of Minakshisundaram and Pleijel; this heat kernel will be obtained as the pointwise limit of a sequence of parametrices of the heat equation. The exposition essentially follows [Rosenberg97] and [BGM71]; several conventions and notations in the present text differ from the corresponding ones in the indicated references.

1.4.2 The heat kernel on non-compact Riemannian manifolds

So far we have constructed, with all the details, the heat kernel on compact manifolds. In the following we shall outline its construction on non-compact manifolds, providing the reader with the bibliographic references necessary to study the details. The strategy will be to build it on relatively compact domains with smooth boundary, and then use an exhaustion with such domains to obtain the heat kernel on the entire manifold. (We shall use the same approach in the next chapter to construct the Wiener measure.)

If U is a relatively compact domain with smooth boundary, the Minakshisundaram-Pleijel method can be adapted to produce the heat kernel on U, i.e. the only elementary solution of the heat equation on U, continuous on $\overline{U}$ and which vanishes on the boundary ∂U (the resulting kernel will be called the "Dirichlet heat kernel on U"). The difference from the method described so far for compact manifolds is the purely technical additional complication of ensuring continuity and vanishing at the boundary, but the idea is the same. Details can be found, somewhat briefly, in ch. VII of [Chavel84], but a full and even more general proof (in which the heat kernel is constructed on differential forms of any degree) is given in section 5 of [RaySi71].

Choosing now an exhaustion with relatively compact domains with smooth boundary $M = \bigcup_{i\ge 0} U_i$, and denoting by h_{U_i} the Dirichlet heat kernel on U_i, we define $h = lim_{i\to\infty} h_{U_i}$ as a pointwise limit. It is shown that this limit exists and does not depend on the chosen exhaustion, and that is satisfies the definition of the heat kernel on M. Details can be found in chap. VIII of [Chavel84] and in [Dodziuk83].

Teorema 1.6. *The heat kernel $h : (0, \infty) \times M \times M \to [0, \infty)$ enjoys the following properties:*

1. *$h > 0$;*
2. *h is smooth;*
3. *$h(\cdot, -, y)$ satisfies the (homogeneous) heat equation for all $y \in M$;*
4. *$\lim_{t\to 0} \int_M h(t, x, y)\, f(y)\, \mathrm{d}y = f(x)$ for all $x \in M$ and all $f \in C_b(M)$;*
5. *h is minimal among the fundamental smooth positive solutions of the heat equation;*
6. *h is unique with the above properties;*
7. *$h(t, x, y) = h(t, y, x)$ for all t, x, y;*
8. *h enjoys the "convolution property"*

$$\int_M h(u, y, p)\, h(v, p, z)\, \mathrm{d}p = h(u + v, y, z) \ ;$$

9. $h(t,x,\cdot) \in L^2(M)$ *for all* $t > 0$ *and* $x \in M$*; for reasons of symmetry, it follows that* $h(t,\cdot,y) \in L^2(M)$ *for all* $t > 0$ *and* $y \in M$*;*

10. $\int_M h(t,x,y)\,\mathrm{d}y \le 1$*; if* M *is complete, with the Ricci curvature bounded below, then* $\int_M h(t,x,y)\,\mathrm{d}y = 1$*;*

11. $h(t,\cdot,-)$ *is the integral kernel of* $\mathrm{e}^{-tH} : L^2(M) \to L^2(M)$ *for all* $t > 0$ *(*H *being the Friedrichs extension of* $-\Delta$*), that is* $(\mathrm{e}^{-tH} f)(x) = \int_M h(t,x,y)\, f(y)\,\mathrm{d}y$ *for all* $f \in L^2(M)$ *and almost all* $x \in M$*.*

1.5 Gaussian upper bounds for the heat kernel

Teorema 1.7. *There exists a continuous and bounded function* $r : M \to (0,\infty)$ *și* $C > 0$ *such that if* $x, y \in M$ *and* $t \ge t_0 > 0$ *then*

$$h(t,x,y) \le \frac{C}{[\min(2t_0, r(x)^2)\min(2t_0, r(y)^2)]^{\frac{n}{4}}} \left[1 + \frac{d(x,y)^2}{t}\right]^{\frac{n}{2}} \exp\left[-\frac{d(x,y)^2}{4t} - \lambda(M)(t-t_0)\right] .$$

1.6 Gaussian lower bounds for the heat kernel

To close this chapter, we shall obtain Gaussian lower bounds for the heat kernel using Harnack's parabolic inequality which we shall prove next. We shall assume that M is compact, so there exists $K > 0$ such that $\mathrm{Ric} \ge -K$, where Ric is the Ricci curvature of M.

Teorema 1.8. *For all* $T > 0$*, if* M *is compact, and* $x, y \in M$ *and* $0 < t < T$*, then*

$$h(t,x,y) \ge t^{-\frac{n}{2}} \exp\left[-\frac{d(x,y)^2}{4t}\left(1 + \frac{2KT}{3}\right) - \frac{nKt}{2}\right].$$

2. A GEOMETRIC CONSTRUCTION OF THE WIENER MEASURE ASSOCIATED TO A RIEMANNIAN MANIFOLD

The aim of this chapter is to construct the Wiener measure on the space of continuous curves in a Riemannian manifold, using only geometric methods and the Kolmogorov extension theorem. This stems from the desire of emphasizing the Riemannian dimension of the problem, thus avoiding the existing probabilistic constructions in the literature, based on stochastic differential equations or on Cartan's rolling map. Although independently conceived, this text is very similar to that of [BP11] which uses much the same ideas and techniques; the cited text sometimes gives simpler solutions (e.g. it does not use Gaussian estimates of the heat kernel but a much simpler estimate), sometimes it is more cumbersome than the one proposed here (the discussion of intrinsic distance on regular domains is unnecessarily complicated).

2.1 Kolmogorov's continuity condition for canonical projective systems

Let $t > 0$, let $\Delta_k = \{\frac{jt}{2^k} \mid j \in \mathbb{N} \cap [0, 2^k]\}$ and let $\Delta = \bigcup_{k\in\mathbb{N}} \Delta_k$. Let (M, d) be a connected metric space. Let $\mathrm{Fin}(0,t) = \{F \subset [0,t] \mid F \text{ is finite}\}$. For $I \in \mathrm{Fin}(0,t)$ let $M^I = \{c : I \to M\}$ and for $J \supseteq I$ let $p_{IJ} : M^J \to M^I$ be the natural restriction. Assume that for any $I \in \mathrm{Fin}(0,t)$ there exists a positive, finite, regular Borel measure w_I on M^I such that $p_{IJ*} w_J = w_I$ for all $J \supseteq I$. Let $M_t = \prod_{s\in[0,t]} M = \{c : [0,t] \to M\}$, endowed with the product topology (that is, that of pointwise convergence) and with the associated Borel σ-algebra, and let $p_I : M_t \to M^I$ be the natural projection for all $I \in \mathrm{Fin}(0,t)$.

Teorema 2.1 (Kolmogorov's extension theorem)**.** *There exists a unique Borel finite measure w_t on M_t such that $p_{I*} w_t = w_I$.*

Teorema 2.2. *1. If there exist $0 < b < a$ and $C > 0$ such that*

$$\int_{M_t} d(c(s), c(s'))^a \, \mathrm{d}w_t(c) \leq C|s - s'|^{1+b}$$

for all $s, s' \in [0,t]$ ("the Kolmogorov continuity condition"), and if $0 < \varepsilon < \frac{b}{a}$, then the subset of the Hölder-continuous curves of expoenent ε defined by $\mathcal{H}_t(\varepsilon) = \{c \in M_t \mid \exists L > 0 \text{ such that } d(c(s), c(s')) \leq L|s-s'|^\varepsilon \; \forall s, s' \in [0,t]\}$ is Borel in M_t and w_t is concentrated on it.

2. If M is separable and $\mathcal{H}_t(\varepsilon)$ is endowed with the topology of uniform convergence (which is the trace of the compact-open topology on the space of continuous curves $\mathcal{C}_t = \{c \in M_t \mid c \text{ continuous}\}$ and is given by the distance $D(c, c') = \max_{s\in[0,t]} d(c(s), c'(s))$), then the restriction of w_t to $\mathcal{H}_t(\varepsilon)$ is Borel and regular with respect to this topology.

2.2 The Wiener measure associated to a Riemannian manifold

We shall now use the above results in the case where the metric space is a Riemann manifold (M, g) of dimension n, and d is the intrinsic distance coming from the Riemannian structure. We shall also fix an arbitrary point $x_0 \in M$ (the start endpoint, at time 0, of the trajectories in the space on which we shall construct the measure w_t).

If $I = \{0 \leq t_1, \dots, t_k \leq t\} \in \mathrm{Fin}(0,t)$, we endow the manifold M^I with the measure

$$w_{x_0,I}(A) = \int_M \mathrm{d}x_1 \, h(t_1, x_0, x_1) \dots \int_M \mathrm{d}x_k \, h(t_k - t_{k-1}, x_{k-1}, x_k) \, 1_A(x_1, \dots, x_k) \ ,$$

where we agree that $\int_M h(0,y,z)f(z)\,\mathrm{d}z = f(y)$ and where 1_A is the characteristic function of the Borel subset $A \subseteq M^I$.

Thanks to the convolution property of the heat kernel, it is easy to show that if $I \subseteq J$, then $p_{IJ*}w_{x_0,J} = w_{x_0,I}$. It is also clear that $w_{x_0,I}$ is a regular positive Borel measure bounded by 1.

2.2.1 The Wiener measure associated to a compact Riemannian manifold

Since r from the theorem 1.7 is continuous and M is compact, there exists $\rho = \min_{x\in M} r(x) > 0$. By taking $s = s_0$ in that formula, we get the more convenient upper bound

$$h(s,y,z) \leq C\left(1 + \frac{d(y,z)^2}{s}\right)^{\frac{n}{2}} \min(2s,\rho^2)^{-\frac{n}{2}} \exp\left(-\frac{d(y,z)^2}{4s}\right) . \tag{2.1}$$

It is shown that the Kolmogorov continuity condition in theorem 2.2.(1) is naturally satisfied in this context. More precisely, if $a > 2$ then

$$\int_{M_t} d(c(u),c(v))^a \,\mathrm{d}w_t(c) \leq K \min(2(v-u),\rho^2)^{-\frac{n}{2}} (v-u)^{\frac{n}{2}+\frac{a}{2}} \leq$$
$$\leq \begin{cases} 2^{-\frac{n}{2}} K (v-u)^{\frac{a}{2}}, & 2(v-u) < \rho^2 \\ K\rho^{-n} t^{\frac{n}{2}} (v-u)^{\frac{a}{2}}, & 2(v-u) \geq \rho^2 \end{cases} ,$$

whence $\int_{M_t} d(c(u),c(v))^a \,\mathrm{d}w_t(c) \leq C(t)(v-u)^{1+b}$, where $C(t) = K\max(2^{-\frac{n}{2}}, \rho^{-n}t^{\frac{n}{2}})$ and $b = \frac{a}{2} - 1$, which shows that Kolmogorov's continuity condition is satisfied. This means that there exists a unique Borel positive regular measure $w_{x_0,t}$ bounded by 1 (because the integral of h is bounded by 1), on $\bigcap_{0<\varepsilon<\frac{1}{2}} \mathcal{H}_t(\varepsilon)$ (namely, the restriction of w_t). This is the Wiener measure on the space of trajectories in M.

2.2.2 The Wiener measure associated to a regular domain

Let now (M,g) be an arbitrary Riemannian manifold.

Definiția 2.3. *A regular domain is a relatively compact connected open subset with smooth boundary.*

We shall assume that x_0 is contained in all the regular domains that we shall talk about.

If $U \subseteq M$ is a regular domain, we endow $\overline{U}$ with the intrinsic distance $d_{\overline{U}}(p,q) = \inf\{L(c) \mid c : [0,1] \to \overline{U}$ is smooth and joins p and $q\}$, where $L(c) = \int_0^1 \sqrt{g(\dot{c}(s),\ dotc(s))}\,\mathrm{d}s$ is the length of c.

Let now (U',g') be a Riemannian dual of $\overline{U}$: this means that $\overline{U}$ embeds smoothly as a submanifold with boundary into U', and that $g'|_{\overline{U}} = g|_{\overline{U}}$. Details of this construction can be found in subsection 5.5 on p.116 of [Duff56]. If $d_{U'}$ is the intrinsic Riemannian distance on U' , then $d_{U'}|_{\overline{U}} \leq d_{\overline{U}}$.

Propoziția 2.4. *There exists $\alpha_U > 0$ such that $d_{\overline{U}} \leq \alpha_U d_{U'}|_{\overline{U}}$.*

Let $h_{\overline{U}}$ be the Dirichlet heat kernel of $\overline{U}$, and $h_{U'}$ be the heat kernel of U'. The idea is to transport the problem to U', where we already know that this condition is satisfied, by replacing $d_{\overline{U}}$ with $\alpha_U\, d_{U'}$ and $h_{\overline{U}}$ with $h_{U'}$ (keeping in mind that the heat kernel of a manifold is minimal, so $h_{\overline{U}} \leq h_{U'}$ on $\overline{U} \subset U'$). Taking $a > 2$ and $0 < u < v < t$ we have:

$$\int_{(\overline{U})_t} d_{\overline{U}}(c(u),c(v))^a \,\mathrm{d}(w_{\overline{U}})_{x_0,t}(c) = \int_{\overline{U}} h_{\overline{U}}(u,x_0,y) \left(\int_{\overline{U}} h_{\overline{U}}(v-u,y,z)\, d_{\overline{U}}(y,z)^a \,\mathrm{d}z\right) \mathrm{d}y$$
$$\leq \alpha_i \int_{U'} h_{U'}(u,x_0,y) \left(\int_{U'} h_{U'}(v-u,y,z)\, d_{U'}(y,z)^a \,\mathrm{d}z\right) \mathrm{d}y =$$
$$= \alpha_i \int_{(U')_t} d_{U'}(c(u),c(v))^a \,\mathrm{d}(w_{U'})_{x_0,t}(c) \leq \alpha_i\, K_i(t)\, (v-u)^{1+(\frac{a}{2}-1)} ,$$

where $(w_{\overline{U}})_{x_0,t}$ and $(w_{U'})_{x_0,t}$ are the measures obtained from the Kolmogorov extension theorem on the spaces $(\overline{U})_t$ and $(U')_t$, respectively, and $K_i(t)$ is the constant $C(t)$ corresponding to U' obtained as in the previous subsection. Since the Kolmogorov continuity condition is satisfied, it follows that $(w_{\bar{U}})_{x_0,t}$ is concentrated on $\mathcal{H}_{t,\overline{U}}(\varepsilon)$ (the space of Hölder-continuous curves of Hölder exponent $\varepsilon > 0$, contained in $\overline{U}$) as a positive, Borel, bounded by 1, regular measure, for any $\varepsilon < \frac{1}{2}$.

2.2.3 The Wiener measure associated to an arbitrary Riemannian manifold

Propoziția 2.5. *There exist exhaustions with closures of regular domains $(\overline{U}_k)_{k\in\mathbb{N}}$; that is, $\overline{U}_k \subset U_{k+1}$ for any $k \in$ mathbbN and $\bigcup_{k\in\mathbb{N}} \overline{U}_k = M$. Moreover, we can take $x_0 \in \overline{U}_k$ for any $k \in \mathbb{N}$.*

Let $0 < \varepsilon < \frac{1}{2}$ and let $(w_k)_{x,t}$ be the Wiener measure on $\mathcal{H}_{t,\overline{U}_k}(\varepsilon)$, as constructed in the previous subsection. The natural embedding $i_k : \overline{U}_k \to M$ induces a natural embedding $i_k : \mathcal{H}_{t,\overline{U}_k}(\varepsilon) \to \mathcal{H}_{t,M}(\varepsilon)$, which allows us to consider the pushed forward measure $(i_k)_*(w_k)_{x,t}$ on $\mathcal{H}_{t,M}(\varepsilon)$. We note that this sequence of measures is bounded by 1 and increasing because $h_{U_k} \le h_{U_l}$ on $\overline{U}_k$ (again, from the minimality property of the heat kernel).

The argument from theorem 2.2.(2) also shows that the Borel σ-algebra of $\mathcal{H}_{t,M}(\varepsilon)$ coincides with the σ-algebra generated by the subsets of the form $p_I^{-1}(B)$ with $I \in \mathrm{Fin}(0,t)$ and $B \subseteq M^I$ Borel. It follows that $(i_k)_*(w_k)_{x_0,t}(B) \le (i_l)_*(w_l)_{x_0,t}(B)$ for any Borel subset $B \subseteq \mathcal{H}_{t,M}(\varepsilon)$, so it makes sense to define

$$w_{x_0,t}(B) = \lim_{k\to\infty} (i_k)_*(w_k)_{x_0,t}(B) = \sup_{k\ge 0} (i_k)_*(w_k)_{x_0,t}(B)$$

for any Borel subset $B \subseteq \mathcal{H}_{t,M}(\varepsilon)$.

Propoziția 2.6. *$w_{x_0,t}$ is a positive Borel regular measure on $\mathcal{H}_{t,M}(\varepsilon)$, bounded by 1.*

Propoziția 2.7. *For any $0 < \varepsilon < \frac{1}{2}$ the Wiener measure $w_{x_0,t}$ is precisely the Wiener measure and is concentrated on $\{c \in \mathcal{H}_t(\varepsilon) \mid c(0) = x_0\}$.*

2.3 The Feynman-Kac formula

One of the classic uses of the Wiener measure is the Feynman-Kac formula, which shows how an initial temperature distribution f propagates in time in the presence of a potential V.

Teorema 2.8. *If $V : M \to \mathbb{R}$ is a lower bounded continuous function, and if $f \in L^2(M)$, then*

$$[\exp(-tL - tV)f](x) = \int_{M_t} \exp\left(\int_0^t -V(c(s))\,\mathrm{d}s\right) f(c(t))\,\mathrm{d}w_{x_0,t}(c) \ .$$

3. A FUNCTIONAL-ANALYTIC CONSTRUCTION OF THE HEAT KERNEL IN HERMITIAN BUNDLES OVER RIEMANNIAN MANIFOLDS

This chapter presents the author's first original contribution, namely the construction of the heat kernel associated with a Laplacian defined by a connection in a Hermitian vector bundle over an arbitrary Riemannian manifold, and the obtaining of an upper bound for it. Such a construction was already known for compact Riemannian manifolds, but at the beginning of the doctoral studies (November 2015) one for arbitrary Riemann manifolds proved impossible to find in the specialized literature, which justified the present approach.

3.1 Motivation and context

Let M be a separable Riemann manifold of dimension n, and let $E \to M$ be a Hermitian bundle over M of complex rank $r < \infty$; we do not impose any other restrictions on M or E.

The fiber of E over $x \in M$ will be denoted E_x, and the Hermitian product on it $\langle \cdot, \cdot \rangle_{E_x}$ (all Hermitian products used in this text will be linear in the first argument). As we shall work with several Hilbert spaces, the norm and the Hermitian product on each of them will carry it as a subscript: if X is a Hermitian space and $v, w \in X$, then $\|v\|_X$ will be the norm of v and $\langle v, w \rangle_X$ the Hermitian product of v and w. The measure on M obtained using the Riemannian metric will be μ_M. If s is a section of E, the notation $\|s\|$ (without any other indices) will mean the function $M \ni x \mapsto \|s(x)\|_{E_x} \in [0, \infty)$. $\Gamma_0(E)$ will be the space of smooth sections in E with compact support. If $1 \le p \le \infty$, $\Gamma^p(E)$ will be the space of classes of measurable sections equal almost everywhere that have the property that $\|s\| \in L^p(M)$. It is known that $\Gamma_0(E)$ is dense in $\Gamma^p(E)$ in the norm topology if $p \neq \infty$, and in the $*$-weak topology if $p = \infty$. The space of equivalence classes of measurable functions equal almost everywhere will be $L^0(M)$ and $\Gamma^0(E)$ will be the analogous space of sections. In concrete calculations (usually integrals on M) involving sections $s \in \Gamma^p(E)$, we shall tacitly understand that we are working with an arbitrary representative of the class s; in such situations it will be immediately clear from the context that the results are independent of the chosen representative. Tildes will always mean extensions by 0: if s is a section of $E|_S$ (or a function) defined on a subset $S \subset M$, then $\widetilde{s}$ will be its extension by 0 to the whole of M. For linear operators between normed spaces, $\|\cdot\|_{op}$ will denote the operator norm, without specifying the spaces anymore when they are clear from the context.

We shall work with a map ("fibered potential") $V : M \to \operatorname{End} E$ such that $V(x) \in \operatorname{End} E_x$. It is general enough to require that the operator norm function $M \ni x \mapsto \|V(x)\|_{op} \in [0, \infty)$ be locally essentially bounded (so $V \in \Gamma^\infty_{loc}(\operatorname{End} E)$, by definition). To be able to use functional calculus methods we shall require that $V(x)$ be self-adjoint for almost any $x \in M$. The minimum of the spectrum of $V(x)$ will be $b(x) = \min \operatorname{spec} V(x)$, for all $x \in M$. For simplicity, we shall impose that $b \ge 0$, but all results remain valid with minimal changes to the proofs in the more general case $\operatorname{ess\,inf}_{x \in M} b(x) \neq -\infty$. The "multiplication" operator $\operatorname{mul}(V) : \Gamma_0(E) \to \Gamma^2(E)$ given by $(\operatorname{mul}(V)s)(x) = V(x)s(x)$ is positive (since $\langle V(x)s(x), s(x) \rangle_{E_x} \ge b(x) \|s(x)\|^2_{E_x} \ge 0$) and essentially self-adjoint.

If ∇ is a Hermitian connection in E (that is, $X\langle s, s' \rangle = \langle \nabla_X s, s' \rangle + \langle s, \nabla_X s' \rangle$ for all $X \in \Gamma(TM)$ and $s, s' \in \Gamma(E)$), the Laplacian associated with the connection $\nabla^* \nabla : \Gamma_0(E) \to \Gamma^2(E)$ is positive-definite and symmetric, so performing the Friedrichs construction on $\nabla^* \nabla + V$ (which will be symmetric and positive) we obtain a densely defined, self-adjoint, positive extension $H_{\nabla, V} : \operatorname{Dom}(H_{\nabla, V}) \to \Gamma^2(E)$ ("fibered Hamiltonian").

Since $\operatorname{spec} H_{\nabla, V} \subseteq [0, \infty)$, we deduce from the spectral theorem that the resolvent at every $\lambda < 0$ has the property that

$$\left\| (H_{\nabla, V} - \lambda)^{-1} \right\|_{op} = \sup \left\{ \left| \frac{1}{\mu - \lambda} \right| \mid \mu \in \operatorname{spec} H_{\nabla, V} \right\} \le$$

$$\le \sup \left\{ \left| \frac{1}{\mu - \lambda} \right| \mid \mu \in [0, \infty) \right\} = \frac{1}{|\lambda|}$$

therefore, using the Hille-Yoshida theorem (corollary 2.22 on p.51 of [Davies80]), we deduce that $-H_{\nabla,V}$ generates a strongly continuous contraction semigroup in $\Gamma^2(E)$. Since every open subset of M is itself a manifold, all the notations and considerations so far hold on such subsets, too.

In this chapter we shall show that $\mathrm{e}^{-tH_{\nabla,V}}$ admits an integral kernel (with various properties that we shall explore further on); more precisely, we shall construct an application $(0,\infty)\times M\times M \ni (t,x,y) \mapsto k_{\nabla,V}(t)(x,y) \in \ mathcalL(E_y,E_x)$ (the last notation representing the space of linear applications from E_y to E_x) such that $(\mathrm{e}^{-tH_{\nabla,V}} s)(x) = \int_M k_{\nabla,V}(t,x,y)\, s(y)\, \mathrm{d}y$ for all $s \in \Gamma^2(E)$. In the particular case $E = M\times\mathbb{C}$, $\nabla = \mathrm{d}$ and $V = 0$, this is exactly the heat kernel of M, a very detailed and copiously commented construction of which can be found in [Grigor'yan09]. If E is no longer trivial, but M is compact, such a result is known as the "Minakshisundaram-Pleijel construction" and can be found, for example, in [BGV92]. On the other hand, at the beginning of his doctoral studies (November 2015), the author was not aware of any other similar work published in the specialized literature dealing with the case of arbitrary manifolds and bundles, which motivated the present research effort. Meanwhile, in 2017 Batu Güneysu published "Covariant Schrödinger Semigroups on Riemannian Manifolds" (cited in the present text as [Güneysu17]), which deals with the same problem, but providing proofs based on completely different techniques than the ones used here which, therefore, retain their originality and interest.

Section 3 and section 4 contain the central results of this chapter, namely the construction of the heat kernel in bundles and the proof of its uniqueness. For this purpose, it will be necessary to prove, also in section 3, the diamagnetic inequality. The heat kernel will be constructed, for technical reasons, in two steps: first on relatively compact domains, then on the entire manifold using an exhaustion with such domains. Once constructed, in Section 5 we study its various integrability properties and find a convenient upper bound to control it. Finally, Section 6 studies its behaviour when the potential V is subjected to bounded perturbations.

The idea of the construction is to obtain the heat kernel in E as the limit of a sequence of approximations. The mathematical tool that will allow this approach is Chernoff's theorem about the approximation of semigroups in Banach spaces.

3.2 Preliminary results

Propoziția 3.1. *The function $b : M \to \mathbb{R}$ defined above by $b(x) = \min \operatorname{spec} V(x)$ is locally essentially bounded.*

Teorema 3.2. *Since M was assumed separable, the space $\Gamma^2(E)$ will be separable.*

We shall need the concept of **outer tensor product** of two bundles. Thus, if M and N are smooth manifolds, and $E \to M$ and $F \to N$ are vector bundles of finite rank, if $p : M\times N \to M$ and $q : M\times N \to N$ are the canonical projections then we can consider the bundle $E \boxtimes F \to M\times N$ defined by $E\boxtimes F = p^*E \otimes q^*F$. The integrable square sections in this bundle will be $\Gamma^2(E\boxtimes F) \simeq \Gamma^2(E)\otimes\Gamma^2(F)$ since we can consider the isomorphism that takes the tensor monomial $(x,y)\mapsto(\sigma\otimes\eta)(x,y)$ into the tensor monomial $(x,y)\mapsto\sigma(x)\otimes\eta(y)$, extended by linearity and then density in the topologies of Hilbert spaces.

Definiția 3.3. *We call **the heat kernel in the bundle** $E \to M$**, corresponding to the connection** ∇ **and the potential** V, the map $k_{\nabla,V} : (0,\infty) \to \Gamma^2_{loc}(E\boxtimes E^*)$ with the property that $\mathrm{e}^{-tH_{\nabla,V}}\sigma = \int_M k_{\nabla,V}(t,\ cdot,y)\,\sigma(y)\,\mathrm{d}y$ for all $\sigma\in\Gamma^2(E)$.*

The purpose of this chapter is to show that such an application exists, is essentially unique, and has good integrability properties. If V is smooth, we shall show that the integral kernel that we shall construct is also smooth. We shall also find an upper bound for it, useful in integral inequalities.

3.3 The construction of the heat kernel in bundles over relatively compact domains

Let $U \subseteq M$ be a regular domain, that is, a relatively compact open subset with smooth boundary ∂U.

Teorema 3.4. *For every $t > 0$ there exists the integral kernel $k^{(U)}_{\nabla,V}(t) \in \Gamma^2(E|_U \boxtimes E^*|_U)$ such that*

$$(\mathrm{e}^{-tH^{(U)}_{\nabla,V}}\sigma)(x) = \int_U k^{(U)}_{\nabla,V}(t)(x,y)\,\sigma(y)\,\mathrm{d}y$$

for almost all $x\in U$ and all $\sigma\in\Gamma^2(E|_U)$.

Teorema 3.5. *The map* $(0,\infty) \ni t \mapsto k^{(U)}_{\nabla,V}(t) \in \Gamma^2(E|_U \boxtimes E^*|_U)$ *is measurable and is the unique map* $k : (0,\infty) \to \Gamma^2(E|_U \boxtimes E^*|_U)$ *with the property that* $\mathrm{e}^{-tH^{(U)}_{\nabla,V}}\sigma = \int_U k(t,\cdot,y)\,\sigma(y)\,\mathrm{d}y$ *for all* $\sigma \in \Gamma^2(E|_U)$.

We are interested in an upper bound for $\|k^{(U)}_{\nabla,V}(t)(x,y)\|_{op}$, and this will be obtained in the next theorem that presents **the integral kernel version of the diamagnetic inequality**.

Teorema 3.6. *For all* $t > 0$*, and almost all* $x, y \in U$, $\|k^{(U)}_{\nabla,V}(t)(x,y)\|_{op} \leq k^{(U)}_{\mathrm{d},b}(t)(x,y)$.

Observația 3.7. *Taking into account that* $b \geq 0$*, the weaker inequality*

$$\|(\mathrm{e}^{-tH^{(U)}_{\nabla,V}}\,\sigma)(x)\|_{E_x} \leq (\mathrm{e}^{-tH^{(U)}_{\mathrm{d},0}}\,\|\sigma\|)(x)$$

can be obtained, whence $\|k^{(U)}_{\nabla,V}\|_{op} \leq k^{(U)}_{\mathrm{d},0}$*, so the heat kernel on functions is a universal upper bound for all heat kernels in bundles over* U*. It follows that* $k^{(U)}_{\nabla,V}$ *inherits all the integrability properties that* $k^{(U)}_{\mathrm{d},0}$ *has. However, since* V *was chosen only in* $\Gamma^\infty_{loc}(\operatorname{End} E)$*, we cannot say anything about the smoothness of* $k^{(U)}_{\nabla,V}$ *in general.*

3.4 The construction of the heat kernel in bundles over arbitrary manifolds

To obtain the heat kernel $k_{\nabla,V}$ in E above M we shall consider an exhaustion $M = \bigcup_{i\geq 0} U_i$ with relatively compact open domains with smooth boundary. Any such open U_i admits a heat kernel $k^{U_i}_{\nabla,V}$ in $E|_{U_i}$ constructed as above, corresponding to the Friedrichs extension $H^{U_i}_{\nabla,V}$ of the operator $\nabla^*\nabla + V|_{U_i}$ (∇ representing, for simplicity of writing, the restriction of the connection ∇ to the bundle $E|_{U_i}$). Intuitively, we will get $k_{\nabla,V}$ as the limit of the kernels $k^{U_i}_{\nabla,V}$, but in an indirect way. The following theorem will obtain, on each open U_i, a map $k^{(i)}_{\nabla,V}$ which will later be proven to be exactly the restriction to U_i of the kernel $k_{\nabla,V}$ we are looking for.

Teorema 3.8. *For all* $i \geq 0$ *and all* $t > 0$*, the sequence* $(k^{U_j}_{\nabla,V}(t)|_{U_i\times U_i})_{j\geq i}$ *is Cauchy in* $\Gamma^2(E|_{U_i} \boxtimes E^*|_{U_i})$*, so it will have a unique limit, denoted* $k^{(i)}_{\nabla,V}(t)$*. This limit will have the property that*

$$\int_{U_i} k^{(i)}_{\nabla,V}(t)(x,y)\,\sigma(y)\,\mathrm{d}y = [\mathrm{e}^{-tH_{\nabla,V}}\,\iota_i(\sigma)](x)$$

for all $\sigma \in \Gamma^2(E)$ *and almost all* $x \in U_i$.

So far, we have obtained a map $k^{(i)}_{\nabla,V}(t)$ on each domain U_i, but is there any relationship between all these maps? The following theorem will show us that, indeed, these maps satisfy a very convenient compatibility relationship that will allow us to define their pointwise limit on the whole manifold.

Teorema 3.9. *If* $i \leq j$ *then the "compatibility relationship"*

$$k^{(j)}_{\nabla,V}(t)|_{U_i\times U_i} = k^{(j)}_{\nabla,V}(t)$$

is true.

For the actual construction of the heat kernel in E, let us denote by $\widetilde{k^{(i)}_{\nabla,V}(t)} \in \Gamma^2_{loc}(E \boxtimes E^*)$ the extension of $k^{(i)}_{\nabla,V}(t)$ by 0 to $M \times M$; we shall choose a measurable representative $l^{(i)}_{\nabla,V}(t)$ of $\widetilde{k^{(i)}_{\nabla,V}(t)} \in \Gamma^2_{loc}(E \boxtimes E^*)$ for each $i \geq 0$ as follows:

- $l^{(0)}_{\nabla,V}(t)$ is chosen arbitrarily;
- $l^{(i+1)}_{\nabla,V}(t)$ is chosen such that $l^{(i+1)}_{\nabla,V}(t)|_{U_i\times U_i} = l^{(i)}_{\nabla,V}(t)$ (such a choice is possible thanks to the compatibility relationship proved above).

We noe define $k_{\nabla,V}(t) = \lim_{i\to\infty} l^{(i)}_{\nabla,V}(t)$. Since the sequence $\left(l^{(i)}_{\nabla,V}(t)\right)_{i\geq 0}$ is constant from some i onward for all $x, y \in M$, the limit exists.

Teorema 3.10. *The measurable map* $(0,\infty) \ni t \mapsto k_{\nabla,V}(t) \in \Gamma^2_{loc}(E \boxtimes E^*)$ *is the heat kernel in* E *corresponding to the connection* ∇ *and to the potential* V*, and is the unique map with these properties.*

Corolarul 3.11. *For all* $t > 0$ *and almost all* $x, y \in M$, $\|k_{\nabla,V}(t)(x,y)\|_{op} \leq k_{\mathrm{d},b}(t)(x,y)$.

3.5 Integrability and smoothness properties

From the weak version $\|k_{\nabla,V}(t)(x,y)\|_{op} \leq h(t,x,y)$ for all $t > 0$ and almost all $x, y \in M$ of the diamagnetic inequality, we immediately conclude that $k_{\nabla,V}$ inherits the same integrability properties that h has (see section 1.4). In particular, as h is jointly continuous and therefore bounded on any compact subset, it follows that $k_{\nabla,V} \in \Gamma^\infty_{loc}(\mathbb{C} \boxtimes E \boxtimes E^*)$. Also, since $(x, y) \mapsto h(t, x, y)$ is integrable and square integrable in each argument, it follows that so will be $(x, y) \mapsto \|k_{\nabla,V}(t)(x, y)\|_{op}$, therefore $k_{\nabla,V}(t)(x, -) \in \Gamma^1(E^*) \cap \Gamma^2(E^*)$ and $k_{\nabla,V}(t)(\cdot, y) \in \Gamma^1(E) \cap \Gamma^2(E)$ for all $t > 0$ and almost all $x, y \in M$.

So far, the mathematical tools at our disposal and the assumptions we have placed ourselves under have allowed us to study only the integrability of $k_{\nabla,V}$. We shall now make the assumption that V is smooth and show that under this assumption $k_{\nabla,V}$ is, in turn, smooth.

Teorema 3.12. *If V is smooth, the differential operator $2\partial_t + L$ is hypoeliptic.*

Teorema 3.13. *The (regular) distribution $u = k_{\nabla,V}|_{(0,\infty)\times U_1\times U_2} \in \mathcal{D}'((0, \infty) \times U_1 \times U_2, \mathbb{C}^{r^2})$ is a solution in the distributional sense of the equation $(2\partial_t + L)u = 0$.*

Corolarul 3.14. *The kernel $k_{\nabla,V}$ is smooth on $(0, \infty) \times M \times M$.*

Under the smoothness assumption on V , we can improve the diamagnetic inequality: it will be true everywhere, not just almost everywhere.

Corolarul 3.15. *If V is smooth and $v : M \to \mathbb{R}$ is a smooth function such that $v \leq b$, then $\|k_{\nabla,V}(t)(x, y)\|_{op} \leq k_{\mathrm{d},v}(t)(x, y)$ for all $t > 0$ and $x, y \in M$.*

3.6 The behaviour under bounded perturbations

Sometimes, in the study of a problem, it may be useful to replace the potential V with some more convenient approximations V_j, to study the problem with the perturbed Hamiltonian $H_{\nabla,V_j} = H_{\nabla,V} + (V_j - V)$ and then try to extend the conclusions to the "original Hamiltonian" $H_{\nabla,V}$ through a passing to the limit. It is useful, then, to study the behaviour of the heat kernel under such bounded perturbations $V_j - V$. Before proceeding to this study, let us remark that the diamagnetic inequality, both in the semigroup and integral kernel versions, is true not only for potentials V for which $\inf b \geq 0$ (as we have assumed for simplicity so far), but also for more general potentials for which $\inf b \neq -\infty$. Indeed, in this last case it is sufficient to redo all the above proofs for the potential $V - \inf b$, whence the conclusion will be immediate.

Teorema 3.16. *Let $B : \Gamma^2(E) \to \Gamma^2(E)$ be a self-adjoint linear operator defined everywhere (hence bounded), and let $\lambda = \inf \operatorname{spec} B$. Then*

$$\|k_{\nabla,V}(t)(x, y) - k_{\nabla,V+B}(t)(x, y)\|_{op} \leq \int_0^t \mathrm{e}^{-(t-\varepsilon)\lambda} \, \mathrm{d}\varepsilon \, \|B\| \, k_{\mathrm{d},b}(t)(x, y)$$

for almost all $x, y \in M$.

Corolarul 3.17. *Let $(V_j)_{j\geq 0}$ and V be locally essentially bounded and self-adjoint fibered potentials, such that $V_j \to V$ in $\Gamma^\infty(\operatorname{End} E)$. Using the notations from the previous theorem, $k_{\nabla,V_j}(t)(x, y) \to k_{\nabla,V}(t)(x, y)$ in $\mathcal{L}(E_y, E_x)$ when $j \to \infty$, for all $t > 0$ and almost all $x, y \in M$.*

4. STOCHASTIC INTEGRATION

This chapter contains the second original result of the thesis: the construction of the concept of "stochastic integral" using exclusively functional-analytic and Riemannian-geometric tools. The stochastic integrals used in the literature are the Itô integral and the Stratonovich integral. Both aim to answer the same question: can we integrate a function along a continuous-time stochastic process on which it depends? The two integrals are two different (but connected by a simple formula) answers to this question. The deep geometric substratum that underlies them tends to be obscured by the probabilistic language in which these concepts were originally formulated. We shall discover in this chapter that, by extracting the problem from the domain of probability theory and placing it in an explicitly geometric one (Riemannian manifolds and integration of 1-forms on curves), we gain a completely new understanding of these stochastic concepts. In particular, we shall see that we can define a general concept of stochastic integral and construct an infinity of such integrals, among which the Itô and Stratonovich ones are special cases. We shall also see why in the usual approximations of the Itô integral the increments are oriented "only toward the future".

In this chapter, $\alpha \in \Omega^1(M)$ will be a smooth, real 1-form; we know what the line integral $\int_c \alpha$ means when $c : [0,1] \to M$ is a smooth curve, and we shall try to make sense of the same integral when c is just continuous (and so the tangent vector $\dot c$ no longer exists).

To orient the reader's expectations a bit, let us briefly review what is known about stochastic integration in $\mathbb{R}^n$. If c is a sufficiently smooth curve, the Riemann sums

$$\sum_{j=0}^{2^k-1} \alpha\left(c\left(\frac{jt}{2^k}\right)\right)\left[c\left(\frac{(j+1)t}{2^k}\right) - c\left(\frac{jt}{2^k}\right)\right]$$

converge to the line integral $\int_c \alpha$. It is interesting to ask: if c is only continuous (or even only an element of $\prod_{s\in[0,t]} M$), do these sums still converge to anything? The answer is known to be affirmative, but in a slightly weaker sense, no longer true for just every curve: *for almost any curve* c (in the sense of the Wiener measure), the limit exists and is called the *Itô integral*. Furthermore, if we symmetrize the Riemann sums above, in the sense of considering now

$$\sum_{j=0}^{2^k-1} \frac{1}{2}\left[\alpha\left(c\left(\frac{jt}{2^k}\right)\right) + \alpha\left(c\left(\frac{(j+1)t}{2^k}\right)\right)\right]\left[c\left(\frac{(j+1)t}{2^k}\right) - c\left(\frac{jt}{2^k}\right)\right] ,$$

they will converge, for almost all curves c, to another limit, called *the Stratonovich integral*.

It is useful to note an interesting fact: if in the formula

$$\sum_{j=0}^{2^k-1} \int_{[0,1]} \alpha_{(1-\tau)\, c(\frac{jt}{2^k}) + \tau\, c(\frac{(j+1)t}{2^k})} \left[c\left(\frac{(j+1)t}{2^k}\right) - c\left(\frac{jt}{2^k}\right)\right] \mathrm{d}P(\tau) ,$$

where P is a probability on $[0,1]$, we consider in turn $P = \delta_0$ (the Dirac measure concentrated at 0) and $P = \frac{1}{2}(\delta_0 + \delta_1)$, then we obtain the sums that converge to the Itô integral and, respectively, the sums that converge to the Stratonovich integral. So, these two stochastic integrals and their approximations are particular cases of a general concept of stochastic integral that we shall highlight below. The generalization of this formula from $\mathbb{R}^n$ to M is straightforward: the line segment $\tau \mapsto \tau\, c(\frac{jt}{2^k}) + (1-\tau)\, c(\frac{(j+1)t}{2^k})$ will be replaced by the unique minimizing geodesic between $c(\frac{jt}{2^k})$ and $c(\ frac(j+1)t2^k)$ (when it exists), and the vector $c(\frac{(j+1)t}{2^k}) - c(\frac{jt}{2^k})$ will be replaced by the tangent vector of this geodesic at the moment τ.

As in chapter 2, let us fix once and for all a point $x_0 \in M$ and for any $t > 0$ consider the space $\mathcal{C}_t = \{c : [0,t] \to M \mid c \text{ is continuous, with } c(0) = x_0\}$ endowed with the Wiener measure w_t.

We consider the trivial bundle $E = M \times \mathbb{C}$ endowed with the trivial Hermitian structure and the connection $\nabla^{(\alpha)} f = \mathrm{d}f + \mathrm{i} f \alpha$ where $\mathrm{i} = \ sqrt{-1}$ is a complex square root of -1. It is easy to see that $\nabla^{(\alpha)}$ is Hermitian and that the operator $-\Delta^{(\alpha)} = (\nabla^{(\alpha)})^* \nabla^{(\alpha)} : C_0^\infty(M) \to C_0^\infty(M)$ is symmetric and positive-definite. Using the Friedrichs construction we obtain a self-adjoint, positive-definite, densely defined extension L_α in $L^2(M)$. Using the results obtained in chapter 3, the semigroup $(\mathrm{e}^{-tL_\alpha})_{t\geq 0}$ will admit a heat kernel h_α associated to the connection $\nabla^{(\alpha)}$. It will be smooth and, from the diamagnetic inequality, will have the property $|h_\alpha(t,x,y)|\leq h(t,x,y)$ for any $t>0$ and almost all $x,y\in M$, where h is the kernel of $(\mathrm{e}^{-tL_0})_{t\geq 0}$ (corresponding to the 1-form $\alpha = 0$), that is, the heat kernel on M.

For every $k \in \mathbb{N}$ we shall consider the manifold $M_k = M^{2^k}$ and the natural projection $\pi_k : \mathcal{C}_t \to M_k$ given by $\pi_k(c) = (c(\frac{t}{2^k}), \dots, c(\frac{2^k t}{2^k}))$. Regardless of whether we endow $\mathcal{C}_t$ with the topology of uniform convergence of curves or with that of pointwise convergence, π_k will be continuous.

To facilitate the reader's orientation in the text that follows, now is the right time to outline the result we are looking for and the strategy by which we shall obtain it. Thus, we shall construct a complex measure density $\rho_{\alpha,t} \in L^\infty(\mathcal{C}_t)$ and seek to show that the map $\mathbb{R} \ni s \mapsto \rho_{s\alpha,t} \in \mathcal{B}(L^2(\mathcal{C}_t))$ (the space of bounded operators on $L^2(\mathcal{C}_t)$) is a 1-parameter strongly continuous unitary group whence, with Stone's theorem, it will have a self-adjoint generator $\mathrm{Strat}_t(\alpha)$. It will be seen that this generator is exactly the Stratonovich integral. The difficulty of proving this assertion comes from the fact that $\rho_{\alpha,t}$ will be obtained by a very abstract procedure, therefore the group structure and unitarity are very hard to prove. To highlight these concrete properties, we shall show that $\rho_{\alpha,t}$ is the limit of a sequence of functions with the desired properties, which will be transfered to the limit by convergence.

More precisely, we shall construct a sequence of measurable real functions $S_{P,t,k}(\alpha)$ linear in $\alpha \in \Omega^1(M)$ such that $\mathrm{e}^{\mathrm{i}S_{P,t,k}(\alpha)} \to \rho_{\alpha,t}$ in $L^2(\mathcal{C}_t)$. Although simple, this idea is complicated by some technical details that we shall point out when we meet them and that force us to approach the problem indirectly: we shall first prove the announced convergence in the space $L^2(\mathcal{C}_t(U))$ associated with an arbitrary relatively compact open subset U with smooth boundary, and then we shall consider an exhaustion of M with such domains, which will allow us to prove the result in the whole $L^2(\mathcal{C}_t)$. The strategy of using exhaustions with relatively compact domains is natural if we remember that we used it both in the construction of the Wiener measure and in that of the heat kernel in bundles.

4.1 A measure density on the space of continuous curves contained in a regular domain

4.1.1 The construction of a measure density

Let $U \subseteq M$ be a connected relatively compact open domain with smooth boundary such that $x_0 \in U$. Let us endow the space

$$\mathcal{C}_t(\bar{U}) = \{c : [0,t] \to \bar{U} \mid c \text{ continuous, with } c(0) = x_0\} ,$$

with the corresponding Wiener measure $w_t^{(U)}$ as we constructed it in subsection 2.2.2. Endowed with the distance $D(c_0,c_1) = \max_{s\in[0,t]} d(c_0(s), c_1(s))$ this becomes a metric space; it is separable (and therefore second-countable), as shown in [Michael61].

Let

$$\mathrm{Cyl}(\mathcal{C}_t(\bar{U})) = \{f \in C_b(\mathcal{C}_t(\bar{U})) \mid \exists k \geq 0 \text{ și } f_k \in C(\bar{U}^{2^k}) \text{ astfel încât } f = f_k \circ \pi_k\}$$

be the algebra of continuous cylindrical functions on $\mathcal{C}_t(\bar{U})$. Clearly, $\mathrm{Cyl}(\mathcal{C}_t(\bar{U})) \subset L^1(\mathcal{C}_t(\bar{U}))$.

Let us define the (obviously linear) functional $W_{\alpha,t}^{(U)} : \mathrm{Cyl}(\mathcal{C}_t(\bar{U})) \to \mathbb{C}$ by

$$W_{\alpha,t}^{(U)}(f_k \circ \pi_k) = \int_M \mathrm{d}x_1\, h_\alpha^{(U)}\left(\frac{t}{2^k}, x_0, x_1\right) \dots \int_M \mathrm{d}x_{2^k}\, h_\alpha^{(U)}\left(\frac{t}{2^k}, x_{2^k-1}, x_{2^k}\right) f_k(x_1, \dots, x_{2^k})$$

for all $f_k \circ \pi_k \in \mathrm{Cyl}(\mathcal{C}_t(\bar{U}))$. In what follows we are going to show that we may extend it to $L^1(\mathcal{C}_t(\bar{U}))$ by continuity.

Propoziția 4.1. *The algebra* $\mathrm{Cyl}(\mathcal{C}_t(\bar{U}))$ *is dense in* $L^p(\mathcal{C}_t(\bar{U}), w_t^{(U)})$ *for all* $p \in [1,\infty)$.

Notice now that

$$|W_{\alpha,t}^{(U)}(f_k \circ \pi_k)| \leq$$
$$\leq \int_U \mathrm{d}x_1 \left| h_\alpha^{(U)}\left(\frac{t}{2^k}, x_0, x_1\right)\right| \dots \int_U \mathrm{d}x_{2^k} \left| h_\alpha^{(U)}\left(\frac{t}{2^k}, x_{2^k-1}, x_{2^k}\right)\right| |f_k(x_1, \dots, x_{2^k})| \leq$$

$$\leq \int_U \mathrm{d}x_1\, h^{(U)}\left(\frac{t}{2^k}, x_0, x_1\right) \dots \int_U \mathrm{d}x_{2^k}\, h^{(U)}\left(\frac{t}{2^k}, x_{2^k-1}, x_{2^k}\right) |f_k(x_1, \dots, x_{2^k})| =$$
$$= \|f_k \circ \pi_k\|_{L^1(\mathcal{C}_t(\bar{U}))} \ ,$$

so $W^{(U)}_{\alpha,t}$ is continuous with respect to the norm $\|\cdot\|_{L^1(\mathcal{C}_t(\bar{U}))}$ on $\mathrm{Cyl}(\mathcal{C}_t(\bar{U}))$ and, since the latter is dense in $L^1(\mathcal{C}_t(\bar{U}))$, it follows that we may extend $W^{(U)}_{\alpha,t}$ to a continuous linear functional on $L^1(\mathcal{C}_t(\bar{U}))$, therefore there exists $\rho^{(U)}_{\alpha,t} \in L^\infty(\mathcal{C}_t(\bar{U}))$ such that $W^{(U)}_{\alpha,t}(f) = \int_{\mathcal{C}_t(\bar{U})} f \rho^{(U)}_{\alpha,t}\, \mathrm{d}w^{(U)}_t$ for all $f \in L^1(\mathcal{C}_t(\bar{U}))$. Moreover, $|W^{(U)}_{\alpha,t}(f)| \leq \|f\|_{L^1(\mathcal{C}_t(\bar{U}))}$ for all $f \in L^1(\mathcal{C}_t(\bar{U}))$, so $\|\rho^{(U)}_{\alpha,t}\|_{L^\infty(\mathcal{C}_t(\bar{U}))} \leq 1$.

4.1.2 A sequence of approximations for the measure density

So far, $\rho^{(U)}_{\alpha,t}$ has been constructed by a very abstract procedure, so its properties are difficult to study. Therefore, in what follows we shall construct a series of concrete approximations of this function, which will enjoy two essential properties: a group property, and the fact of being of absolute value 1. We will then show that this sequence converges to $\rho^{(U)}_{\alpha,t}$, and so these two properties will also be transmitted to $\rho^{(U)}_{\alpha,t}$.

Let P be a regular Borel probability on the interval $[0,1]$; we shall see later that the role of P will be to completely classify the stochastic integrals. We shall denote by $M_1(P)$ the moment of P of order 1, i.e

$$M_1(P) = \int_{[0,1]} \tau\, \mathrm{d}P(\tau) \ .$$

If the points x and y may be joined by a unique minimizing geodesic, we shall denote it by $\gamma_{x,y} : [0,1] \to M$, where we understand that $\gamma(0) = x$ and $\gamma(1) = y$. We then define $I_P(\alpha) : M \times M \to \mathbb{R}$ as follows:

- $I_P(\alpha)(x,y) = \int_{[0,1]} \alpha_{\gamma_{x,y}(\tau)}(\dot{\gamma}_{x,y}(\tau))\, \mathrm{d}P(\tau)$, if there exists a unique minimizing geodesic $\gamma_{x,y}$ between x and y;
- $I_P(\alpha)(x,y) = 0$, otherwise.

For all $k \in \mathbb{N}$, let us define now the "approximations" $S_{P,t,k}(\alpha) : \mathcal{C}_t \to \mathbb{R}$ by

$$S_{P,t,k}(\alpha)(c) = \sum_{j=0}^{2^k-1} I_P(\alpha)\left(c\left(\frac{jt}{2^k}\right), c\left(\frac{(j+1)t}{2^k}\right)\right) +$$
$$+ \frac{t}{2^k}(\mathrm{d}^*\alpha)\left(c\left(\frac{jt}{2^k}\right)\right) \int_{[0,1]} (2\tau - 1)\, \mathrm{d}P(\tau) \ .$$

So far, $\rho^{(U)}_{\alpha,t}$ has been obtained by a highly abstract procedure (section 4.1), which makes it extremely difficult to study its properties and use it. The following theorem fixes this situation, giving us a concrete understanding of the measure density of this object as the limit of a sequence of explicitly constructed approximations.

Teorema 4.2. $\lim_{k\to\infty} \mathrm{e}^{\mathrm{i}S_{P,t,k}(\alpha)}|_{\mathcal{C}_t(\bar{U})} = \rho^{(U)}_{\alpha,t}$ *în* $L^2(\mathcal{C}_t(\bar{U}), w^{(U)}_t)$*, uniformly with respect to* t *from compact subsets of* $(0,\infty)$*, and uniformly with respect to* $x_0 \in U$*.*

4.2 A measure density on the space of continuous curves contained in the whole manifold

Let us now consider an exhaustion $M = \bigcup_{j\geq 0} U_j$ of M with regular domains as above. To make the notation a little easier, let us write $\rho^{(j)}_{\alpha,t}$ instead of $\rho^{(U_j)}_{\alpha,t}$, $h^{(j)}_\alpha$ instead of $h^{(U)j)}_\alpha$ and $w^{(j)}_t$ instead of $w^{(U_j)}_t$. We know for now that $\mathrm{e}^{\mathrm{i}S_{P,t,k}}|_{\mathcal{C}_t(\bar{U}_j)} \to \rho^{(j)}_{\alpha,t}$ in $L^2(\mathcal{C}_t(\bar{U}_j), w^{(j)}_t)$.

The following lemma is as important as it is trivial.

Lema 4.3. *If* $i \leq j$ *then* $\rho^{(j)}_{\alpha,t}|_{\mathcal{C}_t(\bar{U}_i)} = \rho^{(i)}_{\alpha,t}$ *almost everywhere with respect to the Wiener measure* $w^{(i)}_t$*.*

From the equality $\rho^{(j)}_{\alpha,t}|_{\mathcal{C}_t(\bar{U}_i)} = \rho^{(i)}_{\alpha,t}$ almost everywhere for $i \leq j$ it follows that there exists the pointwise limit $\lim_{j\to\infty} \tilde{\rho}^{(j)}_{\alpha,t}$, denoted $\rho_{\alpha,t}$. It will be a measurable function (as the pointwise limit of measurable functions), and bounded by 1 almost everywhere, since all functions in the sequence are so. It will be therefore a function from $L^\infty(\mathcal{C}_t, w_t)$. With the argument from the lemma above it can be shown that $\rho_{\alpha,t}|_{\mathcal{C}_t(\bar{U}_j)} = \rho_{\alpha,t}^{(j)}$ for all $j \geq 0$, as elements of $L^\infty(\mathcal{C}_t, w_t^{(j)})$.

After all these preparatory results, we can finally prove the central result we were after.

Teorema 4.4.

$$\lim_{k\to\infty} \|\mathrm{e}^{\mathrm{i}S_{P,t,k}(\alpha)} - \rho_{\alpha,t}\|_{L^2(\mathcal{C}_t)} = 0$$

uniformly with respect to $t \in (0, T]$*, for every* $T > 0$.

Corolarul 4.5. $\rho_{\alpha,t}$ *does not depend on the exhaustion with regular domains that we have employed.*

4.3 The Stratonovich integral

We have obtained that $\rho_{\alpha,t}$ is the limit of a series of exponentials with imaginary exponents. It is reasonable to ask whether $\rho_{\alpha,t}$ has such a form, and if so it is interesting to study its exponent. The answer to this question (and the moral justification of all the effort made in order to obtain the above technical results) is given by the following theorem.

Teorema 4.6. *There exists a unique real function* $\mathrm{Strat}_t(\alpha) \in L^0(\mathcal{C}_t)$ *such that* $\rho_{\alpha,t} = \mathrm{e}^{\mathrm{i}\,\mathrm{Strat}_t(\alpha)}$.

When we constructed the functions $S_{P,t,k}(\alpha)$, we did it so that the functions $\mathrm{e}^{\mathrm{i}S_{P,t,k}(\alpha)}$ approximate $\rho_{\alpha,t} = \mathrm{e}^{\mathrm{i}\,\mathrm{Strat}_t(\alpha)}$ in $L^2(\mathcal{C}_t)$. We shall see that this approximation property extends, albeit more weakly, to the exponents.

Teorema 4.7. $\lim_{k\to\infty} S_{P,t,k}(\alpha) = \mathrm{Strat}_t(\alpha)$ *in measure, uniformly with respect to* t *from bounded subsets of* $(0, \infty)$.

We shall see in detail in the next section that Strat_t is the Stratonovich integral. That this is the limit in measure of a certain series of approximations was already known; what is new (and unexpected) is that it can be obtained as the generator of the unitary group discussed above (or, abandoning rigour, that it is the "logarithm" of the function $\rho_{\alpha,t}$). This suggests that $\rho_{\alpha,t}$, being the imaginary exponential of a kind of curvilinear integral, can in turn be interpreted as a kind of parallel transport - namely the stochastic parallel transport in the bundle $M \times \ mathbbC$. These considerations, however, will be the subject of another chapter.

Corolarul 4.8. *The map* $\Omega^1(M) \ni \alpha \mapsto \mathrm{Strat}_t(\alpha) \in L^0(\mathcal{C}_t)$ *is* $\mathbb{R}$*-linear.*

4.4 A general concept of stochastic integral

In order to be able to derive a general concept of stochastic integral, let us return to the approximations $S_{P,t,k}(\alpha)$ constructed above and let us define the approximations

$$\begin{aligned} A_{P,t,k}(\alpha)(c) = S_{P,t,k}(\alpha)(c) - \frac{t}{2^k}\sum_{j=0}^{2^k-1} (\mathrm{d}^*\alpha)\left(c\left(\frac{jt}{2^k}\right)\right)\int_{[0,1]}(2\tau-1)\,\mathrm{d}P(\tau) = \\ = \sum_{j=0}^{2^k-1} I_P(\alpha)\left(c\left(\frac{jt}{2^k}\right), c\left(\frac{(j+1)t}{2^k}\right)\right) \end{aligned} \tag{4.1}$$

for every curve $c \in \mathcal{C}_t$ (that is, we only drop the term containing $\mathrm{d}^*\alpha$ și $M_1(P)$). We shall study the behaviour of these approximations on smooth curves, and this "classical" behavior will guide us towards the understanding of "stochastic" behavior.

4.4.1 An approximation of the line integral along differentiable curves

Teorema 4.9. *If* $c : [0, t] \to M$ *is a twice continuously differentiable curve, then* $\int_c \alpha = \lim_{k\to\infty} A_{P,t,k}(\alpha)(c)$.

4.4.2 A geometric definition and a classification of the stochastic integrals

We shall draw inspiration from the similarity between theorem 4.9 and theorem 4.7 in order to produce a concept of stochastic integral. Let $\mathrm{Prob}([0,1])$ be the space of regular Borel probabilities on the interval $[0,1]$.

Definiția 4.10. *We shall say that* $\mathrm{Int}_t : \Omega^1(M) \to L^0(\mathcal{C}_t)$ *is a stochastic integral if and only if there exists* $P \in \mathrm{Prob}([0,1])$ *such that* $\mathrm{Int}_t(\alpha)$ *is the limit in measure of the approximation sequence* $A_{P,t,k}(\alpha)$ *for all* $\alpha \in \ Omega^1(M)$*. In this case, we shall denote this stochastic integral by* $\mathrm{Int}_{P,t}$*, to highlight the dependence on* P*.*

Although in theorem 4.7 the convergence in measure was uniform with respect to t from compact subsets, we did not include this property in the definition of stochastic integrals, it not being clear at the time of writing whether this is an essential ingredient of the concept or, on the contrary, an accidental one without major consequences.

Observația 4.11. *Considering the nature of the concept of convergence in measure, we emphasize that* $\mathrm{Int}_t(\alpha)(\cdot)$ *should be understood not as a function defined for each curve in* $\mathcal{C}_t$*, but as an element of* $L^0(\mathcal{C}_t)$*. This is the major difference from the curvilinear integral, which is defined for every piecewise-smooth curve.*

Let $P \in \mathrm{Prob}([0,1])$. We are interested in studying whether there is any relationship between $\mathrm{Int}_{P,t}(\alpha)$ and the function $\mathrm{Strat}_t(\alpha)$ constructed in section 4.3. We note that for any $c \in \mathcal{C}_t$,

$$\lim_{k\to\infty} \frac{t}{2^k} \sum_{j=0}^{2^k-1} (\mathrm{d}^*\alpha)\left(c\left(\frac{jt}{2^k}\right)\right) = \int_0^t (\mathrm{d}^*\alpha)(c(s))\,\mathrm{d}s$$

as a limit of Riemann sums associated to the continuous function $(\mathrm{d}^*\alpha)\circ c$, the equidistant division with 2^k steps of the interval $[0,t]$ and the system of intermediate points $(c(\frac{jt}{2^k}))_{0\le j\le 2^k-1}$. Moreover, then, the above convergence is also in measure (pointwise convergence involving the one in measure). If we take the limit in measure in formula (4.1) defining the approximations $A_{P,t,k}$, we get

$$\mathrm{Int}_{P,t}(\alpha)(c) = \mathrm{Strat}_t(\alpha)(c) - \int_{[0,1]} (2\tau - 1)\,\mathrm{d}P(\tau) \int_0^t (\mathrm{d}^*\alpha)(c(s))\,\mathrm{d}s$$

which shows that even though the probability P can be extremely complicated, the corresponding stochastic integral $\mathrm{Int}_{P,t}$ retains from it only its moment of order 1, and that any two probabilities in $\mathrm{Prob}([0,1])$ with the same moment of order 1 produce the same stochastic integral. Also, since P appears only in the term containing $\mathrm{d}^*\alpha$, and therefore any stochastic integral is essentially the function Strat_t, we conclude that $\mathrm{Int}_{P,t}$ exists for any $P \in \mathrm{Prob}([0,1])$. Since the function $2\tau - 1$ has a minimum of -1 and a maximum of 1 on $[0,1]$, and since P is a probability, it follows that $\int_{[0,1]} (2\tau-1)\,\mathrm{d}P(\tau) \in [-1,1]$, and so that any stochastic integral Int_t on $\mathcal{C}_t$ is of the form $\mathrm{Int}_t(\alpha) = \mathrm{Strat}_t(\alpha) + \theta \int_0^t (\mathrm{d}^*\alpha)(c(s))\,\mathrm{d}s$ with $\theta \in [-1,1]$.

Moreover, if $P, Q \in \mathrm{Prob}([0,1])$, then

$$\mathrm{Int}_{P,t}(\alpha)(c) = \mathrm{Int}_{Q,t}(\alpha)(c) - 2\int_{[0,1]} \tau\,\mathrm{d}(P-Q)(\tau) \int_0^t (\mathrm{d}^*\alpha)(c(s))\,\mathrm{d}s\ ,$$

and so any two stochastic integrals differ only by a multiple of the integral of $\mathrm{d}^*\alpha$.

This is the right moment to see some concrete examples of such stochastic integrals and to compare the results obtained so far with those already obtained in the probabilistic literature.

- If $P = \delta_0$ (the Dirac measure concentrated at 0), then

$$\mathrm{Int}_{\delta_0,t}(\alpha)(c) = \mathrm{Strat}_t(\alpha)(c) + \int_0^t (\mathrm{d}^*\alpha)(c(s))\,\mathrm{d}s\ .$$

Comparing the approximations $A_{\delta_0,k,t}(\alpha)$ of $I_{\delta_0,t}(\alpha)$ with those of Theorem 7.37 on p.110 of [Émery89] (or with theorem A from [Darling84], which is however stated in some slightly more restrictive assumptions), we immediately recognize that $\mathrm{Int}_{\delta_0,t}(\alpha)$ is the Itô integral of α, therefore from now on we shall denote it $\mathrm{Ito}_t(\alpha)$.

- If $P = \mathrm{Leb}_{[0,1]}$ (the Lebesgue measure on $[0,1]$), or $P = \delta_{\frac{1}{2}}$ (the Dirac measure concentrated at $\frac{1}{2}$), or $P = \frac{1}{2}\delta_1$, or $P = \frac{1}{2}(\delta_0 + \delta_1)$, then the corresponding stochastic integral is

$$\mathrm{Int}_{\mathrm{Leb}_{[0,1]},t}(\alpha) = \mathrm{Strat}_t(\alpha)\ .$$

Comparing the approximations $A_{\mathrm{Leb}_{[0,1]},k,t}(\alpha)$ of $\mathrm{Int}_{\mathrm{Leb}_{[0,1]},t}(\alpha)$ with those of Theorem 7.14 on p.96 of [Émery89], we immediately recognize that $\mathrm{Int}_{\mathrm{Leb}_{[0,1]},t}(\alpha)$ is the Stratonovich integral of α. We also note that $S_{\mathrm{Leb}_{[0,1]},k,t} = A_{\mathrm{Leb}_{[0,1]},k,t}$ for every $k \geq 0$. (The reader is invited to compare these results also with section 6 of [Norris92].)

• In general, if $M_1(P) = \int_{[0,1]} \tau \, \mathrm{d}P(\tau)$, then the stochastic integral $\mathrm{Int}_{P,t}(\alpha)$ corresponding to $P \in \mathrm{Prob}([0,1])$ coincides with the one produced by the probabilities $\delta_{M_1(P)}$ (the Dirac measure concentrated at $M_1(P) \in [0,1]$) and $(1 - M_1(P))\delta_0 + M_1(P)\delta_1$, all these probabilities having the same moment of order 1, namely $M_1(P)$. But, although in principle we could study stochastic integrals by just using these very simple measures, some results are easier to prove using more complicated measures having the same moment of order 1. For example, for the Stratonovich integral it is sometimes more convenient to use the Lebesgue measure on $[0,1]$.

• The case $M_1(P) = 0$ is much simpler than all the others, since the only measure P with this property is δ_0. Indeed, if $[\varepsilon, 1] \subset (0,1]$ then

$$0 = \int_{[0,1]} \tau \, \mathrm{d}P(\tau) \geq \int_{[\varepsilon,1]} \tau \, \mathrm{d}P(\tau) \geq \varepsilon \, P([\varepsilon,1]) \geq 0 \, ,$$

where $P([\varepsilon,1]) = 0$, so $P((0,1]) = 0$, that is, P is concentrated at 0. Since P is a probability, it follows that $P = \delta_0$. Therefore, the only integral corresponding to the situation $M_1(P) = 0$ is the Itô integral.

Observația 4.12. *It follows from the above examples that the Stratonovich and Itô integrals of α are equal if and only if $\mathrm{d}^*\alpha = 0$. Compare this result with the one in lemma 8.24 on p.120 of [Émery89], where only a sufficient condition (hard to verify in practice) is given in order to have this equality. More precisely, Émery first introduces the concept of stochastic parallel transport in TM and in T^*M, starting from which he builds some semimartingales that depend on α, and if these have finite variation then the two stochastic integrals of α are equal.*

Propoziția 4.13. *The Stratonovich integral has the property that $\mathrm{Strat}_t(\mathrm{d}f)(c) = f(c(1)) - f(x_0)$ for every continuously differentiable real function f and for almost every c in$\mathcal{C}_t$.*

Corolarul 4.14 (Itô's lemma)**.** *If $f : M \to \mathbb{R}$ is continuously twice-differentiable, and if Δ is the Laplace-Beltrami operator on M, then*

$$f(c(1)) = f(x_0) + \mathrm{Ito}_t(\mathrm{d}f)(c) + \int_0^t (\Delta f)(c(s)) \, \mathrm{d}s$$

for almost all $c \in \mathcal{C}_t$.

As always, when we study an object that depends on some parameters it is interesting to study the properties of this dependence. In particular, it is interesting to study how the stochastic integral $\mathrm{Int}_{P,t}(\alpha)$ depends on $t \in (0,T]$, with $T > 0$. Since these integrals live in different spaces for different values of t (namely the spaces $L^0(\mathcal{C}_t)$, which depend on t), they must first be embedded in some same space in order for us to be able to compare them, which we shall do in the next theorem. To state it, we recall that the natural topology in $L^0(\mathcal{C}_t)$ is that of convergence in the measure w_t (which is easily shown to be equal to the measure $(\mathrm{res}_{[0,t]})_* w_T$, where $\mathrm{res}_{[0,t]} : \mathcal{C}_T \to \mathcal{C}_t$ is the restriction $\mathrm{res}_{[0,t]}(c) = c|_{[}0,t])$. This topology is metrizable by any distance of the form

$$d_t(f,g) = \int_{\mathcal{C}_t} \varphi(|f-g|) \, \mathrm{d}w_t =$$

$$= \int_{\mathcal{C}_T} \varphi(|f \circ \mathrm{res}_{[0,t]} - g \circ \mathrm{res}_{[0,t]}|) \, \mathrm{d}w_T = d_T(f \circ \mathrm{res}_{[0,t]}, g \circ \mathrm{res}_{[0,t]})$$

where $\varphi : [0,\infty) \to [0,\infty)$ must be continuous, bounded, concave, increasing, with $\varphi(0) = 0$ and $\varphi > 0$ on $(0,\infty)$. Any such distance is called a "Lévy distance".

For the line integral, if $c : [0,T] \to M$ is continuously differentiable, then we have

$$\left| \int_c \alpha - \int_{\mathrm{res}_{[0,t]}(c)} \alpha \right| = \left| \int_{\mathrm{res}_{[t,T]}(c)} \alpha \right| \leq \sup_{s \in [0,T]} |\alpha_{c(s)}(\dot{c}(s))| \, (T-t) \, .$$

In particular, the map $[0,T] \ni t \mapsto \int_{\mathrm{res}_{[0,t]}(c)} \alpha \in \mathbb{R}$ is continuous. In the next theorem we shall present a weaker analogue of this conclusion for stochastic integrals.

Teorema 4.15. *For all $\alpha \in \Omega^1(M)$, the map $(0,T] \ni t \mapsto \mathrm{Int}_{P,t}(\alpha) \circ \mathrm{res}_{[0,t]} \in L^0(\mathcal{C}_T)$ is continuous.*

4.5 The Feynman-Kac-Itô formula

In the following we shall see that the Stratonovich integral emerges absolutely naturally when we try to derive the analogue of the Feynman-Kac formula in the presence of a magnetic potential represented by the 1-form α. Theorem 15.3 on p.162 of [Simon79b] presents an alternative perspective on this problem, but only in the case $M = \mathbb{R}^n$. The following theorem needs an extension of some of the notations used so far: thus, instead of $\mathcal{C}_t$ which denotes a space of curves starting from x_0, we shall use the space

$$\mathcal{C}_{t,x} = \{c : [0,t] \to M \text{ continuous}; c(0) = x\}$$

which we shall endow with the Wiener measure $w_{t,x}$ and the Stratonovich integral $\mathrm{Strat}_{t,x}$.

Teorema 4.16 (The Feynman-Kac-Itô formula). *If $V : M \to \mathbb{R}$ is continuous and $\inf V > -\infty$, and if $f \in L^2(M)$, then*

$$(\mathrm{e}^{-tH_{\mathrm{d}+\mathrm{i}\alpha,V}} f)(x) = \int_{\mathcal{C}_{t,x}} \mathrm{e}^{\mathrm{i}\,\mathrm{Strat}_{t,x}(\alpha) - \int_0^t V(c(s))\,\mathrm{d}s} f(c(t))\,\mathrm{d}w_{t,x}(c)$$

for all $t > 0$ and almost all $x \in M$.

We choose to end here the presentation of results concerning the stochastic calculus on manifolds. The main goal was to show that this chapter of mathematics can be built entirely using only tools of functional analysis, without having to resort to concepts or techniques of probability theory. Also, another aim was to propose a point of view from which the whole of stochastic calculus is seen as arising from a small number of fundamental ideas, as an analogue of line integration. The strategy adopted in this text allowed the classification of stochastic integrals and the highlighting of the simple relationship between them. In particular, we have seen that the Stratonovich integral is the natural stochastic tool in problems with a pronounced geometric component, as it allows many ideas of differential geometry to be carried over almost unchanged to the stochastic context. In the next chapter we shall see that the Itô integral is the preferred stochastic tool in problems with a more pronounced probabilistic component. Neither of these two integrals is ”better” than the other, the choice of using any one of them being made according to the specifics of the problem studied.

From a technical point of view, we managed to perform all the constructions intrinsically, i.e. without resorting to embedding the manifold into Euclidean spaces with Whitney's theorem, as is done in most texts on stochastic calculus on manifolds. We also did not need Cartan's rolling map, which is used for example in [Elworthy82]. It was also not necessary to choose an interpolation rule ([Émery89], p.94, where it is necessary to use the measurable selection theorem, the verification of the hypotheses of which is completely non-trivial because it requires the study of the properties of the Whitney topology on the space of differentiable curves from $[0,1]$ to M - technical details expedited very quickly by Émery), its role being taken over by the cut-off function χ as well as by the extension by 0 of the expression $\int_{[0,1]} \alpha_{\gamma_{x,y}(\tau)}(\dot\gamma_{x,y}(\tau))\,\mathrm{d}P(\tau)$ whenever y is far from x. We also did not need second-order tangent vectors and Schwartz's second-order differential geometry, which Émery needs. This parsimony of fundamental concepts and technical means was an intended goal of the present text which stems from the belief that conceptual and technical minimality must be an imperative of any intellectual construction.

Let us end by quickly discussing the orientation of the tangent vectors in the sequence of approximations of the Itô integral, which we recall corresponds to $P = \delta_0$. It is reasonable that upon first contact with this integral we ask: why should the tangent vectors (”increments”) be directed toward the future and not toward the past? If we understand the stochastic integrals in the sense presented so far, the orientation of the tangent vectors $\exp_x^{-1} y$ (that is, $y - x$ in $\mathbb{R}^n$) is given by the expression $\int_{[0,1]} \alpha_{\gamma_{x,y}(\tau)}(\dot\gamma_{x,y}(\tau))\,\mathrm{d}P(\tau)$ (taking then $P = \delta_0$), i.e. essentially by the expression $\alpha_{c(s)}(\dot c(s))$ that appears in the usual path integral. We thus understand that the orientation of the tangent vectors is not something to be chosen, but is merely an automatic, inevitable consequence of the analogy between stochastic integration and integration along smooth curves.

5. THE SQUARE INTEGRABILITY OF THE STOCHASTIC INTEGRALS

We saw in the previous chapter that stochastic integrals are functions from $L^0(\mathcal{C}_t)$. It is reasonable to ask whether they belong to more convenient function spaces, for example to $L^2(\mathcal{C}_t)$. Unfortunately, our efforts seem to indicate that without imposing additional assumptions this question proves to be intractable; we shall therefore assume in this chapter that α has compact support. Under this assumption the answer to our question is affirmative and, moreover, we shall see that stochastic integrals can be extended from 1-forms with compact support to much larger spaces of forms.

5.1 Topological preliminaries

The function $I_P(\alpha)$, although not smooth on $M \times M$, was regular enough to allow us to obtain the results of the previous chapter. In this chapter, however, we shall need to use it in calculations involving derivatives, so we will adjust it with a cut-off function to avoid its singularities and turn it into a smooth function. Thus, if $\chi : M \times M \to [0,1]$ is the cut-off function from chapter 3, we define $I_P^{(\chi)}(\alpha)$ by $I_P^{(\chi)}(\alpha) = \chi\, I_P(\alpha)$ and we quickly note that $I_P^{(\chi)}(\alpha)$ is smooth. Corresponding to this function, we define the approximations $A_{P,t,k}^{(\chi)}(\alpha) \in L^0(\mathcal{C}_t)$ by

$$A_{P,t,k}^{(\chi)}(\alpha)(c) = \sum_{j=0}^{2^k-1} I_P^{(\chi)}(\alpha) \left(c\left(\frac{jt}{2^k}\right), c\left(\frac{(j+1)t}{2^k}\right)\right) .$$

To begin with, let us prove that the approximations $A_{P,t,k}^{(\chi)}$ also converge in measure to $\operatorname{Int}_{P,t}(\alpha)$, so the introduction of χ does not change anything from this point of view.

Teorema 5.1. *The functions $A_{P,t,k}^{(\chi)}$ converge in measure to $\operatorname{Int}_{P,t}(\alpha)$.*

5.2 The square integrability of the Itô integral

Teorema 5.2. *If α has compact support then $\operatorname{Ito}_t(\alpha) \in L^2(\mathcal{C}_t)$. Furthermore, there exists a measurable and bounded function $N : M \to [0,\infty)$ such that*

$$\|\operatorname{Ito}_t(\alpha)\|^2_{L^2(\mathcal{C}_t)} \le \int_{\mathcal{C}_t} \int_0^t \|\alpha_{c(s)}\|^2_{T^*_{c(s)}M}\, N(c(s))\, \mathrm{d}s\, \mathrm{d}w_t(c) \ .$$

Corolarul 5.3. *If α has compact support, then $\operatorname{Int}_{P,t}(\alpha) \in L^2(\mathcal{C}_t)$. Furthermore,*

$$\|\operatorname{Int}_{P,t}(\alpha)\|_{L^2(\mathcal{C}_t)} \le \sqrt{\int_{\mathcal{C}_t} \int_0^t \|\alpha_{c(s)}\|^2_{T^*_{c(s)}M}\, N(c(s))\, \mathrm{d}s\, \mathrm{d}w_t(c) +}$$
$$+ 2M_1(P) \sqrt{\int_{\mathcal{C}_t} \left(\int_0^t (\mathrm{d}^*\alpha)(c(s))\, \mathrm{d}s \right)^2 \mathrm{d}w_t(c)}$$

A consequence of this last formula is what we might call ”the stochastic embedding”: if we endow the space of smooth 1-forms with compact support $\Omega^1_0(M)$ with the norm

$$\|\alpha\|_{(P,\chi)} = \sqrt{\int_{\mathcal{C}_t} \int_0^t \|\alpha_{c(s)}\|^2_{T^*_{c(s)}M}\, N(c(s))\, \mathrm{d}s\, \mathrm{d}w_t(c) +}$$

$$+ 2M_1(P)\sqrt{\int_{\mathcal{C}_t} \left(\int_0^t (\mathrm{d}^* \alpha)(c(s)) \, \mathrm{d}s \right)^2 \mathrm{d}w_t(c)}$$

then the map $\Omega_0^1(M) \ni \alpha \mapsto \mathrm{Int}_P(\alpha) \in L^2(\mathcal{C}_t)$ is a continuous embedding with norm at most 1. Moreover, if we denote by $\widehat{\Omega_0^1(M)}^{(P,\chi)}$ the completion of $\Omega_0^1(M)$ in this norm, we may extend the stochastic integral by continuity to a continuous linear map $\mathrm{Int}_P : \widehat{\Omega_0^1(M)}^{(P,\chi)} \to L^2(\mathcal{C}_t)$, so that we may define the stochastic integral on a significantly wider space than $\Omega_0^1(M)$. When $P = \delta_0$ (and therefore we work with the Itô integral), and $M = \mathbb{R}^n$, the interaction between the heat kernel and the distance function becomes very simple, a situation in which we find "Itô's isometry", i.e.

$$\|\mathrm{Ito}_t(\alpha)\|^2_{L^2(\mathcal{C}_t)} \leq 2n \int_{\mathcal{C}_t} \int_0^t \|\alpha_{c(s)}\|^2_{T^*_{c(s)}\mathbb{R}^n} \, \mathrm{d}s \, \mathrm{d}w_t(c) \ ,$$

known in the stochastic calculus on $\mathbb{R}^n$ (with inequality, however, instead of equality).

We also note that if $0 < M_1(P) \leq M_1(Q)$, then

$$\|\alpha\|_{(P,\chi)} \leq \|\alpha\|_{(Q,\chi)} \leq \frac{M_1(Q)}{M_1(P)} \|\alpha\|_{(P,\chi)} \ ,$$

so the spaces $\widehat{\Omega_0^1(M)}^{(P,\chi)}$ and $\widehat{\Omega_0^1(M)}^{(Q,\chi)}$ coincide as topological vector spaces and their norms are quasi-isometric.

Corolarul 5.4. *If* $\alpha \in \widehat{\Omega_0^1(M)}^{(\delta_0)}$ *then* $\int_{\mathcal{C}_t} \mathrm{Ito}_t(\alpha) \, \mathrm{d}w_t = 0$.

The reader is invited to compare the above corollary with theorem 3.2.1.(iii) on p.30 of [Øksendal13], which states a similar result in the usual stochastic formalism in $\mathbb{R}^n$. Likewise, one is invited to compare the theorem 5.2 with the statement and proof of Itô's isometry from the stochastic calculus in $\mathbb{R}^n$ (lemma 3.1.5 on p.26 and corollary 3.1.7 on p.29 of the same book), for a better understanding of the results obtained in this text. Also, the reader is invited to notice that in this chapter the main character was the Itô integral, of all the stochastic integrals it lending itself best to the type of mathematical analysis reasoning constructed above. This situation is in contrast to the more geometric one in chapter 4, where the Stratonovich integral proved to be the main tool. This suggests that in applications the choice of one among all the stochastic integrals is one of convenience: where geometric reasoning prevails, the use of the Stratonovich integral is preferable, while in reasonings with a strong analytical character, the Itô integral is recommended.

6. THE STOCHASTIC PARALLEL TRANSPORT

The purpose of this chapter is the construction of the concept of stochastic parallel transport using only functional analysis tools and concepts. The overall idea of the text and even some proof techniques will resemble those used in constructing the Stratonovich stochastic integral; the non-triviality of the bundle in which we shall work and the non-trivial dimension of its fiber will, however, complicate the problem from a technical point of view.

6.1 Motivation and the outline of the chapter

The concept of "stochastic parallel transport" in a vector bundle E over a Riemannian manifold M is usually presented as a byproduct of the concept of "stochastic differential equation"; this is the approach chosen in most texts, for example [IW89] and [Meyer82]. Despite this, K. Itô had conceived this concept differently ([Itô63], [Itô75a], [Itô75b]): for any continuous curve $c : [0, t] \to M$, we consider the unique geodesic segment that joins the consecutive "dyadic" points $c(\frac{jt}{2^k})$ and $c(\frac{(j+1)t}{2^k})$, we these assemble these 2^k geodesic segments into a single piecewise geodesic zigzag curve, and we parallel transport the vector $v \in E_{c(0)}$ along this curve to $E_{c(t)}$; for almost all continuous curves c, the limit when $k \to \infty$ will exist and will be called "the stochastic parallel transport of v along c". The two approaches are equivalent, as shown in [Meyer82] and [Émery90], and both are constructed in the context of probability theory, thus being more accessible to probabilists. The purpose of this chapter is to reconstruct the concept of "stochastic parallel transport" using only the tools and techniques of functional analysis, thus making it accessible to a much wider class of mathematicians.

Since the constructions in this text will be quite technical, let us outline the intuition that underlies them. Let $D_t = \{\frac{jt}{2^k} \mid k \in \mathbb{N},\ j \in \mathbb{N} \cap [0, 2^k]\}$ - the "dyadic" numbers between 0 and t. Following Itô's idea, the parallel transport of $v \in E_{c(0)}$ along the zigzag line determined by the points $\{c(0), c(\frac{t}{2^k}), \dots, c(\frac{(2^k-1)t}{2^k}), c(t)\}$ is the parallel transport $T_{k,0}$ from $c(0)$ to $c(\frac{t}{2^k})$, followed by the parallel transport $T_{k,1}$ from $c(\frac{t}{2^k})$ to $c(\ frac2t2^k)$ and so on, ending with the parallel transport $T_{k,2^k-1}$ from $c(\frac{(2^k-1)t}{2^k})$ to $c(t)$; formally, it is $T_{k,2^k-1} \dots T_{k,0} v$. Now follows the central insight of this chapter: the composition of operators $T_{k,2^k-1} \dots T_{k,0} v$ can be seen as the "contraction" of all the tensor products in

$$
\begin{aligned}
T_{k,2^k-1} \otimes \dots \otimes T_{k,0} \otimes v \in \left(E_{c(t)} \otimes E^*_{c(\frac{(2^k-1)t}{2^k})}\right) \otimes \dots \otimes \left(E_{c(\frac{t}{2^k})} \otimes E^*_{c(0)}\right) \otimes E_{c(0)} \simeq \\
\simeq E_{c(t)} \otimes \left(E^*_{c(\frac{(2^k-1)t}{2^k})} \otimes E_{c(\frac{(2^k-1)t}{2^k})}\right) \otimes \dots \otimes \left(E^*_{c(0)} \otimes E_{c(0)}\right) \simeq \\
\simeq E_{c(t)} \otimes \operatorname{End} E^*_{c(\frac{(2^k-1)t}{2^k})} \otimes \dots \otimes \operatorname{End} E^*_{c(0)} \ .
\end{aligned}
$$

Let us now see what this "contraction" means. If U and V are finite-dimensional vector spaces, if $u \in U$ and $\omega \in V^*$, and $A : U \to V$ is a linear operator, then $\omega \otimes A \otimes u \in V^* \otimes V \otimes U^* \otimes U \simeq \operatorname{End} V^* \otimes \operatorname{End} U^*$; if Id_U and Id_V are the identity operators on U and V, then $\mathrm{Id}_U \otimes \mathrm{Id}_V \in \operatorname{End} V \otimes \operatorname{End} U$, so it makes sense to apply $\omega \otimes A \otimes u$ to $\mathrm{Id}_U \otimes \mathrm{Id}_V$, the result being $\omega(Au)$. We see that to perform this type of contraction in the composition of parallel transports considered above, we need an additional factor $E^*_{c(t)}$ with which to pair the factor $E_{c(t)}$ in order to obtain $\operatorname{End} E^*_{c(t)}$. This means that if $\eta_{c(t)} \in E_{c(t)}$, then

$$\eta_{c(t)} \otimes T_{k,2^k-1} \otimes \dots \otimes T_{k,0} \otimes v \in \operatorname{End} E^*_{c(t)} \otimes \dots \otimes \operatorname{End} E^*_{c(0)}$$

and

$$\eta_{c(t)}(T_{k,2^k-1} \dots T_{k,0} v) = (\eta_{c(t)} \otimes T_{k,2^k-1} \otimes \dots \otimes T_{k,0} \otimes v)(\mathrm{Id}_{E_{c(t)}} \otimes \dots \otimes \mathrm{Id}_{E_{c(0)}}) \ .$$

Following now Itô, we let $k \to \infty$; what we get, then, will be a contraction between two tensor products with an infinite number of factors; the rigorous construction of these tensor products will be our first task, but we can say for now that these tensor product spaces will be $\mathcal{E}_c = \otimes_{s\in D_t} \operatorname{End} E_{c(s)}$ and its dual. If we denote the space of continuous curves by $\mathcal{C}_t$, the fact that $\mathcal{E}_c$ depends on $c \in \mathcal{C}_t$ suggests that the disjoint union $\coprod_{c\in\mathcal{C}_t} \mathcal{E}_c$ will be a (topological) vector bundle of infinite rank over $\mathcal{C}_t$. Since $\eta_{c(t)} \otimes T_{k,2^k-1} \otimes \ldots \otimes T_{k,0} \otimes v$ takes values in the fiber $\mathcal{E}_c^*$ for any $k \in \mathbb{N}$ and any $c \in \mathcal{C}_t$, we understand that these tensor products of parallel transports will be sections of some type in $\mathcal{E}^*$, which makes it reasonable to assume that their limit when $k \to \infty$ (the stochastic parallel transport from which we removed η) will be a section of the same type. Indeed, this will be the case, and to prove it we shall resort to Chernoff's theorem on the approximation of contraction semigroups.

An unexpected consequence of the construction in this chapter is a new version of the Feynman-Kac formula in vector bundles: not only will its proof be completely new, but also its assumptions appear to be the most general considered so far in the literature, to the best of the author's knowledge ; more precisely, the potential will be chosen only locally integrable and lower bounded, and no restriction will be imposed on the manifold.

The plan of this chapter is as follows, with notations to be explained as they become necessary:

- we shall construct a bundle $\mathcal{E}$ over $\mathcal{C}_t$, the fibers of which will be Hilbert spaces of infinite dimension;

- we shall consider spaces of p-integrable sections in $\mathcal{E}$ and $\mathcal{E}^*$ and, in particular, we shall obtain by an abstract argument a particular essentially bounded section $\rho_{t,\omega,\eta}$, which will be the mean square limit of a sequence of sections $(P_{t,\omega,\eta,k})_{k\in\mathbb{N}}$ that will be constructed explicitly;

- we shall highlight a continuous conjugate-linear map $\mathcal{P}^p_{t,v} : \Gamma^p(\mathcal{E}) \to \Gamma^p(p_t^* E)$, which we shall see to enclose a lot of information both about the geometry of the bundle $E \to M$ and about the Wiener measure;

- using the map $\mathcal{P}^2_{t,v}$ we shall be able to give a functional-analytic meaning to the concept of stochastic parallel transport;

- finally, using the same map $\mathcal{P}^2_{t,v}$, we shall see an extension in the bundle E of the Feynman-Kac formula.

6.2 A bundle of infinite rank

Let $E \to M$ be a Hermitian bundle of complex rank $r \in \mathbb{N}$, endowed with a Hermitian connection ∇. Let $D_t = \{\frac{jt}{2^k} \mid k \in \mathbb{N},\ j \in \mathbb{N} \cap [0, 2^k]\}$ - the "dyadic" numbers between 0 and t. We are interested in making sense of the fibration described intuitively, not rigorously, by $\mathcal{E} = \boxtimes_{s\in D_t} \operatorname{End} E \to \mathcal{C}_t$.

If $c \in \mathcal{C}_t$, we define the fiber $\mathcal{E}_c$ of $\mathcal{E}$ over c as $\bigotimes_{s\in D_t} \operatorname{End} E_{c(s)}$. This is a tensor product with infinitely many factors the definition of which, in turn, is non-trivial and requires discussion. Thus, for any $x \in M$, we endow the space $\operatorname{End} E_x$ with the Hermitian product given by $\langle A, B\rangle_{\operatorname{End} E_x} = \frac{1}{r} \operatorname{Trace}(AB^*)$ when $A, B \in \operatorname{End} E_x$. We notice that $\langle \cdot, -\rangle_{\operatorname{End} E_x} = \frac{1}{r} \langle \cdot, -\rangle_{E_x\otimes E_x^*}$, the Hermitian product in the right member being the natural one on the space $E_x \otimes E_x^*$. If $\operatorname{Id}_{E_x} \in \operatorname{End} E_x$ is the identity operator, then $\|\operatorname{Id}_{E_x}\|_{\operatorname{End} E_x} = 1$. This allows us to rigorously construct the tensor product that gives the fiber $\mathcal{E}_c$ as follows. If $D_{t,k} = \{\frac{jt}{2^k} \mid j \in \mathbb{N} \cap [0, 2^k]\}$ for any $k \in \mathbb{N}$, then for any $k \le k'$, we identify the tensor monomial $\otimes_{s\in D_{t,k}} e_{c(s)} \in \bigotimes_{s\in D_{t,k}} \operatorname{End} E_{c(s)}$ with the monomial $\otimes_{s\in D_{t,k'}} e'_{c(s)} \in \bigotimes_{s\in D_{t,k'}} \operatorname{End} E_{c(s)}$ where $e'_{c(s)} = e_{c(s)}$ for $s \in D_{t,k}$ and $e'_{c(s)} = \operatorname{Id}_{E_{c(s)}}$ for $s \in D_{t,k'} \setminus D_{t,k}$. This identifies the space $\bigotimes_{s\in D_{t,k}} \operatorname{End} E_{c(s)}$ with a subspace of the space $\bigotimes_{s\in D_{t,k'}} \operatorname{End} E_{c(s)}$, which allows us to consider the algebraic inductive limit $\varinjlim_{k\in\mathbb{N}} \bigotimes_{s\in D_{t,k}} \operatorname{End} E_{c(s)}$. This limit space is naturally endowed with a Hermitian product obtained from the Hermitian product on every space $\operatorname{End} E_{c(s)}$ as discussed above. Finally, the algebraic inductive limit obtained is completed under this Hermitian product, in the sense of Hilbert spaces, the result being the Hilbert space denoted $\mathcal{E}_c$. It is important to note that $\mathcal{E}_c$ is separable because the set of indices in the inductive limit is $\mathbb{N}$ and every space in the inductive limit is finite-dimensional.

We now define the total space of the fibration as $\mathcal{E} = \bigcup_{c\in\mathcal{C}_t} \{c\} \times \mathcal{E}_c$. The natural projection of $\mathcal{E}$ onto $\mathcal{C}_t$ will be $\operatorname{pr}_{\mathcal{E}} : \mathcal{E} \to \mathcal{C}_t$. For now, $\mathcal{E}$ has only been constructed fiberwise as a set; with the help of explicitly constructed mutually compatible local trivializations, the space $\mathcal{E}$ is endowed with a topology.

Unlike the chapter where we construct the stochastic integrals, we shall use here the notation $\pi_k : \mathcal{C}_t \to M^{2^k+1}$ given by

$$\pi_k(c) = \left(c(0), c\left(\frac{t}{2^k}\right), \ldots, c\left(\frac{(2^k-1)t}{2^k}\right), c(t)\right) .$$

Propoziția 6.1. *The projections $\pi_k : \mathcal{C}_t \to M^{2^k+1}$ and the projection $\operatorname{pr}_{\mathcal{E}} : \mathcal{E} \to \mathcal{C}_t$ are continuous.*

6.3 Integrable sections in bundles of infinite rank

Definiția 6.2. *We shall say that the section $\sigma : \mathcal{C}_t \to \mathcal{E}$ is a* ***cylindrical section*** *if and only if there exists a section $s \in \Gamma^\infty \left((\mathrm{End}\, E)^{\boxtimes (2^k+1)} \right)$ such that $\sigma = s \circ \pi_k$.*

Definiția 6.3. *We define the Lebesgue space $\Gamma^2(\mathcal{E})$ of square-integrable sections as the space of measurable sections σ, identified under equality almost everywhere, with the property that the function $\mathcal{C}_t \ni c \mapsto \|\sigma(c)\|_{\mathcal{E}_c} \in [0, \infty)$ is in $L^2(\mathcal{C}_t, w_t)$.*

Teorema 6.4. *The space $\Gamma^2(\mathcal{E})$ endowed with the scalar product*

$$\langle \sigma_1, \sigma_2 \rangle_{\Gamma^2(\mathcal{E})} = \int_{\mathcal{C}_t} \langle \sigma_1(c), \sigma_2(c) \rangle_{\ mathcalE_c} \, \mathrm{d}w_t(c)$$

is a Hilbert space. Its dual is $\Gamma^2(\mathcal{E}^*)$, where $\mathcal{E}^*$ is the dual bundle of $\mathcal{E}$ in which the fiber $\mathcal{E}^*_c$ is the dual space of $\mathcal{E}_c$ for any $c \in \mathcal{C}_t$.

More generally, and along the same lines of thought, one can introduce the spaces $\Gamma^p(\mathcal{E})$ for all $p \in [1, \infty]$, which will be Banach spaces. In particular, $\Gamma^q(\mathcal{E}) \subseteq \Gamma^p(\mathcal{E})$ if $p \le q$, since the Wiener measure is finite. Also, $\Gamma^p(\mathcal{E}^*)$ is the dual of $\Gamma^{\frac{p}{p-1}}(\mathcal{E})$ for any $p \in (1, \infty]$. The proofs are analogous to those for p-integrable function spaces, the latter being available, for example, in chapter 4 of [Brezis11].

Teorema 6.5. *The space $\mathrm{Cyl}_t(\mathcal{E})$ of essentially bounded cylindrical sections is dense in the space $\Gamma^2(\mathcal{E})$.*

We shall consider, as in the other chapters, an exhaustion of M with relatively compact domains with smooth boundary $M = \bigcup_{i \in \mathbb{N}} U_i$, such that $x_0 \in U_0$. In particular, these will be Riemannian manifolds, so all the above considerations apply to them. The geometric objects extrinsic to the domain U_i will be visually represented by the restriction symbol (for example, the fibrate $E|_{U_i}$), and the objects intrinsically associated with U_i will carry the index (i) (the heat kernel associated with the connection ∇ in $E|_{U_i}$ will be $h^{(i)}_\nabla$, the Laplacian understood as the generator of the heat semigroup in $C(\overline{U_i})$ will be $L^{(i)}$ etc.).

We shall further define a linear functional on $\Gamma^2(\mathcal{E}|_{\mathcal{C}_t(\overline{U_i})})$, which we shall then show to be continuous, so it will correspond to a section of $\Gamma^2(\mathcal{E}^*|_{\mathcal{C}_t(\overline{U_i})})$ which will be intimately related to the stochastic parallel transport. Let us fix $\omega \in E^*_{x_0}$ and $\eta \in \Gamma_{cb}(E)$, and define the functional $W^{(i)}_{t,\omega,\eta}$ on essentially bounded cylindrical sections as follows: if $s : \overline{U_i}^{2^k+1} \to (\mathrm{End}\, E)^{\boxtimes(2^k+1)}|_{U^{2^k+1}}$ is an essentially bounded section, we define

$$W^{(i)}_{t,\omega,\eta}(s \circ \pi_k) = \int_{U_i} \mathrm{d}x_1 \dots \int_{U_i} \mathrm{d}x_{2^k} \left[\omega \otimes h^{(i)}_\nabla \left(\frac{t}{2^k}, x_0, x_1 \right) \otimes \dots \right.$$
$$\left. \dots \otimes h^{(i)}_\nabla \left(\frac{t}{2^k}, x_{2^k-1}, x_{2^k} \right) \otimes \eta(x_{2^k}) \right] \cdot s(x_0, x_1, \dots, s_{x_{2^k}}) \ .$$

The dot in the integrand denotes a contraction of tensors which, to be understood, requires a little comment. The term $\omega \otimes h_\nabla \left(\frac{t}{2^k}, x_0, x_1 \right) \otimes \dots \otimes h_\nabla \left(\frac{t}{2^k}, x_{2^k-1}, x_{2^k} \right) \otimes \eta(x_{2^k})$ belongs to the space $E^*_{x_0} \otimes (E_{x_0} \otimes E^*_{x_1}) \otimes \dots \otimes (E_{x_{2^k-1}} \otimes E^*_{x_{2^k}}) \otimes E_{x_{2^k}}$ which is naturally isomorphic to $(E^*_{x_0} \otimes E_{x_0}) \otimes \dots \otimes (E^*_{x_{2^k}} \otimes E_{x_{2^k}})$, which in turn is isomorphic to $(\mathrm{End}\, E^*)_{x_0} \otimes \dots \otimes (\mathrm{End}\, E^*)_{x_{2^k}}$ (we emphasize that the latter isomorphism is not the natural one but is multiplied by a norming factor, because the scalar product of two endomorphisms was chosen so that the identity has norm 1). In turn, $s(x_0, \dots, x_{2^k})$ belongs to the space $(\mathrm{End}\, E)_{x_0} \otimes \dots \otimes (\mathrm{End}\, E)_{x_{2^k}}$, which clarifies the tensor contraction in the integral.

$W^{(i)}_{t,\omega,\eta}$ is well-defined, trivially linear, and

$$|W^{(i)}_{t,\omega,\eta}(s \circ \pi_k)| \le \frac{1}{\sqrt{r}} \|\omega\|_{E^*_{x_0}} \int_{\mathcal{C}_t(\overline{U_i})} \|(\eta(c(t))\| (s \circ \pi_k)(c)\| \, \mathrm{d}w^{(i)}_t(c) \le$$
$$\le \frac{1}{\sqrt{r}} \|\omega\|_{E^*_{x_0}} [(\mathrm{e}^{-tL^{(i)}} \|\eta\|^2)(x_0)]^{\frac{1}{2}} \, \|(s \circ \pi_k)(c)\|_{\Gamma^2(\mathcal{E}|_{\mathcal{C}_t(\overline{U_i})})} \ ,$$

where $\|\eta\|$ denotes the function $M \ni x \mapsto \|\eta(x)\|_{E_x} \in [0, \infty)$.

Since the essentially bounded sections are dense in $\Gamma^2(\mathcal{E}|_{\mathcal{C}_t(\overline{U_i})})$, it follows that $W^{(i)}_{t,\omega,\eta}$ may uniquely extend to a continuous linear functional on this space, so there exists a unique $\rho^{(i)}_{t,\omega,\eta} \in \Gamma^2(\mathcal{E}^*|_{\mathcal{C}_t(\overline{U_i})})$ such that

$$W^{(i)}_{t,\omega,\eta}(\sigma) = \int_{\mathcal{C}_t(\overline{U_i})} \rho^{(i)}_{t,\omega,\eta}(c)(\sigma(c)) \, \mathrm{d}w^{(i)}_t(c)$$

for all $\sigma \in \Gamma^2(\mathcal{E}|_{\mathcal{C}_t(\overline{U_i})})$. Furthermore, $\|\rho^{(i)}_{t,\omega,\eta}\|_{\Gamma^2(\mathcal{E}^*|_{\mathcal{C}_t(\overline{U_i})})} \leq \frac{1}{\sqrt{r}} \|\omega\|_{E^*_{x_0}} [(\mathrm{e}^{-tL^{(i)}} \|\eta\|^2)(x_0)]^{\frac{1}{2}}$.

Next, we shall seek to understand what kind of geometric object $\rho^{(i)}_{t,\omega,\eta}$ is; more precisely we shall investigate its connection with the parallel transport in E. To this end, for any $(x, y) \in M \times M$ let us define $P(x, y) : E_y \to E_x$ by:

- $P(x, y) =$ the parallel transport from y to x, if there exists a unique minimizing geodesic (in M) between x and y defined on $[0, 1]$,
- $P(x, y) = 0$ otherwise.

We note that P thus defined is a section in the bundle $E \boxtimes E^*$. As the subset

$$\{(x, y) \in M \times M \mid \text{there is a unique minimizing geodesic between } x \text{ and } y \text{ defined on } [0, 1]\}$$

is open in $M \times M$, in particular it will be measurable. Since $(x, y) \mapsto P(x, y)$ is a continuous section in $E \boxtimes E^*$ on this subset, P will be a measurable section in this bundle.

With P thus defined, for any curve $c \in \mathcal{C}_t$ and any $k \in \mathbb{N}$ we define

$$P_{t,\omega,\eta,k}(c) = \omega \otimes P\left(c(0), c\left(\frac{t}{2^k}\right)\right) \; otimes \ldots \otimes P\left(c\left(\frac{(2^k-1)t}{2^k}\right), c(t)\right) \otimes \eta(c(t))$$

and, as P is measurable and bounded by 1 in the pointwise operator norm, we conclude that $P_{t,\omega,\eta,k}$ is a measurable and bounded cylindrical section in the bundle $\mathcal{E}^*$. We shall show that $\rho_{t,\omega,\eta}$ is the limit of the sequence $(P_{t,\omega,\eta,k})_{k\in\mathbb{N}}$ in $\Gamma^2(\mathcal{E}^*)$.

Teorema 6.6. *The sequence* $(P_{t,\omega,\eta,k}|_{\mathcal{C}_t(\overline{U_i})})_{k\in\mathbb{N}}$ *converges to* $\rho^{(i)}_{t,\omega,\eta}$ *in* $\Gamma^2(\mathcal{E}^*|_{\mathcal{C}_t(\overline{U_i})})$*, uniformly with respect to* t *from bounded subsets of* $(0, \infty)$.

So far, we have worked on the curve spaces $\mathcal{C}_t(\overline{U_i})$ associated with the domains U_i in the exhaustion of M. We did this only for technical reasons. The time is now to remove this exhaustion and obtain global objects and relationships between them.

Teorema 6.7. *If* $i \leq j$ *then* $\rho^{(j)}_{t,\omega,\eta}|_{\mathcal{C}_t(\overline{U_i})} = \rho^{(i)}_{t,\omega,\eta}$.

This compatibility property between the sections $(\rho^{(j)}_{t,\omega,\eta})_{j\in\mathbb{N}}$ ensures that the section defined by $\rho_{t,\omega,\eta} = \lim_{j\to\infty} \rho^{(j)}_{t,\omega,\eta}$ is well defined and that $\rho_{t,\omega,\eta}|_{\mathcal{C}_t(\overline{U_j})} = \rho^{(j)}_{t,\omega,\eta}$. In the definition of $\rho_{t,\omega,\eta}$ we understand that we work with measurable representatives of the classes of sections $\rho^{(j)}_{t,\omega,\eta} \in \Gamma^\infty(\mathcal{E}^*|_{\mathcal{C}_t(\overline{U_j})}) \subseteq \Gamma^\infty(\mathcal{E}^*)$, so that changing these representatives changes the limit only on some null subset.

Teorema 6.8. *The section* $\rho_{t,\omega,\eta}$ *thus defined is measurable and essentially bounded. In addition,* $\|\rho_{t,\omega,\eta}(c)\|_{\mathcal{E}^*_c} = \frac{1}{\sqrt{r}} \|\omega\|_{E^*_{x_0}} \|\eta_{c(t)}\|_{E_{c(t)}}$ *for almost any* $c \in \mathcal{C}_t$.

We have seen in theorem 6.6 that $P_{t,\omega,\eta,k}|_{\mathcal{C}_t(\overline{U_j})} \to \rho_{t,\omega,\eta}|_{\mathcal{C}_t(\overline{U_j})}$ in square mean with respect to the measure $w^{(j)}_t$, for any $j \in \mathbb{N}$. We shall now prove that all these convergences on the spaces $\mathcal{C}_t(\overline{U_j})$ lead to a global convergence, on the space $\mathcal{C}_t$.

Teorema 6.9. *The sequence* $(P_{t,\omega,\eta,k})_{k\in\mathbb{N}}$ *converges to* $\rho_{t,\omega,\eta}$ *in* $\Gamma^2(\mathcal{E}^*)$*, uniformly with respect to* $t \in (0, T]$ *for every* $T > 0$.

6.4 A continuous map between spaces of integrable sections

The object $\rho_{t,\omega,\eta}$, although it contains a lot of information, does not have a clear geometric interpretation. In what follows we shall obtain a new mathematical object from it which will turn out to be the stochastic parallel transport. Furthermore, the section η should play no role in defining the stochastic parallel transport, since the sections $\mathcal{P}_{t,v,k}$ below (which will be seen to approximate the stochastic parallel transport) do not depend on it. Indeed, the use of η was only dictated by the technical details of chosen proof strategy, with η having a purely auxiliary role, and our efforts in what follows will be focused on removing it from the results obtained.

Let $p_t : \mathcal{C}_t \to M$ be the projection given by $p_t(c) = c(t)$. For all $v \in E_{x_0}$ we shall denote by $v^\flat \in E^*_{x_0}$ the linear form defined by $v^\flat = \sqrt{r}\, \langle \cdot, v \; rangle_{E_{x_0}} \in E^*_{x_0}$. The notation $p_t^* E$ denotes the bundle over $\mathcal{C}_t$ obtained as the pull-back under p_t of the bundle $E \to M$. The fiber $(p_t^* E)_c$ will, by definition, be $E_{c(t)}$, and we shall prefer the second notation for its simplicity. We shall next construct, for any $p \in (1, \infty]$, a continuous conjugate-linear map $\Gamma^p(\mathcal{E}) \ni \xi \mapsto \mathcal{P}^p_{t,v}(\xi) \in \Gamma^p(p_t^* E)$ such that

$$[\rho_{t,v^\flat,\eta}(c)]\,[\xi(c)] = \langle \mathcal{P}^p_{t,v}(\xi)(c),\, \eta(c(t)) \rangle_{E_{c(t)}}$$

for all $\eta \in \Gamma^\infty(E)$.

Let $M = \bigcup_{i \in \mathbb{N}} V_i'$ be a cover of M with open trivialization domains for E. Let $V_0 = V_0'$ and $V_i = V_i' \setminus (V_0 \cup \ldots \cup V_{i-1})$ for $i \geq 1$; these subsets will be mutually disjoint measurable trivialization domains. Above each V_i we shall consider a measurable orthonormal frame $\{\eta_i^1, \ldots, \eta_i^r\}$. Defining η^l by $\eta^l|_{V_i} = \eta_i^l$ for any $1 \leq l \leq r$ and $i \in \mathbb{N}$, we obtain a measurable orthonormal global frame $\{\eta^1, \ldots, \eta^r\}$ in E, consisting of sections from $\Gamma^\infty(E) \subseteq \Gamma_t$, so from sections η^l for which $\rho_{t,v\ flat,\eta^l}$ is a well-defined object as we saw above. Let $\{\eta_1, \ldots, \eta_r\}$ be the dual frame in E^* defined by $\eta_k(\eta^l) = \delta_k^l$ (Kronecker's symbol).

It is shown that $\rho_{t,\omega,\eta}$, which has so far been constructed under the assumption that $\eta \in \Gamma_{cb}(E)$, can be defined for a significantly wider class of sections η that includes the space $\Gamma^\infty(E)$.

If $\sigma \in \Gamma^{\frac{p}{p-1}}(p_t^* E^*)$, then for every $c \in \mathcal{C}_t$ we may write

$$\sigma(c) = \sum_{l=1}^r \{\sigma(c)\ \ [\eta^l(c(t))]\}\, \eta_l(c(t)) \in E^*_{c(t)}\ .$$

We then define the function $\mathcal{F}^p_{t,v,\xi} : \Gamma^{\frac{p}{p-1}}(p_t^* E^*) \to \mathbb{C}$ by

$$\mathcal{F}^p_{t,v,\xi}(\sigma) = \sum_{l=1}^r \int_{\mathcal{C}_t} \{[\sigma(c)]\,[\eta^l(c(t))]\}\, \overline{\{[\rho_{t,v^\flat,\eta^l}(c)]\,[\xi(c)]\}}\, \mathrm{d}w_t(c)$$

and we immediately see that it is linear. It is also shown to be continuous, so there exists a unique section $\mathcal{P}^p_{t,v}(\xi) \in \Gamma^p(p_t^* E)$ such that

$$\mathcal{F}^p_{t,v,\xi}(\sigma) = \int_{\mathcal{C}_t} [\sigma(c)]\,[\mathcal{P}^p_{t,v}(\xi)(c)]\, \mathrm{d}w_t(c)$$

for any $\sigma \in \Gamma^{\frac{p}{p-1}}(p_t^* E^*)$, and

$$\|\mathcal{P}^p_{t,v}(\xi)\|_{\Gamma^p(p_t^* E)} \leq r\, \|v\|_{E_{x_0}}\, \|\xi\|_{\Gamma^p(\mathcal{E})}\ .$$

The continuity and conjugate-linearity of $\xi \mapsto \mathcal{P}^p_{t,v}(\xi)$ are obvious.

Corolarul 6.10. *In the above notations,* $\langle \eta_{c(t)},\, \mathcal{P}^p_{t,v}(\xi)(c) \rangle_{E_{c(t)}} = [\rho_{t,v^\flat,\eta}(c)]\,[\xi(c)]$ *for all* $\eta \in \Gamma^\infty(E)$ *and almost all* $c \in \mathcal{C}_t$.

The linearity of the dependence of $\mathcal{P}^p_{t,v}(\xi)$ on $v \in E_{x_0}$ allows us to obviously define a section $\mathcal{P}^p_t(\xi) \in \Gamma^p(p_t^* E) \otimes E^*_{x_0}$ so that $\mathcal{P}^p_t(\xi)(c)(v) = \mathcal{P}^p_{t,v}(\xi)(c)$ for any $v \in E_{x_0}$ and $c \in \mathcal{C}_t$. Furthermore,

$$\|\mathcal{P}^p_t(\xi)(c)\|_{E_{c(t)} \otimes E^*_{x_0}} = \sup_{\|v\|_{E_{x_0}} = 1} \|\mathcal{P}^p_{t,v}(\xi)(c)\|_{E_{c(t)}} \leq r\, \|v\|_{E_{x_0}}\, \|\xi\|_{\Gamma^p(\mathcal{E})} = r\, \|\xi\|_{\Gamma^p(\mathcal{E})}\ .$$

The map $\mathcal{P}^p_t$ encloses a lot of information about the differential geometry and the stochastic calculus associated with the bundle E. Below we shall see only two uses of it, enough hopefully to convince the reader of its importance.

6.5 The first application: the stochastic parallel transport

We start by defining the objects $\mathcal{P}_{t,v,k}$ by the explicit formula

$$\mathcal{P}_{t,v,k}(c) = P\left(c(t), c\left(\frac{(2^k-1)t}{2^k}\right)\right) \ldots P\left(c\left(\frac{t}{2^k}\right), c(0)\right) v$$

where $v \in E_{x_0}$ is arbitrary, $c \in \mathcal{C}_t$ and $k \in \mathbb{N}$. We note that $\mathcal{P}_{t,v,k}(c)$ belongs to the fiber $E_{c(t)} = (p_t^* E)_c$. Since we have shown that P is a measurable map, it means that $\mathcal{P}_{t,v,k}$ is a measurable section in $p_t^* E$. Since, in addition, $\|\mathcal{P}_{t,v,k}\|_{E_{c(t)}} \leq \|v\|_{E_{x_0}}$, we deduce that $\mathcal{P}_{t,v,k} \in \Gamma^\infty(p_t^* E) \subseteq \Gamma^2(p_t^* E)$.

Let us define the section $\mathrm{Id} : \mathcal{C}_t \to \mathcal{E}$ by $\mathrm{Id}(c) = \otimes_{s \in D_t} \mathrm{Id}_{E_{c(s)}} \in \mathcal{E}_c$; more precisely, $\mathrm{Id}(c)$ is the equivalence class (in the sense of constructing the inductive limit as a space of equivalence classes), for instance, of the element $\mathrm{Id}_{E_{x_0}}$, and the map $\mathcal{C}_t \ni c \mapsto \mathrm{Id}_{E_{x_0}} \in \mathcal{E}_c$ is obviously continuous. Moreover, it is obvious that $\|\mathrm{Id}(c)\|_{\mathcal{E}_c} = \|\mathrm{Id}(E_{x_0})\|_{\mathrm{End}\, E_{x_0}} = 1$, so $\mathrm{Id} \in \Gamma^\infty(p_t^* E) \subseteq \Gamma^2(p_t^* E)$. We then notice that

$$[P_{t,v^\flat,\eta,k}(c)]\,[\mathrm{Id}(c)] = [P_{t,v^\flat,\eta,k}(c)]\,[\mathrm{Id}_{E_{c(0)}} \otimes \ldots \otimes \mathrm{Id}_{E_{c(t)}}] = \langle \eta(c(t)),\, \mathcal{P}_{t,v,k}(c) \rangle_{E_{c(t)}}\ .$$

Teorema 6.11. *$\mathcal{P}_{t,v,k} \to \mathcal{P}^2_{t,v}(\mathrm{Id})$ in $\Gamma^2(p_t^* E)$ for all $v \in E_{x_0}$, uniformly with respect to $t \in (0,T]$ for every $T > 0$.*

Comparing this result with the one obtained by probabilistic methods ([Itô63], [Itô75a], [Itô75b]), we conclude that $\mathcal{P}^2_{t,v}(\mathrm{Id})$ is the stochastic parallel transport in E of the vector $v \in E_{x_0}$. In particular, $\mathcal{P}^2_{t,v}(\mathrm{Id})$ does not depend on the choices made in its construction (trivialization domains, orthonormal local frames above them, etc.), being the limit of a sequence of sections that do not depend on these choices.

6.6 The second application: the Feynman-Kac formula in vector bundles

In the following, we shall present an extension of the Feynman-Kac formula in Hermitian bundles, one of the motivations being the desire to convince the reader that the new formalism and methods constructed in this chapter allow one to obtain all the already known results related to the stochastic parallel transport, sometimes even under better assumptions than those currently in use. We shall work with a "potential" $V \in \Gamma^1_{loc}(\mathrm{End}\, E)$ with the property that $\operatorname{ess\,inf}_{x \in M} \min \operatorname{spec} V(x) = \beta > -\infty$ (in short: $V \geq \beta$), and with $V(x)$ self-adjoint for almost all $x \in M$. The quadratic form $\Gamma_0(E) \ni \eta \mapsto \int_M \langle V(x)\eta_x, \eta_x\rangle_{E_x} \, \mathrm{d}x \in \mathbb{R}$ will give rise to a self-adjoint densely defined in $\Gamma^2(E)$, which we shall continue to denote V, for simplicity. Indeed, the quadratic form is well-defined because

$$\left| \int_M \langle V(x)\eta_x, \eta_x\rangle_{E_x} \, \mathrm{d}x \right| \leq \sup_{x \in M} \|\eta_x\|^2 \int_{\operatorname{supp} \eta} \|V(x)\|_{op} \, \mathrm{d}x < \infty \ .$$

It is also lower bounded by β because if $\{e_{1,x}, \dots, e_{r,x}\}$ is an orthonormal basis in E_x made of eigenvectors of $V(x)$ corresponding to the eigenvalues $\lambda_{1,x} \leq \dots \leq \lambda_{r,x} \subset [\beta, \infty)$ for almost all $x \in M$, and if $\eta_x = \sum_{i=1}^r \alpha_{i,x}\, e_{i,x}$, we have that

$$\langle V(x)\eta_x, \eta_x\rangle_{E_x} = \left\langle \sum_{i=1}^r \alpha_{i,x}\lambda_{i,x} e_{i,x}, \sum_{j=1}^r \alpha_{j,x} e_{j,x} \right\rangle_{E_x} =$$
$$= \sum_{i=1}^r \lambda_{i,x} |\alpha_{i,x}|^2 \geq \sum_{i=1}^r \lambda_{1,x} |\alpha_{i,x}|^2 = \lambda_{1,x} \|\eta\|^2_{E_x} \geq \beta \|\eta\|^2_{E_x} \ .$$

In the same way (using quadratic forms) one can construct the self-adjoint densely defined operator corresponding to the sum of $\nabla^* \nabla$ and V; we shall denote it $H_{\nabla, V}$. Of course, this construction can be performed not only on M but also on any relatively compact open domain with smooth boundary.

When the point from which the curves start is no longer the fixed $x_0 \in M$, as before, but some variable $x \in M$, all the objects that depend on it will get it as an additional lower index, i.e. the space of continuous curves what start from x will be $\mathcal{C}_{t,x}$, on which we shall have the Wiener measure $w_{t,x}$ (precisely the notations used in theorem 4.16), and all objects constructed in this text will accordingly acquire an additional index x, that is, we shall have the sections $\rho_{t,\omega,\eta,x}$, $\mathcal{P}_{t,v,x}$ and $\mathcal{P}_{t,x}$ etc.

For each $k \in \mathbb{N}$ let us denote by $\mathrm{V}_{t,x,k} \in \Gamma^\infty(\mathcal{E})$ the section given by

$$\mathrm{V}_{t,x,k}(c) = \mathrm{e}^{-\frac{t}{2^k} V\left(c\left(\frac{t}{2^k}\right)\right)} \otimes \dots \otimes \mathrm{e}^{-\frac{t}{2^k} V(c(t))} \ .$$

Since $V \geq \beta$ and $t \geq 0$, it is immediate that $\|\mathrm{V}_{t,x,k}(c)\|_{\mathcal{E}_c} \leq \mathrm{e}^{-t\beta}$ for almost all $c \in \mathcal{C}_{t,x}$, so by the Banach-Alaoglu theorem we conclude that there exists a subsequence $(k_l)_{l \in \mathbb{N}} \subseteq \mathbb{N}$ such that the subsequence $(\mathrm{V}_{t,x,k_l})_{l \in \mathbb{N}}$ has a weak accumulation point $\mathrm{V}_{t,x} \in \Gamma^2(\mathcal{E})$. We conclude, in particular, that the section $\mathcal{P}^2_{t,x}(\mathrm{V}_{t,x})$ exists in $\Gamma^2(p_t^* E) \otimes E_x^*$.

Teorema 6.12 (The Feynman-Kac formula). *If $\eta \in \Gamma^2(E)$ then*

$$(\mathrm{e}^{-tH_{\nabla,V}} \eta)(x) = \int_{\mathcal{C}_{t,x}} [\mathcal{P}^2_{t,x}(\mathrm{V}_{t,x})(c)]^* \, \eta_{c(t)} \, \mathrm{d}w_{t,x}(c)$$

for all $t > 0$ and almost all $x \in M$.

We notice that if we define the map $\mathcal{V}_{t,x} : \mathcal{C}_{t,x} \to \mathrm{End}\, E_x$ by

$$\mathcal{V}_{t,x}(c) = [\mathcal{P}^2_{t,x}(\mathrm{V}_{t,x})(c)]^* \, [\mathcal{P}^2_{t,x}(\mathrm{Id})(c)]$$

the Feynman-Kac formula can be rewritten, in a trivial way, in the equivalent form

$$(\mathrm{e}^{-tH_{\nabla,V}}\eta)(x) = \int_{\mathcal{C}_{t,x}} [\mathcal{V}_{t,x}(c)]\, [\mathcal{P}^2_{t,x}(\mathrm{Id})(c)]^{-1}\, \eta_{c(t)}\, \mathrm{d}w_{t,x}(c) \ ;$$

thus rewritten, the Feynman-Kac formula in bundles was also obtained by other authors, but in other contexts and under different assumptions:

- in [BP08] functional analysis techniques are used (also based on Chernoff's theorem) but the potential is assumed to be smooth and M is closed;

- in [DT01] probabilistic techniques are used to give in Proposition 4.5 very abstract conditions under which the Feynman-Kac formula is valid, after which Proposition 5.1 shows that these assumptions are satisfied when M is closed, and the potential (denoted there $\mathcal{R}$) is assumed to be smooth (p.48);

- in [Güneysu10] the Feynman-Kac formula is proved using functional analysis techniques, but the existence of the stochastic parallel transport is accepted without proof, under the assumption that the manifold is both metrically and stochastically complete, and under very generous assumptions on the potential (in Theorem 3.1 it is assumed to be essentially bounded, and in theorem 3.3 the result is extended to the more general situation when the potential is locally square-integrable); in remark 1.4 the author outlines how the proof would have to be modified if the assumption of metric completeness were to be dropped, but without elaborating;

- in [BG20] (not yet published, existing only as a preprint at the time of writing) the potential V, which can be understood as a differential operator of order 0, is now assumed to be a differential operator of order 0 or 1 acting on smooth sections in E (so in particular V has smooth coefficients), so that the operator $\nabla^*\nabla + V$ is sectorial; this potential naturally gives rise to a stochastic differential equation the unique solution of which is assumed to be locally square-integrable, uniformly with respect to $x \in M$ (in our notations); this assumption guarantees that the equality in the Feynman-Kac formula will hold everywhere, not just almost everywhere. No restrictions are imposed on M.

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