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George Georgescu

The purpose of this paper is to prove a version of the Chang's omitting types theorem [2] for the Boolean valued models. The proof combines the Chang's arguments with some technics of Shorb [6] and Loullis [3].

Let L be a countable first-order language having the variables $v_0, v_1, \ldots$.

Throughout this paper we shall suppose that L has at least one constant. We shall write $\varphi(\vec{x})$ (resp. $\varphi(\vec{c})$) instead of $\varphi(x_1,\ldots,x_n)$ (resp. $\varphi(c_1,\ldots,c_n)$). The set of formulas (resp. sentences) of L will be denoted by $\mathcal{F}(L)$ (resp. $\mathcal{S}(L)$). For any set C of new constants. L(C) is the language obtained from L by adding the constants of C.

If T is a consistent set of sentences of L we call two formulas φ and ψ equivalent with respect to T if $T \vdash \varphi \leftrightarrow \psi$. Let ${}^*\varphi$ be the equivalence class of $\varphi \in \mathcal{F}(L)$ and $\mathcal{F}(L)/_T$ the Lindenbaum-Tarski algebra

$$\mathcal{F}(L)/_T = \{ {}^*\varphi \mid \varphi \in \mathcal{F}(L) \}.$$

The Boolean algebra $\mathcal{S}(L)/_T = \{ {}^*\varphi \mid \varphi \in \mathcal{S}(L) \}$ is a subalgebra of $\mathcal{F}(L)/_T$.

In the usual way, we define by induction the Σ_n - formulas and Π_n - formules of L. The set of Σ_n - formulas (resp. Π_n - formulas) of L will be denoted by $\Sigma_n(L)$(resp. $\Pi_n(L)$).

Let B be a complete Boolean algebra. A B-valued structure M is defined by a function $\| \;\|_M : \mathcal{S}(L(M)) \longrightarrow B$ which satisfies the following conditions:

(B_1) $\|a = a\|_M = 1$

(B_2) $\|a = b\|_M \leq \|b = a\|_M$

(B_3) $\|a = b\|_M \wedge \|b = c\|_M \leq \|a = c\|_M$

(B_4) For any atomic sentence $\varphi(a_1,\ldots,a_n)$:

$$\|\varphi(a_1,\ldots,a_n)\|_M \wedge \bigwedge_{i=1}^{n} \|a_i = b_i\|_M \leq \|\varphi(b_1,\ldots,b_n)\|_M$$

(B_5) $\|\varphi \vee \psi\|_M = \|\varphi\|_M \vee \|\psi\|_M$

(B_6) $\|\exists x \varphi(x)\|_M = \bigvee_{a \in M} \|\varphi(a)\|_M$.

We shall denote with the same symbol the B-valued structure M as well as its universe.

For any two B-valued structures M, N we shall write $M \subseteq N$ if the universe of M is included in the universe of N and for any atomic sentence $\varphi(\vec{a})$ of L(M) we have $\|\varphi(\vec{a})\|_M = \|\varphi(\vec{a})\|_N$.

If $\{M_\nu \mid \nu < \mu\}$ is a direct family of B-structures then the union $\bigcup_{\nu<\mu} M_\nu$ is a B-valued structure which is defined in an obvious way.

Now we shall recall some results of A.Shorb [6] in the form given in [3]:

Let T be a consistent set of L. A <u>B-assignment</u> of T is a function h: $\mathcal{S}(L)/_T \to B$ and a partial <u>B-assignment</u> of T is a partial function from $\mathcal{S}(L)/_T$ to B. A B-assignment is <u>consistent</u> if it is a morphism of Boolean algebras and a partial B-assignment is consistent if it can be extended to a consistent B-assignment.

For any partial map h from $\mathcal{S}(L)/_T$ to B we shall denote by $\bar{\bar{h}}$ the function with the domain $dom(\bar{\bar{h}}) = \{ {}^*\varphi \in \mathcal{S}(T)/_T \mid \varphi^* \in dom(h)$ or $\neg^*\varphi \in dom(h)\}$ and defined by

$$\bar{\bar{h}}\,({}^{*}\varphi) = \begin{cases} h\,({}^{*}\varphi), \text{ if } {}^{*}\varphi \in \text{dom}\,(h) \\ \neg\, h\,(\neg^{*} h), \text{ if } \neg^{*}\varphi \in \text{dom}\,(h). \end{cases}$$

<u>Lemma 1</u>. ([3],[6]). A partial B-assignment h of T is consistent iff

(i) $\bigwedge_{i=1}^{n} {}^{*}\varphi_i = 0$ implies $\bigwedge_{i=1}^{n} \bar{\bar{h}}\,({}^{*}\varphi_i) = 0$ for any $n > 0$

and ${}^{*}\varphi_1, \dots, {}^{*}\varphi_n$ dom(h), and

(ii) $\bar{\bar{h}}$ is well defined.

The following result is the Shorb's completeness theorem:

<u>Lemma 2</u>. ([3],[6]). Let T be a consistent set of sentences of L and let h be a consistent partial B-assignment of T. Then there is a B-valued structure M such that $\|\varphi\|_M = h\,({}^{*}\varphi)$, for any sentence φ of L(M).

The following definitions were given by C.C.Chang in [2].

Let T be a consistent set of sentences of L. A <u>type</u> is a non empty subset Γ of $\mathcal{F}(L)/_T$ such that $0 \notin \Gamma$ and Γ is closed under $\wedge$. A type Γ is a $\sum_n^{*}$ - type (resp. $\prod_n^{*}$ - type) if

$$\Gamma \subset \textstyle\sum_n^{*} = \{{}^{*}\varphi \mid \varphi \in \sum_n (L)\}$$

$$(\text{resp. } \Gamma \subset \textstyle\prod_n^{*} = \{{}^{*}\varphi \mid \varphi \in \prod_n (L)\}).$$

For any two types Γ and Δ we shall write $\Gamma \leq \Delta$ if for any ${}^{*}\varphi \in \Delta$, there is a ${}^{*}\psi \in \Gamma$ such that ${}^{*}\psi \leq {}^{*}\varphi$.

A $\sum_{n+1}^{*}$ - type Γ is (n+1) - <u>existential</u> if there is no $\sum_n^{*}$ - type Δ such that $\Delta \leq \Gamma$.

A B-valued structure B is a <u>model</u> of T if $\|\varphi\|_M = 1$ for any $\varphi \in T$.

A type Γ is <u>realized</u> by a sequence $a_o, a_1, \dots$ of elements of a model M of T if

$$\|\varphi(a_o, a_1, \dots)\|_M = 1, \text{ for any } {}^{*}\varphi \in \Gamma,$$

where a_i is the interpretation of the vaviable a_i for any $i < \omega$.

M *omits* Γ if Γ is not realized by any sequence of elements of M.

For any B-valued structure M we shall denote

$$D_n(M) = \sum_n(L(M)) \cup \prod_n(L(M)), \text{ for any } n<\omega$$

$$D_n^o(M) = \{\varphi \in D_n(M) \mid \|\varphi\|_M = 1\}, \text{ for any } n<\omega .$$

If $\vec{a} = (a_1,\dots,a_n)$ we shall write $\vec{a} \subset M$ instead of $a_1,\dots,a_n \in M$.

The following result is a generalisation of the Chang's omitting types theorem (see [2], p.66). The proof is directly inspirated from [2].

Theorem. Let T be a consistent set of sentences of L such that $T \subset \prod_{n+1}(L)$ and $n \geq 0$. For any complete Boolean algebra B and for any B-valued model M of T there exists a B-valued model N of T such that $M \subset N$ and

(a) N realizes every $\sum_n^*$ - type realized by M;

(b) N omits every (n+1) - existential type.

Proof. We shall define by induction a sequence of B-valued models of T:

$$M = M_o \subset M_1 \subset \dots \subset M_k \subset \dots$$

Suppose that M_k is constructed. Since $\|\varphi\|_{M_k} = 1$ for any $\varphi \in T \cup D_{n-1}^o(M_k)$ it results that $T \cup D_{n-1}^o(M_k)$ is consistent in $L(M_k)$. Consider a set Λ_k of sentences of $L(M_k)$ wich is maximal to respect the following conditions

(1) $\quad D_{n-1}^o(M_k) \subset \Lambda_k \subset \sum_n(L(M_k))$

$T \cup \Lambda_k$ is consistent in $L(M_k)$.

Denote by W the following subset of $\mathcal{S}(L(M_k))/_T$:

$$W = \{{}^*\varphi \mid \varphi \in T \cup \Lambda_k \text{ or } \neg\varphi \in T \cup \Lambda_k\} \cup \{{}^*\varphi \mid \varphi \in D_{n-1}(M_k)\}$$

and define a map $h : W \longrightarrow B$:

$$(2)\quad h({}^{*}\varphi) = \begin{cases} 1, & \text{if } \varphi \in T \cup \Lambda_k \\ 0, & \text{if } \neg\varphi \in T \cup \Lambda_k \\ \|\varphi\|_{M_k}, & \text{if } \varphi \in D_{n-1}(M_k). \end{cases}$$

We shall prove that h is well-defined. Consider $\varphi(\vec{a})$, $\psi(\vec{a}) \in D_{n-1}(M_k)$ such that $T \vdash \varphi(\vec{a}) \leftrightarrow \psi(\vec{a})$, therefore $T \vdash \forall \vec{x}\, (\varphi(\vec{x}) \longleftrightarrow \psi(\vec{x}))$. It results that $\|\forall \vec{x}\, (\varphi(\vec{x}) \leftrightarrow \psi(\vec{x}))\|_{M_k} = 1$ then

$$h({}^{*}\varphi) = \|\varphi(\vec{a})\|_{M_k} = \|\psi(a)\|_{M_k} = h({}^{*}\psi).$$

The other cases will be treated in an obvious way (see the proof of the theorem 7.1 of [3]).

For any ${}^{*}\varphi_1, \dots, {}^{*}\varphi_\ell \in W$ the following implication holds:

$$\bigwedge_{i=1}^{\ell} {}^{*}\varphi_i = 0 \Rightarrow \bigwedge_{i=1}^{\ell} h({}^{*}\varphi_i) = 0$$

As in the proof of theorem 7.1 of [3] we can consider that $\varphi_1, \dots, \varphi_\ell \in D_{n-1}(M_k)$.

If $a_1, \dots, a$ are the constants of $L(M_k) - L$ which appear in $\varphi_1, \dots, \varphi_\ell$ and $\vec{a} = (a_1, \dots, a_\ell)$, then we have the following implications:

$$\bigwedge_{i=1}^{\ell} \varphi_i(\vec{a}) = 0 \Rightarrow T \vdash \bigvee_{i=1}^{\ell} \neg\varphi_i(\vec{a})$$

$$\Rightarrow T \vdash \forall \vec{x}\, (\bigvee_{i=1}^{\ell} \neg\varphi_i(\vec{a}))$$

$$\Rightarrow \|\forall \vec{x}\, (\bigvee_{i=1}^{\ell} \neg\varphi_i(\vec{a}))\|_{M_k} = 1$$

$$\Rightarrow \|\bigvee_{i=1}^{\ell} \neg\varphi_i(\vec{a})\|_{M_k} = 1$$

$$\Rightarrow \bigwedge_{i=1}^{\ell} h({}^{*}\varphi_i(\vec{a})) = 0$$

From the Lemma 1 we can deduce that h is a partial B-assignment of T.

By the Shorb's completeness theorem there exists a B-valued structure M_{k+1} (for $L(M_k)$) such that

(3) $\quad \|\varphi\|_{M_{k+1}} = h({}^{*}\varphi)$, for any ${}^{*}\varphi \in W$.

From (3) it results that M_{k+1} is a B-valued model of T. Since $\|\varphi\|_{M_{k+1}} = \|\varphi\|_{M_k}$ for any $\varphi \in D_{n-1}(M_k)$ we can assume that $M_k \subset M_{k+1}$. Let us denote $N = \bigcup_{k<\omega} M_k$.

We shall prove by induction on the complexity of formulas that for any $k<\omega$, $\varphi(x) \in D_{n-1}(M_k)$ and $\vec{a} \in M_k$ we have

(4) $\quad \|\varphi(\vec{a})\|_{M_k} = \|\varphi(\vec{a})\|_N$

We shall consider only the case when $\varphi(\vec{x})$ has the form $\exists v\, \psi(v,\vec{x})$. The inductive hypothesis is that $\|\psi(b,\vec{a})\|_{M_k} = \|\psi(b,\vec{a})\|_N$ for any $b \in M_k$, then we have

$$\|\exists v\,\psi(v,\vec{a})\|_N = \bigvee_{\ell \geq k} \bigvee_{b \in M_\ell} \|\psi(b,\vec{a})\|_N$$

$$= \bigvee_{\ell \geq k} \bigvee_{b \in M_\ell} \|\psi(b,\vec{a})\|_{M_\ell}$$

$$= \bigvee_{\ell \geq k} \|\exists v\,\psi(v,\vec{a})\|_{M_\ell}$$

$$= \|\exists v\,\psi(v,\vec{a})\|_{M_k}.$$

We claim that

(5) $\quad \|\varphi\|_N = 1$ for any $\varphi \in T$.

From $T \subset \Pi_{n+1}(L)$ it follows that φ has the form $\forall\vec{x}\,\exists\vec{y}\,\psi(\vec{x},\vec{y})$, where $\varphi(\vec{x},\vec{y}) \in \Pi_{n-1}(L)$. It results for any $\ell \in \omega$:

$$\bigwedge_{\vec{a} \in M_\ell} \bigvee_{\vec{b} \in M_\ell} \|\psi(\vec{a},\vec{b})\|_{M_\ell} = \quad {}_M = 1,$$

therefore for any $\vec{a} \in M_\ell$ we have

$$\bigvee_{\vec{b} \in M_\ell} \| \psi(\vec{a},\vec{b}) \|_{M_\ell} = 1.$$

Let $\vec{a} \in N$ be then there is $k \in \omega$ such that $\vec{a} \in M_k$. In accordance to (4) we obtain

$$\| \exists \vec{y}\, \psi(\vec{a},\vec{y}) \|_N = \bigvee_{\ell \geq k} \bigvee_{\vec{b} \in M_\ell} \| \psi(\vec{a},\vec{b}) \|_N$$

$$= \bigvee_{\ell \geq k} \bigvee_{\vec{b} \in M_\ell} \| \psi(\vec{a},\vec{b}) \|_{M_\ell} = 1.$$

From this it follows

$$\| \varphi \|_N = \| \forall \vec{x}\, \exists \vec{y}\; \psi(\vec{x},\vec{y}) \|_N$$

$$= \bigwedge_{\vec{a} \in N} \| \exists \vec{y}\; \psi(\vec{a},\vec{y}) \|_N = 1.$$

Now we suppose that Γ is a $\sum_n^*$ - type which is realized in M by the sequence $a_o, a_1, \ldots$. Consider $^* \varphi(v_o,\ldots,v_m) \in \Gamma$ with $\varphi \in \sum_n (L(M))$, then $\| \varphi(a_o,\ldots,a_m) \|_M = 1$. But $\varphi(v_o,\ldots,v_m)$ has the form

$$\exists x_1 \ldots \exists x_t\; \psi(v_o,\ldots,v_m, x_1,\ldots,x_t),\ \psi \in \prod_{n-1}(L(M))$$

then it results from (4):

$$\| \varphi(a_o,\ldots,a_m) \|_N = \bigvee_{b_1,\ldots,b_t \in N} \| \psi(a_o,\ldots,a_m, b_1,\ldots,b_t) \|_N \geq$$

$$\geq \bigvee_{b_1,\ldots,b_t \in N} \| \psi(a_o,\ldots,a_m, b_1,\ldots,b_t) \|_M =$$

$$= \| \varphi(a_o,\ldots,a_m) \|_M = 1.$$

Then the sequence $a_o, a_1, \ldots$ realizes the type Γ in N.

Now we shall prove the second condition of the theorem. Suppose that there exists a (n+1) - existential type Γ which is realized in N by a sequence $a_o, a_1, \ldots$.

Let $^{*}\gamma$ be an element of Γ where the formula $\gamma(v_o,\ldots,v_m)$ is in $\Sigma_{n+1}(L(N))$, then γ has the form $\exists\vec{x}\,\varphi(\vec{v},\vec{x})$ with $\varphi(\vec{v},\vec{x})\in\Pi_n(L(M))$.

Let us consider $k\in\omega$ such that $a_o,\ldots,a_m\in M_k$. We shall prove that there exist $\ell\geq k$ and $\vec{b}\in M_\ell$ such that $\neg\varphi(\vec{a},\vec{b})\notin\Lambda_\ell$, where $\vec{a}=(a_o,\ldots,a_m)$. Suppose that for any $\ell\geq k$ and $\vec{b}\in M_\ell$ we have $\neg\varphi(\vec{a},\vec{b})\in\Lambda_\ell$.

By (3) it results

$$\|\varphi(\vec{a},\vec{b})\|_{M_{\ell+1}} = 0 \text{ for any } \ell\geq k \text{ and } \vec{b}\in M_\ell.$$

But $\varphi(\vec{a},\vec{b})\in\Pi_n(L(M))$ then it follows from (3):

$$\|\varphi(\vec{a},\vec{b})\|_N = 0 \text{ for any } \ell\geq k \text{ and } \vec{b}\in M_\ell.$$

We obtain the contradiction

$$\|\exists\vec{x}\,\varphi(\vec{a},\vec{x})\|_N = \bigvee_{\ell\geq k}\ \bigvee_{\vec{b}\in M}\|\varphi(\vec{a},\vec{b})\|_N = 0,$$

then there exist $\ell\geq k$ and $\vec{b}\in M_\ell$ such that $\neg\varphi(\vec{a},\vec{b})\notin\Gamma$, But $\neg\varphi(\vec{a},\vec{b})$ is logically equivalent to a sentence in $\Sigma_n(L(M_\ell))$, then it follows from the maximality of Λ_ℓ that $T\cup\Lambda_\ell\cup\{\neg\varphi(\vec{a},\vec{b})\}$ is inconsistent.

Then there exist $\sigma(\vec{a},\vec{b},\vec{c})\in\Lambda_\ell$ such that

$$T\vdash\sigma(\vec{a},\vec{b},\vec{c})\longrightarrow\varphi(\vec{a},\vec{b})$$

where $\vec{c}=(c_1,\ldots,c_s)$ and the constants $c_1,\ldots,c_s$ of $L(M_\ell)$ do not occur in T or φ. Exactly as in [2], p.68 we have

$$T\vdash\delta_\gamma(\vec{v})\longrightarrow\gamma(\vec{v}),$$

where $\delta_\gamma(\vec{v})$ is the formula $\exists\vec{x}\,\exists\vec{y}\,\sigma(\vec{v},\vec{x},\vec{y})\in\Sigma_n(L)$.

But $\sigma(\vec{a},\vec{b},\vec{c})\in\Lambda_\ell$, then we get from (3) that $\|\sigma(\vec{a},\vec{b},\vec{c})\|_{M_{\ell+1}}=1$. Since $\sigma(\vec{a},\vec{b},\vec{c})\in\Sigma_n(L(M_\ell))$ we can deduce by (4) that $\|\sigma(\vec{a},\vec{b},\vec{c})\|_N=1$. In accordance to

$$\| \delta_\gamma(\vec{a}) \|_N = \bigvee_{\vec{b}' \in N} \bigvee_{\vec{c}' \in N} \| \sigma(\vec{a}, \vec{b}', \vec{c}') \|_N \geq \| \sigma(\vec{a}, \vec{b}, \vec{c}) \|_N$$

it follows that $\| \delta_\gamma(\vec{a}) \|_N = 1$.

Exactly as in [2], we consider the Σ_n^* - type generated by the set $\{ {}^*\delta_\gamma \mid {}^*\gamma \in \Gamma \}$. Since $\Delta \leq \Gamma$, the contradiction is obvious.

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