Logic Seminar: Proof Interpretations IV

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In this seminar we will conclude our study of *monotone modified realizability*, with the following:

1. Corollary of the Soundness theorem for monotone modified realizability: in $\mathbf{E}-\mathbf{HA}^{\omega}+\mathrm{AC}+\mathrm{IP}^{\omega}_{\neg}+\mathrm{CA}^{\omega}_{\neg}$, from a proof of

   $$\forall x^1 \forall y \leq_\rho sx \exists z^\tau A(x,y,z)$$

   we can extract a term t such that we can prove

   $$\forall x^1 \forall y \leq_\rho sx \exists z \leq_\tau tx A(x,y,z)$$

   for any formula A, closed term s and types ρ, τ with $\deg(\tau)=2$.

2. *Main theorem on program extraction by* monotone *modified realizability*: in $\mathbf{E}-\mathbf{HA}^{\omega}+\mathrm{AC}+\mathrm{IP}^{\omega}_{\neg}+\Omega$, from a proof of

   $$\forall x^1 \forall y \leq_\rho sx \exists z^\tau A(x,y,z)$$

   we can extract a term t such that we can prove

   $$\forall x^1 \forall y \leq_\rho sx \exists z^\tau A(x,y,tx)$$

   for any formula $A(x,y,z)$ in $\mathcal{L}(\mathbf{E}-\mathbf{HA}^{\omega})$, where Ω is a set of sentences of the type $\forall \underline{u}^{\underline{\delta}}(C(\underline{u}) \to \exists \underline{v} \leq_{\underline{\sigma}} \underline{r}\, \underline{u} \neg B(\underline{u},\underline{v}))$, with C and B arbitrary formulas of $\mathbf{E}-\mathbf{HA}^{\omega}$, and ρ, τ, $\underline{\sigma}$, $\underline{\delta}$ are arbitrary (tuples of) types.

3. Applications of (monotone) modified realizability:

   - If $\Phi^{1(1)(0)}_{(\cdot)}$, $\Phi^{1(1)}$ are closed terms of $\mathbf{E}-\mathbf{HA}^{\omega}$ satisfying

     $$\forall x,y \leq_1 N \forall n^0(\tilde{x} =_{\mathbb{R}} \tilde{y} \to \Phi_n(\tilde{x}) =_{\mathbb{R}} \Phi_n(\tilde{y}) \wedge \Phi(\tilde{x}) =_{\mathbb{R}} \Phi(\tilde{y}))$$

     then provable pointwise convergence of Φ_n towards Φ on $[0,1]$ implies provable uniform convergence on $[0,1]$.

   - The *Weak Markov Principle* (WMP for short), is not provable in $\mathbf{E}-\mathbf{HA}^{\omega}+\mathrm{AC}+\mathrm{CA}^{\omega}_{\neg}$, where WMP:"*Every pseudo-positive real number is positive*", and $a \in \mathbb{R}$ is pseudo-positive if $\forall x \in \mathbb{R}(\neg\neg(0<x) \vee \neg\neg(x<a))$.