Logical Metatheorems for Metric Spaces I

Speaker: Daniel Ivan, IMAR

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Given a set X and symbols b_X and d_X, we present the theory $\mathcal{A}_i^\omega[X,d]$, an extension of the weakely extensional 'Heyting' arithmetic theory that also contains the axiom of choice principle. $\mathcal{A}_i^\omega[X,d]$ is defined over the set of all finite types generated by the ground types $\mathbb{N}$ and X, denoted by $\mathbf{T}^X$.

The goal of this seminar is to prove, using the monotone functional interpretation, the following logical metatheorem.
Let σ, ρ, τ be types of $\mathbf{T}^X$, such that σ is a type of degree 1, ρ is a type of degree $(\cdot, 0)$, and τ is a type of degree $(\cdot, X)$. Let $s^{\rho\sigma}$ be a closed term of $\mathcal{A}_i^\omega[X,d]$, $B_\forall(x^\sigma, y^\rho, z^\tau, u^\mathbf{0})$ be a $\forall$-formula with only x, y, z, u free, and $C(x^\sigma, y^\rho, z^\tau, v^\mathbf{0})$ a general formula with only x, y, z, v free. Both B and C are formulas of $\mathscr{L}(\mathcal{A}_i^\omega[X,d])$. If

$$\mathcal{A}_i^\omega[X,d] + \Delta \vdash \forall x^\sigma \, \forall y \leq_\rho sx \, \forall z^\tau (\forall u^\mathbf{0} \, B_\forall(x,y,z,u) \rightarrow \exists v^\mathbf{0} \, C(x,y,z,v)),$$

then one can extract a function $\Phi : \mathcal{S}_\sigma \times \mathbb{N} \rightarrow \mathbb{N}$, such that for all $x \in \mathcal{S}_\sigma$ and all $b \in \mathbb{N}$,

$$\forall x^\sigma, y \leq_\rho sx \, \forall z^\tau [\forall u \leq_\mathbf{0} \Phi(x,b) \, B_\forall(x,y,z,u) \rightarrow \exists v \leq_\mathbf{0} \Phi(x,b) \, C(x,y,z,v)]$$

holds in any nonempty metric space whose metric is bounded by $b \in \mathbb{N}$ and that satisfies Δ, where

$$\Delta :\equiv \{\forall \underline{a}^{\underline{\delta}} \, \exists \underline{b} \leq_{\underline{\sigma}} \underline{r} \, \underline{a} \, \forall \underline{c}^{\underline{\gamma}} \, F_0(\underline{a}, \underline{b}, \underline{c})\},$$

and F_0 is a quantifier-free formula of $\mathcal{L}(\mathcal{A}_i^\omega[X,d])$. The theorem still holds if we add to $\mathcal{A}_i^\omega[X,d]$:

- Independence of premise principle, for universal formulas:

  $$\mathrm{IP}_\forall^\omega :\equiv (\forall \underline{x} A_0(\underline{x}) \rightarrow \exists y^\rho B(y)) \rightarrow \exists y^\rho (\forall \underline{x} A_0(\underline{x}) \rightarrow B(y)),$$

  where y is not free in $\forall \underline{x} A_0(\underline{x})$ and A_0 is a quantifier free formula.

- Markov Principle:

  $$\mathrm{M}^{\underline{\rho}} :\equiv \neg\neg \exists \underline{x}^{\underline{\rho}} A_0(\underline{x}) \rightarrow \exists \underline{x}^{\underline{\rho}} A_0(\underline{x}),$$

  where A_0 is an arbitrary quantifier-free formula.

The theorem also holds for tuples of variables $\underline{x}, \underline{y}, \underline{z}, \underline{u}, \underline{v}$, of appropriate types.