

# Braided spaces with dilations and sub-riemannian symmetric spaces

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## Abstract

Braided sets which are also spaces with dilations are presented and explored in this paper, in the general frame of emergent algebras. Examples of such spaces are the sub-riemannian symmetric spaces.

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# 1 Introduction

In the previous paper [6] we introduced and studied emergent algebras, as a generalization of differentiable algebras. An emergent algebra is a uniform idempotent right quasigroup, definition 5.2.

In this paper we explain with details previous results concerning conical groups, section 6, dilatation structures (metric spaces with dilations), section 7 as well as new results concerning braided sets which are also dilatation structures, section 9, in the frame of emergent algebras. Finally, we show that sub-riemannian symmetric spaces (which are not Loos symmetric spaces) can be seen as braided dilatation structures.

There is another, but related, line of research concerning symmetric spaces as emergent algebras, based on the notion of a "approximate symmetric space". We postpone the presentation of this for a future paper.

For yet another research line concerning the generalization of spaces with dilations to deformations of normed groupoids see the paper [7].

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## 2 Quandles, Loos symmetric spaces, contractible groups

For braided sets see the paper [10].

**Definition 2.1** *Let  $X$  be a non-empty set and  $S : X \times X \rightarrow X \times X$ ,  $S(x_1, x_2) = (S_1(x_1, x_2), S_2(x_1, x_2))$  be a bijection. We define for  $i = 1, 2$  the maps  $S^{ii+1} : X^3 \rightarrow X^3$ ,  $S^{12} = S \times id_X$ ,  $S^{23} = id_X \times S$ .*

*(i) A map  $S$  is called non-degenerate if for any fixed  $y, z \in X$  the maps  $x \mapsto S_2(x, y)$  and  $x \mapsto S_1(z, x)$  are bijections.*

*(ii) A pair  $(X, S)$  is a braided set (and  $S$  is a braided map) if  $S$  satisfies the braid relation*

$$S^{12}S^{23}S^{12} = S^{23}S^{12}S^{23} \quad (2.0.1)$$

*(iii)  $S$  is involutive if  $S^2 = id_{X \times X}$ .*

*A braided set which is involutive is called symmetric set.*

A large class of braided sets is made by pairs  $(X, S)$ , with

$$S(x, y) = (x * y, x)$$

and  $(X, *)$  is a rack. Racks and quandles are right quasigroups, a notion that we shall use further, so here is the definition.

**Definition 2.2** A right quasigroup is a set  $X$  with a binary operation  $*$  such that for each  $a, b \in X$  there exists a unique  $x \in X$  such that  $a * x = b$ . We write the solution of this equation  $x = a \setminus b$ .

A quasigroup is a set  $X$  with a binary operation  $*$  such that for each  $a, b \in X$  there exist unique elements  $x, y \in X$  such that  $a * x = b$  and  $y * a = b$ . We write the solution of the last equation  $y = b / a$ .

An idempotent right quasigroup (irq) is a right quasigroup  $(X, *)$  such that for any  $x \in X$   $x * x = x$ . Equivalently, it can be seen as a set  $X$  endowed with two operations  $\circ$  and  $\bullet$ , which satisfy the following axioms: for any  $x, y \in X$

$$(P1) \quad x \circ (x \bullet y) = x \bullet (x \circ y) = y$$

$$(P2) \quad x \circ x = x \bullet x = x$$

The correspondence between notations using  $*$ ,  $\setminus$  and those using  $\circ$ ,  $\bullet$ , is:  $* = \circ$ ,  $\setminus = \bullet$ .

In knot theory, J.C. Conway and G.C. Wraith, in their unpublished correspondence from 1959, used the name "wrack" for a self-distributive right quasigroup generated by a link diagram. Later, Fenn and Rourke [11] proposed the name "rack" instead. Quandles are particular case of racks, namely self-distributive idempotent right quasigroups. They were introduced by Joyce [15], as a distillation of the Reidemeister moves. More precisely, the axioms of a (rack ; quandle ; irq) correspond respectively to the (2,3 ; 1,2,3 ; 1,2) Reidemeister moves.

We are interested in two particular cases of quandles. The first is related to symmetric spaces in the sense of Loos [16] chapter II, definition 1.

**Definition 2.3**  $(X, inv)$  is a Loos algebraic symmetric space if  $inv : X \times X \rightarrow X$  is an operation which satisfies the following axioms:

(L1)  $inv$  is idempotent: for any  $x \in X$  we have  $inv(x, x) = x$ ,

(L2) distributivity: for any  $x, y, z \in X$  we have

$$inv(x, inv(y, z)) = inv(inv(x, y), inv(x, z))$$

(L3) for any  $x, y \in X$  we have  $inv(x, inv(x, y)) = y$ ,

(L4) for every  $x \in X$  there is a neighbourhood  $U(x)$  such that  $inv(x, y) = y$  and  $y \in U(x)$  then  $x = y$ .

If  $X$  is a manifold,  $inv$  is smooth (of class  $C^\infty$ ) and (L4) is true locally then  $(X, inv)$  is a symmetric space as defined by Loos [16] chapter II, definition 1.

Remark that if  $(X, inv)$  is a Loos symmetric space then it is clearly a quandle, therefore  $(X, Inv)$  is a braided symmetric set, where:

$$Inv(x, y) = (inv^x y, x)$$

**Definition 2.4** A contractible group is a pair  $(G, \alpha)$ , where  $G$  is a topological group with neutral element denoted by  $e$ , and  $\alpha \in \text{Aut}(G)$  is an automorphism of  $G$  such that:

- $\alpha$  is continuous, with continuous inverse,
- for any  $x \in G$  we have the limit  $\lim_{n \rightarrow \infty} \alpha^n(x) = e$ .

If  $(G, \alpha)$  is a contractible group then  $(G, *)$  is a quandle, with:

$$x * y = x\alpha(x^{-1}y)$$

Contractible groups are particular examples of conical groups. In [6] we proved that conical groups, as well as some symmetric spaces, can be described as emergent algebras, coming from uniform idempotent right quasigroups.

### 3 Motivation: emergent algebras

A differentiable algebra, is an algebra (set of operations  $\mathcal{A}$ ) over a manifold  $X$  with the property that all the operations of the algebra are differentiable with respect to the manifold structure of  $X$ . Let us denote by  $\mathcal{D}$  the differential structure of the manifold  $X$ .

From a more computational viewpoint, we may think about the calculus which can be done in a differentiable algebra as being generated by the elements of a toolbox with two compartments:

- $\mathcal{A}$  contains the algebraic information, that is the operations of the algebra, as well as algebraic relations (like for example "the operation  $*$  is associative", or "the operation  $*$  is commutative", and so on),
- $\mathcal{D}$  contains the differential structure informations, that is the information needed in order to formulate the statement "the function  $f$  is differentiable",
- the compartments  $\mathcal{A}$  and  $\mathcal{D}$  are compatible, in the that any operation from  $\mathcal{A}$  is differentiable according to  $\mathcal{D}$ .

In the paper [6] we proposed the notion of a emergent algebra as a generalization of a differentiable algebra. Computations in a emergent algebra (short name for a uniform idempotent right quasigroup, definition 5.2) are generated by a class  $\mathcal{E}$  of operations and relations from which a algebra  $\mathcal{A}$  and a generalization of a differentiable structure  $\mathcal{D}$  "emerge". The meaning of this emergence is the following: all elements of  $\mathcal{A}$  and  $\mathcal{D}$  (algebraic operations, relations, differential operators, ...) are constructed by finite or virtually infinite "recipes", which can be implemented by some class of circuits made by very simple gates (the operations in the uniform idempotent right quasigroup). An emergent algebra space is then described by:

- a class of transistor-like gates (that is binary operations), with in/out ports labeled by points of the space and a internal state variable which can be interpreted as "scale".
- a class of elementary circuits made of such gates (these are the "generators" of the emergent algebra). The elementary circuits are in fact certain ternary operations constructed from the operations in the uniform idempotent right quasigroup. They have the property that the output converges as the scale goes to zero, uniformly with respect to the input.
- a class of equivalence rules saying that some simple assemblies of elementary circuits have equivalent function (these are the "relations" of the emergent algebra).

We shall explore in more detail this point of view, concentrating on braided sets which are also spaces with dilations (dilatation structures).

## 4 $\Gamma$ -idempotent right quasigroups

**Definition 4.1** *We use the operations of a irq to define the sum, difference and inverse operations of the irq: for any  $x, u, v \in X$*

- (a) *the difference operation is  $(xuv) = (x \circ u) \bullet (x \circ v)$ . By fixing the first variable  $x$  we obtain the difference operation based at  $x$ :  $v \overset{-x}{-} u = dif^x(u, v) = (xuv)$ .*
- (b) *the sum operation is  $(xuv) = x \bullet ((x \circ u) \circ v)$ . By fixing the first variable  $x$  we obtain the sum operation based at  $x$ :  $u \overset{x}{+} v = sum^x(u, v) = (xuv)$ .*
- (a) *the inverse operation is  $inv(x, u) = (x \circ u) \bullet x$ . By fixing the first variable  $x$  we obtain the inverse operator based at  $x$ :  $\overset{-x}{-} u = inv^x u = inv(x, u)$ .*

For any  $k \in \mathbb{Z}^* = \mathbb{Z} \setminus \{0\}$  we define also the following operations:

- $x \circ_1 u = x \circ u$ ,  $x \bullet_1 u = x \bullet u$ ,
- for any  $k > 0$  let  $x \circ_{k+1} u = x \circ (x \circ_k u)$  and  $x \bullet_{k+1} u = x \bullet (x \bullet_k u)$ ,
- for any  $k < 0$  let  $x \circ_k u = x \bullet_{-k} u$  and  $x \bullet_k u = x \circ_{-k} u$ .

For any  $k \in \mathbb{Z}^*$  the triple  $(X, \circ_k, \bullet_k)$  is a irq. We denote the difference, sum and inverse operations of  $(X, \circ_k, \bullet_k)$  by the same symbols as the ones used for  $(X, \circ, \bullet)$ , with a subscript "k".

In [6] we introduced idempotent right quasigroups and then iterates of the operations indexed by a parameter  $k \in \mathbb{N}$ . This was done in order to simplify the notations mostly. Here, in the presence of the group  $\Gamma$ , we might define a  $\Gamma$ -irq.

**Definition 4.2** Let  $\Gamma$  be a commutative group. A  $\Gamma$ -idempotent right quasigroup is a set  $X$  with a function  $\varepsilon \in \Gamma \mapsto \circ_\varepsilon$  such that  $(X, \circ_\varepsilon)$  is a irq and moreover for any  $\varepsilon, \mu \in \Gamma$  and any  $x, y \in X$  we have

$$x \circ_\varepsilon (x \circ_\mu y) = x \circ_{\varepsilon\mu} y$$

It is then obvious that if  $(X, \circ)$  is a irq then  $(X, k \in \mathbb{Z} \mapsto \circ_k)$  is a  $\mathbb{Z}$ -irq (we define  $x \circ_0 y = y$ ).

The following is a slight modification of proposition 3.4 and point (k) proposition 3.5 [6], for the case of  $\Gamma$ -irqs (the proof of this proposition is almost identical, with obvious modifications, with the proof of the original proposition).

**Proposition 4.3** Let  $(X, \circ_\varepsilon)_{\varepsilon \in \Gamma}$  be a  $\Gamma$ -irq. Then we have the relations:

$$(a) (u +_\varepsilon^x v) -_\varepsilon^x u = v$$

$$(b) u +_\varepsilon^x (v -_\varepsilon^x u) = v$$

$$(c) v -_\varepsilon^x u = (-_\varepsilon^x u) +_\varepsilon^{x \circ u} v$$

$$(d) -_\varepsilon^{x \circ u} (-_\varepsilon^x u) = u$$

$$(e) u +_\varepsilon^x (v +_\varepsilon^{x \circ u} w) = (u +_\varepsilon^x v) +_\varepsilon^x w$$

$$(f) -_\varepsilon^x u = x -_\varepsilon^x u$$

$$(g) x +_\varepsilon^x u = u$$

(k) for any  $\varepsilon, \mu \in \mathbb{Z}^*$  and any  $x, u, v \in X$  we have the distributivity property:

$$(x \circ_\mu v) -_\varepsilon^x (x \circ_\mu u) = (x \circ_{\varepsilon\mu} u) \circ_\mu (v -_\varepsilon^x u)$$

## 5 Uniform idempotent right quasigroups

Let  $\Gamma$  be a topological commutative group. We suppose that  $\Gamma$  as a topological space is separable.

**Definition 5.1** Let  $(X, \tau)$  be a topological space.  $\tau$  is the collection of open sets in  $X$ . A filter in  $(X, \tau)$  is a function  $\mu : \tau \rightarrow \{0, 1\}$  such that:

$$(a) \mu(X) = 1,$$

$$(b) \text{ for any } A, B \in \tau, \text{ if } A \subset B \text{ then } \mu(A) \leq \mu(B),$$

$$(c) \text{ for any } A, B \in \tau \text{ we have } \mu(A \cup B) + \mu(A \cap B) \geq \mu(A) + \mu(B).$$

An absolute of a separable topological commutative group  $\Gamma$  is a class  $Abs(\Gamma)$  of filters  $\mu$  in  $\Gamma$  with the properties:

- (i) for any  $\varepsilon \in \Gamma$  there are  $A \in \tau$  and  $\mu \in Abs(\Gamma)$  such that  $\mu(A) = 1$  and  $x \notin A$ ,
- (ii) for any  $\mu, \mu' \in Abs(\Gamma)$  there is  $A \in \tau$  such that  $\mu(A) > \mu'(A)$ ,
- (iii) for any  $\varepsilon \in \Gamma$  and  $\mu \in Abs(\Gamma)$  the transport of  $\mu$  by  $\varepsilon$ , defined as  $\varepsilon\mu(A) = \mu(\varepsilon A)$ , belongs to  $Abs(\Gamma)$ .

Let  $f : \Gamma \rightarrow (X, \tau)$  be a function from  $\Gamma$  to a separable topological space, let  $Abs(\Gamma)$  be an absolute of  $\Gamma$ , and  $\mu \in Abs(\Gamma)$ . We say that  $f$  converges to  $z \in X$  as  $\varepsilon$  goes to  $\mu$  if for any open set  $A$  in  $X$  with  $z \in A$  we have  $\mu(f^{-1}(A)) = 1$ . We write:

$$\lim_{\varepsilon \rightarrow \mu} f(\varepsilon) = z$$

For example, if  $\Gamma = (0, +\infty)$  with multiplication, then  $Abs(\Gamma) = \{0\}$  is an absolute, where "0" is the filter defined by  $0(A) = 1$  if and only if the number 0 belongs to the closure of  $A$  in  $\mathbb{R}$ . Also,  $Abs(\Gamma) = \{0, \infty\}$  is an absolute, where " $\infty$ " is the filter defined by:  $\infty(A) = 1$  if and only if  $A$  is unbounded.

Let  $\Gamma$  be a commutative separable topological group,  $\chi : \Gamma \rightarrow (0, +\infty)$  a continuous morphism and  $Abs((0, +\infty))$  an absolute of  $(0, +\infty)$ . Let  $Abs(\Gamma)$  be the class of filters on  $\Gamma$  constructed like this:  $\mu \in Abs(\Gamma)$  if there exists  $\alpha \in Abs((0, +\infty))$  such that for any open set  $A$  in  $\Gamma$ ,  $\mu(A) = 1$  if there is an open set  $B \subset (0, +\infty)$  with  $\chi^{-1}(B) \subset A$  and  $\alpha(B) = 1$ . Then  $Abs(\Gamma)$  is an absolute of  $\Gamma$ .

Another example: let  $\Gamma_0$  be a topological separable commutative group, let  $G$  be a finite commutative group and let  $\Gamma = \Gamma_0 \times G$ . We think now about  $G$  and  $\Gamma_0$  as being subgroups of  $\Gamma$ . Let  $Abs(\Gamma_0)$  be an absolute of  $\Gamma_0$ . We construct  $Abs(\Gamma)$  as the collection of all filters  $\mu$  on  $\Gamma$  such that there is  $g \in G$  with  $g\mu \in Abs(\Gamma_0)$ . Then  $Abs(\Gamma)$  is an absolute of  $\Gamma$ .

**Definition 5.2** A  $\Gamma$ -uniform irq  $(X, *, \setminus)$  is a separable uniform space  $X$  endowed with continuous irq operations  $*$ ,  $\setminus$  such that:

- (C) the operation  $*$  is compactly contractive: for each compact set  $K \subset X$  and open set  $U \subset X$ , with  $x \in U$ , there is an open set  $A(K, U) \subset \Gamma$  with  $\mu(A) = 1$  for any  $\mu \in Abs(\Gamma)$  and for any  $u \in K$  and  $\varepsilon \in A(K, U)$ , we have  $x *_{\varepsilon} u \in U$ ;
- (D) the following limits exist for any  $\mu \in Abs(\Gamma)$

$$\lim_{\varepsilon \rightarrow \mu} v -_{\varepsilon}^x u = v -_{\infty}^x u \quad , \quad \lim_{\varepsilon \rightarrow \mu} u +_{\varepsilon}^x v = u +_{\infty}^x v$$

and are uniform with respect to  $x, u, v$  in a compact set.

The main property of a uniform irq is the following. It is a consequence of relations from proposition 4.3.

**Theorem 5.3** *Let  $(X, *, \setminus)$  be a uniform irq. Then for any  $x \in X$  the operation  $(u, v) \mapsto u +_{\infty}^x v$  gives  $X$  the structure of a conical group with the dilatation  $u \mapsto x *_\varepsilon u$ .*

Conical groups are described in the next section.

## 6 Conical groups are distributive uniform irqs

For a dilatation structure (see section 7) the metric tangent spaces have a group structure which is compatible with dilatations. This structure, of a group with dilatations, is interesting by itself. The notion has been introduced in [8], [2]; we describe it further.

Let  $\Gamma$  be a topological commutative groups with an absolute  $Abs(\Gamma)$ .

**Definition 6.1** *A group with dilatations  $(G, \delta)$  is a topological group  $G$  with an action of  $\Gamma$  (denoted by  $\delta$ ), on  $G$  such that for any  $\mu \in Abs(\Gamma)$*

*H0. the limit  $\lim_{\varepsilon \rightarrow \mu} \delta_\varepsilon x = e$  exists and is uniform with respect to  $x$  in a compact neighbourhood of the identity  $e$ .*

*H1. the limit*

$$\beta(x, y) = \lim_{\varepsilon \rightarrow \mu} \delta_\varepsilon^{-1} ((\delta_\varepsilon x)(\delta_\varepsilon y))$$

*is well defined in a compact neighbourhood of  $e$  and the limit is uniform.*

*H2. the following relation holds*

$$\lim_{\varepsilon \rightarrow \mu} \delta_\varepsilon^{-1} ((\delta_\varepsilon x)^{-1}) = x^{-1}$$

*where the limit from the left hand side exists in a neighbourhood of  $e$  and is uniform with respect to  $x$ .*

**Definition 6.2** *A conical group  $(N, \delta)$  is a group with dilatations such that for any  $\varepsilon \in \Gamma$  the dilatation  $\delta_\varepsilon$  is a group morphism.*

A conical group is the infinitesimal version of a group with dilatations ([2] proposition 2).

**Proposition 6.3** *Under the hypotheses H0, H1, H2,  $(G, \beta, \delta)$ , is a conical group, with operation  $\beta$ , dilatations  $\delta$ .*

One particular case is the one of contractible groups, definition 2.4, which are also normed groups. Indeed, in this case we may take  $\Gamma = \mathbb{Z}$ .

Locally compact conical groups are locally compact groups admitting a contractive automorphism group. We begin with the definition of a contracting automorphism group [20], definition 5.1.



**Definition 6.4** Let  $G$  be a locally compact group. An automorphism group on  $G$  is a family  $T = (\tau_t)_{t>0}$  in  $\text{Aut}(G)$ , such that  $\tau_t \tau_s = \tau_{ts}$  for all  $t, s > 0$ .

The contraction group of  $T$  is defined by

$$C(T) = \left\{ x \in G : \lim_{t \rightarrow 0} \tau_t(x) = e \right\} .$$

The automorphism group  $T$  is contractive if  $C(T) = G$ .

Next is proposition 5.4 [20], which gives a description of locally compact groups which admit a contractive automorphism group.

**Proposition 6.5** For a locally compact group  $G$  the following assertions are equivalent:

- (i)  $G$  admits a contractive automorphism group;
- (ii)  $G$  is a simply connected Lie group whose Lie algebra admits a positive graduation.

The proof of the next proposition is an easy application of the previously explained facts.

**Proposition 6.6** Let  $(G, \delta)$  be a locally compact conical group. Then the associate irq  $(G, *)$  is an uniform irq.

A particular class of locally compact groups which admit a contractive automorphism group is made by Carnot groups. They are related to sub-riemannian or Carnot-Carathéodory geometry, which is the study of non-holonomic manifolds endowed with a Carnot-Carathéodory distance. Non-holonomic spaces were discovered in 1926 by G. Vrăncăanu [22], [23]. The Carnot-Carathéodory distance on a non-holonomic space is inspired by Carathéodory [9] work from 1909 on the mathematical formulation of thermodynamics. Such spaces appear in applications to thermodynamics, to the mechanics of non-holonomic systems, in the study of hypo-elliptic operators cf. Hörmander [14], in harmonic analysis on homogeneous cones cf. Folland, Stein [12], and as boundaries of CR-manifolds.

The following result is a slight modification of [6], theorem 6.1, consisting in the replacement of "contractible" by "conical" in the statement of the theorem.

**Theorem 6.7** Let  $(G, \alpha)$  be a locally compact conical group and  $G(\alpha)$  be the associated uniform irq. Then the irq is distributive, namely it satisfies the relation: for any  $\varepsilon, \lambda \in \Gamma$

$$x *_{\varepsilon} (y *_{\lambda} z) = (x *_{\varepsilon} y) *_{\lambda} (x *_{\varepsilon} z) \tag{6.0.1}$$

Conversely, let  $(G, *)$  be a distributive uniform irq. Then there is a group operation on  $G$  (denoted multiplicatively), with neutral element  $e$ , such that:

- (i)  $xy = x +_{\infty}^e y$  for any  $x, y \in G$ ,

(ii) for any  $x, y, z \in G$  we have  $(xyz)_\infty = xy^{-1}z$ ,

(iii) for any  $x, y \in G$  we have  $x *_\varepsilon y = x(e *_\varepsilon (x^{-1}y))$ .

In conclusion there is a bijection between distributive  $\Gamma$ -uniform irqs and conical groups.

## 7 Normed uniform irqs are dilatation structures

For simplicity we shall list the axioms of a dilatation structure  $(X, d, \delta)$  without concerning about domains and codomains of dilatations. For the full definition of dilatation structures, as well as for their main properties and examples, see [2], [3], [4]. The notion appeared from my efforts to understand the last section of the paper [1] (see also [19], [13], [17], [18]).

Let  $\Gamma$  be a topological commutative groups with an absolute  $Abs(\Gamma)$  and with a morphism  $|\cdot|: \Gamma \rightarrow (0, +\infty)$  such that for any  $\mu \in Abs(\Gamma)$

$$\lim_{\varepsilon \rightarrow \mu} |\varepsilon| = 0$$

**Definition 7.1** A triple  $(X, d, \delta)$  is a dilatation structure if  $(X, d)$  is a locally compact metric space and the dilatation field

$$\delta: \Gamma \times \{(x, y) \in X \times X : y \in \text{dom}(\varepsilon, x)\} \rightarrow X \quad , \quad \delta(\varepsilon, x, y) = \delta_\varepsilon^x y$$

gives to  $X$  the structure of a uniform idempotent right quasigroup over  $\Gamma$  (definition 5.2), with the operation: for any  $\varepsilon \in \Gamma$

$$x *_\varepsilon y = \delta_\varepsilon^x y$$

Moreover, the distance is compatible with the dilatations, in the sense:

- A1. the uniformity on  $(X, \delta)$  is the one induced by the distance  $d$ ,
- A2. There is  $A > 1$  such that for any  $x$  there exists a function  $(u, v) \mapsto d^x(u, v)$ , defined for any  $u, v$  in the closed ball (in distance  $d$ )  $\bar{B}(x, A)$ , such that for any  $\mu \in Abs(\Gamma)$

$$\lim_{\varepsilon \rightarrow \mu} \sup \left\{ \left| \frac{1}{|\varepsilon|} d(\delta_\varepsilon^x u, \delta_\varepsilon^x v) - d^x(u, v) \right| : u, v \in \bar{B}_d(x, A) \right\} = 0$$

uniformly with respect to  $x$  in compact set. Moreover the uniformity induced by  $d^x$  is the same as the uniformity induced by  $d$ , in particular  $d^x(u, v) = 0$  implies  $u = v$ .

In order to make connection with the original definition of a dilatation structure introduced and studied in [2], [3], we shall relate the notations used here

with the original ones. The operations induced by the uniform irq structure on  $X$  are:

$$\begin{aligned} v -_{\varepsilon}^x u &= \Delta_{\varepsilon}^x(u, v) = \delta_{\varepsilon^{-1}}^{\delta_{\varepsilon}^x u} \delta_{\varepsilon}^x v \\ u +_{\varepsilon}^x v &= \Sigma_{\varepsilon}^x(u, v) = \delta_{\varepsilon^{-1}}^x \delta_{\varepsilon}^{\delta_{\varepsilon}^x u} v \end{aligned}$$

where  $\Delta_{\varepsilon}^x, \Sigma_{\varepsilon}^x$  are the approximate difference, respectively approximate sum operations induced by dilatation structures. Similarly we have the following correspondence of notations:

$$\begin{aligned} v -_{\infty}^x u &= \Delta^x(u, v) = \lim_{\varepsilon \rightarrow \mu} \delta_{\varepsilon^{-1}}^{\delta_{\varepsilon}^x u} \delta_{\varepsilon}^x v \\ u +_{\varepsilon}^x v &= \Sigma^x(u, v) = \lim_{\varepsilon \rightarrow \mu} \delta_{\varepsilon^{-1}}^x \delta_{\varepsilon}^{\delta_{\varepsilon}^x u} v \end{aligned}$$

The conclusion is therefore that adding a distance in the story of uniform irqs gives us the notion of a dilatation structure.

We go a bit into details.

**Proposition 7.2** *Let  $(X, d, \delta)$  be a dilatation structure,  $x \in X$ , and let*

$$\delta_{\varepsilon}^x d(u, v) = \frac{1}{|\varepsilon|} d(\delta_{\varepsilon}^x u, \delta_{\varepsilon}^x v)$$

*Then the net of metric spaces  $(\bar{B}_d(x, A), \delta_{\varepsilon}^x d)$  converges in the Gromov-Hausdorff sense to the metric space  $(\bar{B}_d(x, A), d^x)$ . Moreover this metric space is a metric cone, in the following sense: for any  $\lambda \in \Gamma$  we have*

$$d^x(\delta_{\lambda}^x u, \delta_{\lambda}^x v) = |\lambda| d^x(u, v)$$

**Proof.** The first part of the proposition is just a reformulation of axiom A2, without the condition of uniform convergence. For the second part remark that

$$\frac{1}{|\varepsilon|} d(\delta_{\varepsilon}^x \delta_{\lambda}^x u, \delta_{\varepsilon}^x \delta_{\lambda}^x v) = |\lambda| \frac{1}{|\varepsilon \lambda|} d(\delta_{\varepsilon \lambda}^x u, \delta_{\varepsilon \lambda}^x v)$$

Therefore if we pass to the limit with  $\varepsilon \rightarrow \mu$  in these two relations we get the desired conclusion.  $\square$

Particular examples of dilatation structures are given by normed groups with dilatations.

**Definition 7.3** *A normed group with dilatations  $(G, \delta, \|\cdot\|)$  is a group with dilatations  $(G, \delta)$  endowed with a continuous norm function  $\|\cdot\| : G \rightarrow \mathbb{R}$  which satisfies (locally, in a neighbourhood of the neutral element  $e$ ) the properties:*

- (a) *for any  $x$  we have  $\|x\| \geq 0$ ; if  $\|x\| = 0$  then  $x = e$ ,*
- (b) *for any  $x, y$  we have  $\|xy\| \leq \|x\| + \|y\|$ ,*

(c) for any  $x$  we have  $\|x^{-1}\| = \|x\|$ ,

(d) the limit  $\lim_{\varepsilon \rightarrow \mu} \frac{1}{|\varepsilon|} \|\delta_\varepsilon x\| = \|x\|^N$  exists, is uniform with respect to  $x$  in compact set,

(e) if  $\|x\|^N = 0$  then  $x = e$ .

In a normed group with dilatations we have a natural left invariant distance given by

$$d(x, y) = \|x^{-1}y\| \quad . \quad (7.0.1)$$

Any normed group with dilatations has an associated dilatation structure on it. In a group with dilatations  $(G, \delta)$  we define dilatations based in any point  $x \in G$  by

$$\delta_\varepsilon^x u = x\delta_\varepsilon(x^{-1}u). \quad (7.0.2)$$

The following result is theorem 15 [2].

**Theorem 7.4** *Let  $(G, \delta, \|\cdot\|)$  be a locally compact normed group with dilatations. Then  $(G, d, \delta)$  is a dilatation structure, where  $\delta$  are the dilatations defined by (7.0.2) and the distance  $d$  is induced by the norm as in (7.0.1).*

The general theorem 5.3 has a stronger conclusion in the case of dilatation structures, namely "conical groups" are replaced by "normed conical groups".

## 8 Differentiability

We have seen that to any uniform irq we can associate a bundle of contractible groups  $x \in X \mapsto (X, +_\infty^x, x*)$ . This bundle can be seen as a tangent bundle, namely: to any  $x \in X$  is associated a conical group with  $x$  as neutral element, which is the tangent space at  $x$ . We shall denote it by  $T^x X$ .

In the particular case of a manifold, this is indeed a correct definition in the following sense: if we look to a small portion of the manifold then we know that there is a chart of this small portion, which puts it in bijection with an open set in  $\mathbb{R}^n$ . We have seen that we can associate to  $\mathbb{R}^n$  a uniform irq by using as the operation  $*$  a homothety with fixed ratio  $\varepsilon < 1$ . This uniform irq is transported on the manifold by the chart. If we ignore the facts that we are working not with the whole manifold, but with a small part of it, and not with  $\mathbb{R}^n$ , but with a open set, then indeed we may identify, for any point  $x$  in the manifold, a neighbourhood of the point with a neighbourhood of the tangent space at the point, such that the operation of addition of vectors in the tangent space at  $x$  is just the operation  $+_\infty^x$  and scalar multiplication by (any integer power of)  $\varepsilon$  is just  $u \mapsto x *_\varepsilon u$ .

The same is true in the more complex situation of a sub-riemannian manifold, as shown in [4], in the sense that (locally) we may associate to each point  $x$  a "dilatation" of ratio  $\varepsilon$ , which in turn gives us a structure of uniform irq. In

the end we get a bundle of Carnot group operations which can be seen as a tangent bundle of the sub-riemannian manifold. (In this case we actually have more structure given by the Carnot-Carathéodory distance, which induces also a "group norm" on each Carnot group.)

A uniform irq can be seen as a generalization of a differential structure. For this we give a definition of differentiable functions between two uniform irqs. This definition corresponds to uniform differentiability in the metric case of dilatation structures, definition 16 and the comments after it in [2]. It is a generalization of Pansu differentiability [19].

**Definition 8.1** *Let  $(X, *, \backslash)$  and  $(Y, \circ, \backslash\backslash)$  be two  $\Gamma$ -uniform irqs. A function  $f : X \rightarrow Y$  is differentiable if there is a function  $Tf : X \times X \rightarrow Y$  such that*

$$\lim_{k \rightarrow \infty} f(x) \backslash\backslash_k f(x *_k u) = Tf(x, u)$$

*uniformly with respect to  $x, u$  in compact sets.*

By abstract nonsense the application  $Tf$  has nice properties, like  $Tf(x, \cdot) : (X, +^x) \rightarrow (Y, +^{f(x)})$  is a morphism of conical groups.

## 9 Sub-riemannian symmetric spaces as braided $\mathbb{R} \times \mathbb{Z}_2$ -dilatation structures

Sub-riemannian symmetric spaces have been introduced in [21], section 9. We shall be interested in the description of sub-riemannian geometry by dilatation structures, therefore we shall use the same notations as in the previous paper [4] (see also the relevant citations in that paper, as well as the long paper [5], where the study of sub-riemannian geometry as a length dilatation structure is completed).

**Definition 9.1** *(adaptation of [21] definition 8.1) Let  $(M, D, g)$  be a regular sub-riemannian manifold. We say that  $\Psi : M \rightarrow M$  is an infinitesimal isometry if  $\Psi$  is  $\mathcal{C}^1$  and  $D\Psi$  preserves the metric  $g$ . An infinitesimal isometry is regular if for any  $x \in M$  and any tangent vector  $u \in T_x M$*

$$\Psi(\exp_x(u)) = \exp_{\Psi(x)}(D\Psi(x)u)$$

By [21] theorem 8.2.,  $\mathcal{C}^1$  isometries are regular infinitesimal isometries and, conversely, regular infinitesimal isometries are isometries.

An equivalent description of regular infinitesimal isometries is the following: they are  $\mathcal{C}^1$  Pansu differentiable isometries.

**Definition 9.2** *([21] definition 9.1) A sub-riemannian symmetric space is a regular sub-riemannian manifold  $(M, D, g)$  which has a transitive Lie group  $G$  of regular infinitesimal isometries acting differentiably on  $M$  such that:*

- (i) there is a point  $x \in X$  such that the isotropy subgroup  $K$  of  $x$  is compact,
- (ii)  $K$  contains an element  $\Psi$  such that  $D\Psi(x)|_{D_x} = -id$  and  $\Psi$  is involutive.

If  $G$  is a group for which (i), (ii) holds then we call  $G$  an admissible isometry group for  $M$ .

**Theorem 9.3** ([21] theorem 9.2) *If  $M$  is a sub-riemannian symmetric space and  $G$  is an admissible isometry group, then there exists an involution  $\sigma$  of  $G$  such that  $\sigma(K) \subset K$  with the following properties (we write  $\mathfrak{g} = \mathfrak{g}^+ + \mathfrak{g}^-$ , where  $\mathfrak{g}^+, \mathfrak{g}^-$  are the subspaces of  $\mathfrak{g}$  on which  $D\sigma$  acts as  $Id, -Id$ ):*

- (a)  $\mathfrak{g}$  is generated as a Lie algebra by a subspace  $\mathfrak{p}$  and the Lie algebra  $\mathfrak{t}$  of  $K$  with  $\mathfrak{p} \subset \mathfrak{g}^-, \mathfrak{t} \subset \mathfrak{g}^+$ ,
- (b) there exists a positive definite quadratic form  $g$  on  $\mathfrak{p}$  and  $ad K$  maps  $\mathfrak{p}$  to itself and preserves  $g$ . Furthermore,  $\mathfrak{p}$  may be identified with  $D_x$  under the exponential map of the Lie algebra  $\mathfrak{g}$ , and  $g$  with the sub-riemannian metric on  $D_x$ .

Conversely, given a Lie group  $G$  and an involution  $\sigma$  such that (a) and (b) hold, then  $G/K$  forms a sub-riemannian symmetric space, where  $D_{x_0} = exp \mathfrak{p}$  for the point  $x_0$  identified with the coset  $K$ , and the sub-riemannian metric on  $D_{x_0}$  is given by  $g$ . The bundle  $D$  and its metric is then uniquely determined by the requirement that elements of  $G$  be infinitesimal isometries.

As a consequence of this theorem we see that we may endow a sub-riemannian symmetric space, with admissible isometry group  $G$ , with a (reflexive space) operation

$$(x, y) \in M^2 \mapsto \Psi(x, y) = \Psi^x y$$

such that  $\Psi$  is distributive, for any  $x \in X$  the map  $\Psi^x$  satisfies (ii) definition 9.2, and for any  $g \in G$  and any  $x, y \in X$  we have

$$g(\Psi^x y) = \Psi^{g(x)} g(y)$$

We explained in [4] that we can construct a dilatation structure over a regular sub-riemannian manifold by using adapted frames.

Let us consider now dilatations structures with  $\Gamma$  isomorphic with  $\mathbb{R} \times \mathbb{Z}_2$ . That means  $\Gamma$  is the commutative group made by two copies of  $(0, +\infty)$ , generated by  $(0, +\infty)$  and an element  $\sigma \notin (0, +\infty)$ , with the properties: for any  $\varepsilon \in (0, +\infty)$  we have  $\varepsilon\sigma = \sigma\varepsilon$  and  $\sigma\sigma = 1$ . The absolute we take has two elements, one corresponding to  $\varepsilon \rightarrow 0$  (we denote it by "0") and the other one is the transport by  $\sigma$  of 0, denoted by "0 $\sigma$ ". The morphism  $|\cdot|$  is defined by

$$|\varepsilon| = |\sigma\varepsilon| = \varepsilon$$

Let  $(X, d, \delta)$  be a dilatation structure with respect to the group  $\Gamma$ , absolute  $Abs(\Gamma)$  and morphism  $|\cdot|$  described previously. Then for any  $\varepsilon \in (0, +\infty)$  and any  $x \in X$  we have the relations:

$$\delta_\sigma^x \delta_\varepsilon^x = \delta_\varepsilon^x \delta_\sigma^x \quad , \quad \delta_\sigma^x \delta_\sigma^x = id$$

**Proposition 9.4** Denote by  $\sigma^x y = \delta_{\sigma^x}^x y$  and suppose that for any  $x \in X$  the map  $\sigma^x$  is not the identity map. Then  $\sigma^x$  is involutive, a isometry of  $d^x$  and an isomorphism of the conical group  $T^x X$ .

**Proof.** For any  $x \in X$  clearly  $\sigma^x$  is involutive, commutes with dilatations  $\delta_{\varepsilon}^x$  and as a consequence of proposition 7.2, is an isometry of  $d^x$ . We need to show that it preserves the operation  $+_{\infty}^x$ . We shall work with the notations from dilatation structures. We have then, for any  $\varepsilon \in (0, +\infty)$ :

$$\sigma^{\delta_{\varepsilon}^x u} \Delta_{\sigma \varepsilon}^x(u, v) = \delta_{\varepsilon^{-1}}^{\delta_{\sigma \varepsilon}^x u} \delta_{\sigma \varepsilon}^x v = \Delta_{\varepsilon}^x(\sigma^x u, \sigma^x v)$$

We pass to the limit with  $\varepsilon \rightarrow 0$  and we get the relation:

$$\sigma^x \Delta^x(u, v) = \Delta^x(\sigma^x u, \sigma^x v)$$

which shows that  $\sigma^x$  is an isomorphism of  $T^x X$ .  $\square$

This proposition motivates us to introduce braided  $\mathbb{R} \times \mathbb{Z}_2$ -dilatation structures.

**Definition 9.5** Let  $(X, d, \delta)$  be a dilatation structure, with respect to the group  $\Gamma$ , absolute  $\text{Abs}(\Gamma)$  and morphism  $|\cdot|$  described previously, and such that for any  $x \in X$  the map  $\sigma^x$  is not the identity map. This dilatation structure is braided if the map

$$(x, y) \in X^2 \mapsto (\sigma^x y, x)$$

is a braided map.

**Theorem 9.6** A sub-riemannian symmetric space  $M$  with admissible isometry group  $G$  can be endowed with a braided  $\mathbb{R} \times \mathbb{Z}_2$ -dilatation structure which is  $G$ -invariant, that is for any  $g \in G$ , for any  $x, y \in M$ , and for any  $\varepsilon \in \Gamma$  we have

$$g(\delta_{\varepsilon}^x y) = \delta_{\varepsilon}^{g(x)} g(y)$$

**Sketch of the proof.** In the particular case of a sub-riemannian symmetric space we may obviously take the adapted frames to be  $G$ -invariant, therefore we may construct a dilatation structure (over the group  $(0, +\infty)$  with multiplication) which is  $G$ -invariant. Because  $\Psi^x$  satisfies (ii) definition 9.2, it follows  $\Psi^x$  is differentiable in  $x$  in the sense of dilatation structures. We extend the dilatation structure to a braided one by defining for any  $x \in X$

$$\sigma^x = T\Psi^x(x, \cdot)$$

By  $G$ -invariance of both the dilatation structure and the operation  $\Psi$  it follows that

$$T\Psi^x(x, \cdot) = \Psi^x$$

therefore  $\sigma^x$  commutes with  $\delta_{\varepsilon}^x$ , which ensures us that we well defined a braided dilatation structure.  $\square$

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