Curvature of sub-Riemannian spaces

Marius Buliga

Institut Bernoulli
Bâtiment MA
École Polytechnique Fédérale de Lausanne
CH 1015 Lausanne, Switzerland
Marius.Buliga@epfl.ch

and

Institute of Mathematics, Romanian Academy
P.O. BOX 1-764, RO 70700
Bucureşti, Romania
Marius.Buliga@imar.ro

This version: 20.01.2004

1 Introduction

To any metric space there is an associated metric profile. The rectifiability of the metric profile gives a good notion of curvature of a sub-Riemannian space. We shall say that a curvature class is the rectifiability class of the metric profile. We classify then the curvatures by looking to homogeneous metric spaces. The classification problem is solved for contact 3 manifolds, where we rediscover a 3 dimensional family of homogeneous contact manifolds, with a distinguished 2 dimensional family of contact manifolds which don't have a natural group structure. The classification of 3 dimensional homogeneous contact manifolds has been done by Hughen [8].

I have discovered metric profiles in various proof of Mitchell theorem 1. Also this is explained in the paper. In my opinion, the use of the notion of metric profile clarifies the question: why several proofs for same result (Mitchell theorem 1) and moreover, any of them equally long and complex?

It has to be mentioned that contrary to other attempts to define the curvature of a sub-Riemannian manifold, here is presented an almost pure metrical construction, not using differential geometry, which is notoriously misleading when used in a sub-Riemannian frame. Once one knows what to look for, then differential geometry (read "Euclidean analytic differential geometry") recovers its well known strength, though.

The structure of the paper is described further. In sections $2-4$ is given a short presentation of sub-Riemannian manifolds, Carnot groups, Pansu derivative and Gromov-Hausdorff distance. For the expert reader these sections serve only to fix notations needed later.

Section 5 is about deformations of sub-Riemannian manifold, seen as curves in the space CMS of isometry classses of compact metric spaces, with the Gromov-Hausdorff distance.

In section 6 can be found a discussion of various proofs of Mitchell [11] theorem 1. This section justifies the notion of metric profile, which is the subject of section 7. In the same section is given the notion of curvature in terms of rectifiability classes of metric profiles.

In order to classify the curvatures homogeneous spaces are used. Section 8 is dedicated to this subject.

As an application, in section 9 are studied the homogeneous contact 3 manifolds, Finally, in section 10 the problem of classification is solved for a large class of contact 3 manifolds.

2 Regular sub-Riemannian manifolds

Classical references to this subject are Bellaïche [1] and Gromov [6]. The interested reader is advised to look also to the references of these papers.

Let M be a connected manifold. A distribution (or horizontal bundle) is a subbundle D of M. To any point $x \in M$ there is associated the vectorspace $D_x \subset T_x M$.

Given the distribution D, a point $x \in M$ and a sufficiently small open neighbourhood $x \in U \subset M$, one can define on U a filtration of bundles as follows. Define first the class of horizontal vectorfields on U:

$$\mathcal{X}^1(U,D) \ = \ \{X \in \Gamma^\infty(TU) \, : \, \forall y \in U \ , \ X(u) \in D_y\}$$

Next, define inductively for all positive integers k:

$$\mathcal{X}^{k+1}(U,D) \ = \ \mathcal{X}^k(U,D) \cup [\mathcal{X}^1(U,D), \mathcal{X}^k(U,D)]$$

Here $[\cdot,\cdot]$ denotes vectorfields bracket. We obtain therefore a filtration $\mathcal{X}^k(U,D) \subset \mathcal{X}^{k+1}(U,D)$. Evaluate now this filtration at x:

$$V^k(x,U,D) \ = \ \left\{X(x) \, : \, X \in \mathcal{X}^k(U,D)\right\}$$

There are $m(x)$, positive integer, and small enough U such that $V^k(x,U,D) = V^k(x,D)$ for all $k \geq m$ and

$$D_x = V^1(x,D) \subset V^2(x,D) \subset ... \subset V^{m(x)}(x,D)$$

We equally have

$$\nu_1(x) = \dim V^1(x,D) < \nu_2(x) = \dim V^2(x,D) < ... < n = \dim M$$

Generally $m(x)$, $\nu_k(x)$ may vary from a point to another.

The number $m(x)$ is called the step of the distribution at x.

Definition 2.1 *The distribution D is regular if $m(x)$, $\nu_k(x)$ are constant on the manifold M.*

The distribution is completely non-integrable if for any $x \in M$ we have $V^{m(x)} = T_x M$.

Definition 2.2 *A sub-Riemannian (SR) manifold is a triple (M,H,g), where M is a connected manifold, H is a completely non-integrable distribution on M, and g is a metric (Euclidean inner-product) on the horizontal bundle H.*

The Carnot-Carathéodory distance associated to the sub-Riemannian manifold is the distance induced by the length l of horizontal curves:

$$d(x,y) \ = \ \inf\{l(c) \, : \, c : [a,b] \to M \ , \ c(a) = x \ , \ c(b) = y\}$$

The Chow theorem ensures the existence of a horizontal path linking any two sufficiently closed points, therefore the CC distance it at least locally finite.

We shall work further only with regular sub-Riemannian manifolds, if not otherwise stated.

Bellaïche introduced the concept of privileged chart around a point $p \in M$.

Let $(x_1,...,x_n) \mapsto \phi(x_1,...,x_n) \in M$ be a chart of M around p (i.e. p has coordinates $(0,....,0)$). Denote by $X_1,...,X_n$ the frame of vectorfields associated to the coordinate chart. The chart is called adapted (or the frame is called adapted) if the following happens: $X_1,...,X_{\nu_1}$ forms a basis of V^1, $X_{\nu_1+1},...,X_{\nu_2}$ form a basis of V^2, and so on.

Suppose that the frame $X_1,...,X_n$ is adapted. The degree of X_i is then k if $X_i \in V^k \setminus V^{k-1}$.

Definition 2.3 *A chart (or a frame) is privileged if moreover the following happens: for any $i = 1, ..., n$ the function*

$$t \mapsto d(p, \phi(..., t, ...))$$

(with t on the position i) is exactly of order deg X_i at $t = 0$.

Privileged charts (frames) always exist, as proved by Bellaïche [1] Theorem 4.15.

A privileged frame transforms the filtration into a direct sum. Define

$$V_i = \textit{ span } \{X_k : \textit{deg } X_k = i\}$$

Then the tangent space decomposes as a direct sum of vectorspaces V_i. Moreover, each space V^i decomposes in a direct sum of spaces V_k with $k \leq i$.

The intrinsic dilatations associated to a privileged frame are defined, in the chart ϕ, for any $\varepsilon > 0$ (sufficiently small if necessary) by

$$\delta_\varepsilon(x_i) = (\varepsilon^{deg \ i} x_i)$$

We may define (locally around p) a Lie bracket associated to the privileged frame, which comes from the vectorfield bracket written in coordinates with respect to the frame (which is a basis of the tangent space).

In terms of vectorfields, the intrinsic dilatation associated to the privileged frame transforms X_i into

$$\Delta_\varepsilon X_i = \varepsilon^{deg \ X_i} X_i$$

and the metric g into $\frac{1}{\varepsilon^2} g$.

The nilpotentization of the distribution with respect to the chosen privileged frame is then the bracket

$$[X, Y]_N = \lim_{\varepsilon \to 0} \Delta_\varepsilon^{-1} [\Delta_\varepsilon X, \Delta_\varepsilon Y]$$

It is very important to notice that the useful part of the nilpotentization bracket is its evaluation at the point p. It is generically false that there are privileged coordinates around an open set in M. This is however true in the particular case of contact manifolds, as a consequence of Frobenius theorem.

3 Carnot groups

Carnot groups are particular examples of sub-Riemannian manifolds. They are especially important because they provide infinitesimal models for any sub-Riemannian manifold.

Definition 3.1 *A Carnot (or stratified nilpotent) group is a connected simply connected group N with a distinguished vectorspace V_1 such that the Lie algebra of the group has the direct sum decomposition:*

$$n = \sum_{i=1}^{m} V_i \ , \quad V_{i+1} = [V_1, V_i]$$

The number m is the step of the group. The number

$$Q \;=\; \sum_{i=1}^{m} i \; dim V_i$$

is called the homogeneous dimension of the group.

Because the group is nilpotent and simply connected, the exponential mapping is a diffeomorphism. We shall identify the group with the algebra, if is not locally otherwise stated.

The structure that we obtain is a set N endowed with a Lie bracket and a group multiplication operation given by the Baker-Campbell-Hausdorff formula.

Any Carnot group admits a one-parameter family of dilatations. For any $\varepsilon > 0$, the associated dilatation is:

$$x \;=\; \sum_{i=1}^{m} x_i \;\mapsto\; \delta_\varepsilon x \;=\; \sum_{i=1}^{m} \varepsilon^i x_i$$

Any such dilatation is a group morphism and a Lie algebra morphism.

In fact the class of Carnot groups is characterised by the existence of dilatations.

Proposition 3.2 *Suppose that the Lie algebra $\mathfrak{g}$ admits an one parameter group $\varepsilon \in (0, +\infty) \mapsto \delta_\varepsilon$ of simultaneously diagonalisable Lie algebra isomorphisms. Then $\mathfrak{g}$ is the algebra of a Carnot group.*

We can always find Euclidean inner products on N such that the decomposition $N \;=\; \sum_{i=1}^{m} V_i$ is an orthogonal sum. Let us pick such an inner product and denote by $\| \cdot \|$ the Euclidean norm associated to it.

We shall endow the group N with a structure of a sub-Riemannian manifold now. For this take the distribution obtained from left translates of the space V_1. The metric on that distribution is obtained by left translation of the inner product restricted to V_1.

The Carnot-Carathéodory distance is

$$d(x,y) \;=\; \inf \left\{ \int_a^b \|c^{-1}\dot{c}\| \,\mathrm{d}t \;:\; c(a) = x, \; c(b) = y, \; c^{-1}\dot{c} \in V_1 \right\}$$

The distance is obviously left invariant.

We collect the important facts to be known about Carnot groups:

(a) If V_1 Lie-generates the whole Lie algebra of N then any two points can be joined by a horizontal path.

(b) The metric topology and uniformity of N are the same as Euclidean topology and uniformity respective.

(c) The ball $B(0,r)$ looks roughly like the box $\left\{ x \;=\; \sum_{i=1}^{m} x_i \;:\; \|x_i\| \leq r^i \right\}$.

(d) the Hausdorff measure $\mathcal{H}^Q$ is group invariant and the Hausdorff dimension of a ball is Q.

(e) there is a one-parameter group of dilatations, where a dilatation is an isomorphism δ_ε of N which transforms the distance d in εd.

In Euclidean spaces, given $f : R^n \rightarrow R^m$ and a fixed point $x \in R^n$, one considers the difference function:

$$X \in B(0,1) \subset R^n \ \mapsto \ \frac{f(x+tX) - f(x)}{t} \in R^m$$

The convergence of the difference function as $t \rightarrow 0$ in the uniform convergence gives rise to the concept of differentiability in it's classical sense. The same convergence, but in measure, leads to approximate differentiability. This and another topologies might be considered (see Vodop'yanov [13], [14]).

In the frame of Carnot groups the difference function can be written using only dilatations and the group operation. Indeed, for any function between Carnot groups $f : G \rightarrow P$, for any fixed point $x \in G$ and $\varepsilon > 0$ the finite difference function is defined by the formula:

$$X \in B(1) \subset G \ \mapsto \ \delta_\varepsilon^{-1} \left(f(x)^{-1} f\left(x \delta_\varepsilon X\right)\right) \in P$$

In the expression of the finite difference function enters δ_ε^{-1} and δ_ε, which are dilatations in P, respectively G.

Pansu's differentiability is obtained from uniform convergence of the difference function when $\varepsilon \rightarrow 0$.

The derivative of a function $f : G \rightarrow P$ is linear in the sense explained further. For simplicity we shall consider only the case $G = P$. In this way we don't have to use a heavy notation for the dilatations.

Definition 3.3 *Let N be a Carnot group. The function $F : N \rightarrow N$ is linear if*

(a) F is a group morphism,

(b) for any $\varepsilon > 0$ $F \circ \delta_\varepsilon \ = \ \delta_\varepsilon \circ F$.

We shall denote by $HL(N)$ the group of invertible linear maps of N, called the linear group of N.

The condition (b) means that F, seen as an algebra morphism, preserves the grading of N.

The definition of Pansu differentiability follows:

Definition 3.4 *Let $f : N \rightarrow N$ and $x \in N$. We say that f is (Pansu) differentiable in the point x if there is a linear function $Df(x) : N \rightarrow N$ such that*

$$\sup \left\{ d(F_\varepsilon(y), Df(x)y) \ : \ y \in B(0,1) \right\}$$

converges to 0 when $\varepsilon \rightarrow 0$. The functions F_ε are the finite difference functions, defined by

$$F_t(y) \ = \ \delta_t^{-1} \left(f(x)^{-1} f(x \delta_t y)\right)$$

For the differentiability notion in a sub-Riemannian manifold the reader can consult Margulis, Mostow [9] [10], Vodop'yanov , Greshnov [15], [16] or Buliga [4]. We shall use further the fact that isometries of a sub-Riemannian manifold are derivable in the sense of Pansu and the derivative is linear in the sense of the definition 3.3.

4 Gromov-Hausdorff distance

The references for this section are Gromov [7], chapter 3, and Burago & al. [5] section 7.4. There are several definitions of distances between metric spaces. The very fertile idea of introducing such distances belongs to Gromov.

In order to introduce the Hausdorff distance between metric spaces, recall the Hausdorff distance between subsets of a metric space.

Definition 4.1 *For any set $A \subset X$ of a metric space and any $\varepsilon > 0$ set the ε neighbourhood of A to be*

$$A_\varepsilon \ = \ \cup_{x \in A} B(x, \varepsilon)$$

The Hausdorff distance between $A, B \subset X$ is defined as

$$d_H^X(A, B) \ = \ \inf \{ \varepsilon > 0 \ : \ A \subset B_\varepsilon \ , \ B \subset A_\varepsilon \}$$

By considering all isometric embeddings of two metric spaces X, Y into an arbitrary metric space Z we obtain the Hausdorff distance between X, Y (Gromov [7] definition 3.4).

Definition 4.2 *The Hausdorff distance $d_H(X, Y)$ between metric spaces X Y is the infimum of the numbers*

$$d_H^Z(f(X), g(Y))$$

for all isometric embeddings $f : X \to Z$, $g : Y \to Z$ in a metric space Z.

If X, Y are compact then $d_H(X, Y) < +\infty$. Indeed, let Z be the disjoint union of X, Y and $M \ = \ \max \{ diam(X), diam(Y) \}$. Define the distance on Z to be

$$d^Z(x, y) \ = \ \begin{cases} d^X(x, y) & x, y \in X \\ d^Y(x, y) & x, y \in Y \\ \frac{1}{2} M & \text{otherwise} \end{cases}$$

Then $d_H^Z(X, Y) < +\infty$.

The Hausdorff distance between isometric spaces equals 0. The converse is also true (Gromov *op. cit.* proposition 3.6) in the class of compact metric spaces.

Theorem 4.3 *If X, Y are compact metric spaces such that $d_H(X, Y) = 0$ then X, Y are isometric.*

5 Deformations of a sub-Riemannian manifold

There are several deformations of a sub-Riemannian manifold around a point, which can be studied in the metric spaces CMS of isometry classes of pointed compact metric spaces, with the Gromov-Hausdorff distance. For the isometry class of the pointed metric space (X, x, d) we shall use the notation $[X, x, d]$ or $[X, d]$ when he marked point is obvious. We shall work only with spaces (X, x, d) such that $X = \bar{B}((x, 1)$.

The Ball-Box theorem (or the theorem concerning the existence of privileged frames) ensures us that small closed balls in sub-Riemannian manifolds are compact.

Definition 5.1 *The metric profile associated to the space (M, d) is the assignment (for small enough $\varepsilon > 0$)*

$$(x \in M, \varepsilon > 0) \ \mapsto \ \mathbb{P}^m_\varepsilon(x) = [\bar{B}(x, 1), \frac{1}{\varepsilon} d] \in CMS$$

The celebrated Mitchell [11] theorem 1 can be formulated in the following way:

Theorem 5.2 *(Mitchell, theorem1) The metric profile of a regular sub-Riemannian manifold can be prolonged by continuity in $\varepsilon = 0$. Moreover*

$$\mathbb{P}^m_0(p) = [\bar{B}(0, 1), d_N]$$

the isometry class of the nilpotentization of the distribution at p.

There are several proofs of this theorem. In order to understand them I shall introduce the notion of metric profile further.

Consider a privileged chart around $p \in M$. With this chart come the associated privileged frame, dilatations δ_ε and Δ_ε. For any Riemannian metric g one can define deformations of this metric by the formula:

$$g_\varepsilon(\Delta_\varepsilon X, \Delta_\varepsilon Y) = g(X, Y)$$

Let us begin by describing deformations induced from a privileged chart. These deformations are seen as curves in CMS, the space of isometry classes of compact metric spaces, with the Gromov-Hausdorff distance.

The dilatation flow δ_ε induces the following deformation:

$$[\delta, \varepsilon](t) = [\bar{B}(p, 1), (\delta, \varepsilon)]$$

where the distance (δ, ε) is given by

$$(\delta, \varepsilon)(x, y) = d(\delta_\varepsilon^{-1} x, \delta_\varepsilon^{-1} y)$$

We can induce another deformation: let $(D_\varepsilon, g_\varepsilon)$ be the pair distribution - metric on the distribution obtained by transport with δ_ε, namely:

$$D_\varepsilon(\delta_\varepsilon x) = D\delta_\varepsilon(x) D(x)$$

$$g_\varepsilon(\delta_\varepsilon x)(D\delta_\varepsilon(x)u, D\delta_\varepsilon(x)v) = g(x)(u,v)$$

for any $u, v \in T_x M$. The deformation associated is

$$[D, g, \varepsilon] = [\bar{B}(p,1), (D, g, \varepsilon)]$$

where the notation (D, g, ε) is used for the CC distance in the sub-Riemannian manifold $(M, D_\varepsilon, g_\varepsilon)$.

It is an important remark that generally

$$[\delta, \varepsilon] \neq [D, g, \varepsilon]$$

This is because the right-handed term is a distance given as the infimum of lengths of some horizontal curves.

A sufficient condition for the equality to happen is that δ_ε has local convex data in the sense of the generalized Local-Global Principe to be found in Buliga [3] Section1.2. In this case (small) CC balls in the manifold $(M, D_\varepsilon, g_\varepsilon)$ are convex (that is there are geodesics connecting any two points in the closure of the ball, inside the closure of tha ball). This is not happening even if M is a Carnot group. For example the closed balls in the Heisenberg group are not convex (with respect to the CC distance).

Finally, a deformation is associated to the dilatations Δ_ε and pairs privileged frame - Riemannian metric g. This is simply

$$[\Delta, \varepsilon] = [\bar{B}(p,1), (\Delta, \varepsilon)]$$

where (Δ, ε) is the Riemannian distance induced by the Riemannian metric g and the privileged frame.

We shall see that all these deformations are particular metric profiles. A general definition of a metric profile will be given further, after the discution of various proofs of Mitchell theorem 1.

6 Mitchell theorem 1

One can identify in the literature several proofs of this theorem. Exactly what is proven in each available variant of proof? The answer is: each proof basically shows that various metric profiles introduced in the previous section can be prolonged to 0. Each of this metric profiles are close in the GH distance to the original metric profile of the CC distance. More precisely:

Lemma 6.1 *Let $\mathbb{P}'(t)$ be any of the previously introduced metric profiles $[\delta, \varepsilon]$, $[D, g, \varepsilon]$, $[\Delta, \varepsilon]$. Then*

$$d_{GH}(\mathbb{P}^m_\varepsilon, \mathbb{P}'(\varepsilon)) = O(\varepsilon)$$

The proof of this lemma reduces to a control problem. In the case of the profile $[\delta, \varepsilon]$, this is Mitchell [11] lemma 1.2.

Mitchell [11] and Bellaïche [1] theorem 5.21, proposition 5.22, proved the following:

Theorem 6.2 *The metric profile $\varepsilon \mapsto [D, g, \varepsilon]$ can be prolonged at $\varepsilon = 0$ by continuity. We have*

$$[D, g, 0] = [\bar{B}(0,1), d_N]$$

The corresponding result of Gromov [6] section 1.4B and Vodop'yanov [16] is:

Theorem 6.3 *The metric profile $\varepsilon \mapsto [\Delta, \varepsilon]$ can be prolonged at $\varepsilon = 0$ by continuity. We have*

$$[\Delta, 0] = [\bar{B}(0,1), d_N]$$

By the use of the Ball-Box theorem and any of the results before, one can obtain an analogous result:

Theorem 6.4 *The metric profile $\varepsilon \mapsto [\delta, \varepsilon]$ can be prolonged at $\varepsilon = 0$ by continuity. We have*

$$[\delta, 0] = [\bar{B}(0,1), d_N]$$

The proofs of these theorems can be described as a manipulation of brackets associated with growth estimates in ε.

Any of these theorems imply the Mitchell theorem 1, with the use of the approximation lemma 6.1. But in fact these theorems are different statements in terms of metric profiles.

I am interested to know if more information can be obtain from the use of a particular metric profile. We shall see that this is indeed the case. For example the curvature is a notion which is associated with a choice of such a profile. In the Riemannian case it does make no difference the choice of a metric profile. The phenomenon of dependence curvature — metric profile is purely non-Riemannian. This path will not be pursued in this paper, where we shall use only the curvature given by the metric profile $\mathbb{P}^m$ associated with a metric space.

7 Metric profile

The purpose of this section is two-folded. It serves as an introduction to the notion of metric profile. It is also written for further reference. For example the notion of approximate metric profile, useful in the understanding of the construction of a tangent bundle to a sub-Riemannian group (see Buliga [4]), will not be used in this paper. It will rather serve as an appetizer for the interested reader.

The class of ε nets (with arbitrary ε) in compact metric spaces will be denoted by NETS. In this paper nets always have positive separation.

Likewise one can consider the classes CMS_a, $NETS_a$, of compact metric spaces (nets in compact metric spaces respectively) of diameter not greater than $a > 0$. The class $[NETS_a]$ with the Lipschitz distance is continuously embedded in $[CMS_b]$ with Gromov-Hausdorff distance, for any $b > a$.

We can define a notion of metric profile regardless to any distance.

Definition 7.1 *A metric profile is a curve $\mathbb{P} : [0, a] \to [CMS]$ such that:*

(a) it is continuous at 0,

(b) for any $b \in [0, a]$ *and fixed* $\varepsilon \in (0, 1]$ *we have*

$$d_{GH}(\mathbb{P}(\varepsilon b), \mathbb{P}_{d_b}(\varepsilon, x)) \;=\; O(b)$$

We used here the notation $\mathbb{P}(b) = [\bar{B}(x, 1), d_b]$ *and* $\mathbb{P}_{d_b}(\varepsilon, x) = [\bar{B}(x, 1), \frac{1}{\varepsilon} d_b]$.

Note that in this definition is not stated that $\mathbb{P}(0) = \mathbb{P}_{d_b}(0)$. Look for example to the metric profile used by Gromov, namely $[\Delta, \varepsilon]$. For this profile we never have the mentioned equality, because $[\Delta, b](0)$ is always the Euclidean unit n dimensional ball. Nevertheless this is a profile in the sense of the previous definition.

Definition 7.2 *The metric profile is nice if for all small enough* b *we have* $\mathbb{P}(0) = \mathbb{P}_{d_b}(0)$.

The metric profile of a homogeneous space is just a curve in the space $[CMS]$, continuous at 0. Likewise, if we look at a homogeneous sub-Riemannian manifold, all metric profiles previously introduced are not depending on points in the manifold.

In order to give the definition of an approximate metric profile, we need a slightly modified version of proposition 3.5, chapter 3, Gromov [7].

Proposition 7.3 *Let* $(X_i)_i$, $(Y_i)_i$ *be two sequences in* CMS *such that*

$$d_{GH}(X_i, Y_i) \to 0$$

as $i \to \infty$. *Then for any* $\eta > 0$ *and for any sequence* $(N_i)_i \subset NETS$ *of* η *nets* $N_i \subset X_i$, *there is a sequence* $(M_i)_i \subset NETS$ *of* $\eta + 2d_{GH}(X_i, Y_i) + d_{GH}^2(X_i, Y_i)$ *nets* $M_i \subset Y_i$ *such that*

$$d_{Lip}(N_i, M_i) \;\leq\; 2d_{GH}(X_i, Y_i) + d_{GH}^2(X_i, Y_i)$$

Corollary 7.4 *Let* $\mathbb{P}$ *be a nice metric profile,* $\eta > 0$ *and* $\tilde{\mathbb{P}}_\eta : [0, 1] \to [NETS]$ *be a curve such that* $\tilde{\mathbb{P}}_\eta(a)$ *is a* η *net in* $\mathbb{P}(a)$ *for all* a.

Then there exists a function $\tilde{\mathbb{P}}_\eta : [0, 1] \times [0, 1] \to [NETS]$ *such that*

i) $\tilde{\mathbb{P}}_\eta(a, 1) = \tilde{\mathbb{P}}_\eta(a)$ *for any* a,

ii) $\tilde{\mathbb{P}}_\eta(a, \varepsilon)$ *is a* $\eta + O(a)o(\varepsilon)$ *net in* $\mathbb{P}(a, \varepsilon)$, *for all* a,

iii) the following estimate holds

$$d_{Lip}(\tilde{\mathbb{P}}_\eta(a, 1), \tilde{\mathbb{P}}_\eta(a, \varepsilon)) \;=\; 2\eta + O(a)O(\varepsilon)$$

The definition of an approximate metric profile follows.

Definition 7.5 *Let* $\mathbb{P}$ *be a nice metric profile. An approximate metric profile of* $\mathbb{P}$ *is a function* $\tilde{\mathbb{P}}$ *which satisfies the conclusions of the previous corollary, with the slight modification consisting in replacement of* η *in the estimates by* $O(\eta)$.

It would be interesting to see what is happening in the case where an approximate metric profile is made by balls in discrete groups.

We shall define further the notion of curvature associated with a given metric profile.

Definition 7.6 *Suppose $\mathbb{P}$ is a nice metric profile. Suppose moreover that*

$$d(\mathbb{P}(\varepsilon)(0), \mathbb{P}(\varepsilon)(a)) = O(\varepsilon a)$$

Then we shall call such a profile rectifiable at $\varepsilon = 0$.

Two metric profiles $\mathbb{P}_1$, $\mathbb{P}_2$ which are rectifiable at $\varepsilon = 0$ are equivalent if

$$d((P_1(\varepsilon)(a), \mathbb{P}_2(\varepsilon)(a)) = o(a)$$

(for fixed ε).

The curvature class of a metric profile $\mathbb{P}$ is the equivalence class of $\mathbb{P}$.

In particular cases we would like to be able to compute the curvature. This can be done by using homogeneous spaces.

8 The homogeneous case

To a homogeneous space we can associate the groups $Isom(X, d)$ and $Isom_p(X, d)$, of isometries (isometries wich fix the point p respectively) of (X, d). The coset class $Isom(X, d)/Isom_p(X, d)$ is homeomorphic with (X, d) by the map

$$\pi : Isom(X, d)/Isom_p(X, d) \rightarrow X$$

The construction of the map π is explained further. Let $p \in X$ be a fixed point. Pick a coset $fIsom_p(X, d)$ and define $\pi(fIsom_p(X, d)) = f(p)$. Obviously the definition is good.

The inner action of $Isom_p(X, d)$ on $Isom(X, d)$ gives an action of $Isom_p(X, d)$ on the coset space $Isom(X, d)/Isom_p(X, d)$. This inner action is compatible with the action of $Isom_p(X, d)$ on X in the sense: for any $h \in Isom_p(X, d)$ and for all $f \in Isom(X, d)$ we have

$$\pi(hfh^{-1}Isom_p(X, D)) = h(\pi(fIsom_p(X, d)))$$

In the case of a regular sub-Riemannian (X, D, g) manifold we can associate to it the triple $(Isom(X, d), Isom_p(X, d), D)$ or an equivalent triple which is described further.

The situation is as follows: note $G = Isom(X, d)$, $G_0 = Isom_p(X, d)$. Then any right-invariant vectorfield on G descends on a vectorfield on left cosets G/G_0. In particular, if we endow G with a right-invariant distribution, then G/G_0 is endowed with a distribution induced by the descent of any right invariant "horizontal" frame. G/G_0 is not usually a regular sub-Riemannian manifold. Look for example to the case: $G = H(1)$, G_0 is the one parameter group generated by an element of the distribution. Then G/G_0 is the Grushin plane, which is not a regular sub-Riemannian manifold.

Consider on G the right invariant distribution

$$D" = Lie\ G_0 + D'$$

such that $Lie\ G_0 \cap D' = 0$ and D' descends on the distribution D on G/G_0 (if $G = Isom(X,d)$, $G_0 = Isom_p(X,d)$) . We have seen that the action of G_0 on G/G_0 which mimicks the action of $Isom_p(X,d)$ on (X,d) is the (descent of) the inner action. We are not wrong if we suppose that isometries preserve the distribution at 0, which translates into the condition: for any $h \in G_0$

$$Ad_h D' \subset D'$$

We know one more thing about the homogeneous metric space (X,d): its tangent cone. Consider on G with given distribution $D"$ the dilatations δ_ε and a privileged right-invariant basis around the neutral element. The knowledge of the tangent cone implies the following:

(a) we know some relations in the algebra $Lie\ G$,

(b) we know that for any $h \in G_0$ $Ad_h \in HL(G,D")$, that is Ad_h commutes with dilatations δ_ε.

In conclusion, we can describe homogeneous metric spaces coming from sub-Riemannian manifolds by looking to triples $(G, G_0, D")$, which satisfy certain relations.

It goes without saying that we have also an Euclidean metric on the distribution $D"$.

Definition 8.1 *Let (X,d) be a metric space and $p \in X$ a point such that the metric profile associated to $X,d)$ and p can be prolonged at $\varepsilon = 0$ and it is rectifiable at $\varepsilon = 0$. We shall say that the curvature of (X,d) at p is $(G, G_0, D")$ if the metric profile at p is equivalent with the metric profile of G/G_0 with respect to the distribution D (the descent of $D"$).*

This definition is insinuating that $(G, G_0, D")$ (and the overlooked metric on $D"$) are uniquely defined up to trivial transformation. We shall explore this issue in the final section, for a particular case.

As an exercise we want to compute all Riemannian homogeneous n manifolds. So we are looking at groups G which contain a subgroup G_0 such that:

$$Lie\ G \ = \ Lie\ G_0 + D'$$

$$[Lie\ G_0, D'] \subset D'$$

$$[D', D'] \subset Lie\ G_0$$

and for any $x \in Lie\ G_0$ the restriction of ad_x on D' is antisymmetric . Moreover, D' has dimension n. For example, when $n = 2$ we have two cases. The first case is G

3 dimensional, with a basis X_0, X_1, X_2 for *Lie G*, such that X_0 generates *Lie* G_0 and X_1, X_2 generate D'. The bracket relations that we know are:

$$[X_0, X_1] = aX_1 + bX_2$$

$$[X_0, X_2] = cX_1 + dX_2$$

$$[X_1, X_2] = eX_0$$

From Jacobi identity we get $e(a + d) = 0$ and from the condition ad_{X_0} restricted to D' antisymmetric we get $a = 0$, $d = 0$, $b + c = 0$. Therefore we have

$$[X_0, X_1] = bX_2$$

$$[X_0, X_2] = -bX_1$$

$$[X_1, X_2] = eX_0$$

We have a one dimensional family of homogeneous Riemannian surfaces, where the curvature can be measured by b/e(except $e = 0$, but the factor space is the Euclidean plane; see also next case).

The second case is $dim\ G_0 = 0$ and G is abelian 2 dimensional. But this is trivial, moreover, it is contained in the previous case.

This is well known and seems to be related to the Cartan method of equivalence.

9 Application: curvature of contact 3 manifolds

Contact manifolds are particular cases of sub-Riemannian manifolds. The contact distribution is completely non-integrable. By using natural normalization of the contact form (see for example Bieliavski, Flbel, Gorodski [2] or Hughen [8]) we can uniquely associate to a contact structure, endowed with a metric on the contact distribution, a sub-Riemannian manifold. The nilpotentization of the contact distribution is always a Heisenberg group.

The horizontal linear maps on the Heisenberg group are known. Moreover, the group of isometries of $H(n))$ which preserve the origin is $SU(n)$.

If we want to look for all homogeneous contact 3 manifolds, we have to consider two cases. The first case is G 4 dimensional, with a basis for *Lie G* given by X_0, X_1, X_2, X_3, such that X_0 is a basis for *Lie* G_0,

$$[X_1, X_2] = X_3 + aX_0$$

$$[X_1, X_3] = bX_0 + AX_1 + BX_2$$

$$[X_2, X_3] = cX_0 + CX_1 + DX_2$$

(which comes from the knowledge of the nilpotentization and from the condition $[D', D'] \subset$ *Lie* G_0),

$$[X_0, X_1] = dX_1 + eX_2$$

$$[X_0, X_2] = fX_1 + gX_2$$

and the ad_{X_0} condition is $d = g = 0$, $e + f = 0$ and $[X_0, X_3] = 0$.

We have to use further the Jacobi identities. We begin with:

$$[X_1, [X_2, X_3]] + [X_2, [X_3, X_1]] + [X_3, [X_1, X_2]] = 0$$

This gives: $A + D = 0$, $eC = 0$, $eb = 0$.

The next relation is:

$$[X_0, [X_2, X_3]] + [X_2, [X_3, X_0]] + [X_3, [X_0, X_2]] = 0$$

This gives: $e(A - D) = 0$, $e(B + C) = 0$.

The relation:

$$[X_0, [X_1, X_2]] + [X_1, [X_2, X_0]] + [X_2, [X_0, X_1]] = 0$$

gives nothing new.

We continue with:

$$[X_0, [X_1, X_3]] + [X_1, [X_3, X_0]] + [X_3, [X_0, X_1]] = 0$$

which lead to nothing new.

If $e = 0$ then we have $f = 0$, $A + D = 0$, and we get the relations:

$$[X_1, X_2] = X_3 + aX_0$$

$$[X_1, X_3] = bX_0 + AX_1 + BX_2$$

$$[X_2, X_3] = cX_0 + CX_1 - AX_2$$

$$[X_0, X_1] = 0$$

$$[X_0, X_2] = 0$$

$$[X_0, X_3] = 0$$

By a change of basis: $X_3' = X_3 + aX_0$, ... , we arrive to the description of G as a direct sum of a 3 dimensional group with $G_0 = S(1)$. This is in reality a singular case (in the sense that G_0 is not really needed in the construction: it is added and after factorized out without any consequences).

If $e \neq 0$ then we have the relations

$$[X_1, X_2] = X_3 + aX_0$$

$$[X_1, X_3] = 0$$

$$[X_2, X_3] = cX_0$$

$$[X_0, X_1] = eX_2$$

$$[X_0, X_2] = -eX_1$$

$$[X_0, X_3] = 0$$

These form a 2 dimensional family of homogeneous spaces which are not groups.

The second case is *dim* $G_0 = 0$ and G is 3 dimensional, with a basis for *Lie* G given by X_1, X_2, X_3, such that:

$$[X_1, X_2] = X_3$$

$$[X_1, X_3] = AX_1 + BX_2$$

$$[X_2, X_3] = CX_1 + DX_2$$

The Jacobi identity

$$[X_1, [X_2, X_3]] + [X_2, [X_3, X_1]] + [X_3, [X_1, X_2]] = 0$$

gives the relation $A + D = 0$, therefore we recover previous case.

We have a 3 dimensional family of regular homogeneous spaces which are also groups. Particular examples are: SO(3), SL(2,R), E(1,1).

10 Classification of curvatures

We shall prove in this section that for any two 3 dimensional homogeneous spaces which are also groups, if they have the same curvature class then they are isometric. This will partially solve the problem of classification of curvatures for 3 dimensional contact manifolds.

More specifically we shall prove the following:

Theorem 10.1 *Let G_1, G_2 be two 3 dimensional groups. We identify the Lie algebras and hence we have two brackets on $\mathbb{R}^3$ denoted by $[\cdot, \cdot]_i$, $i = 1, 2$.*

Define also $\delta_\varepsilon(X_1) = \varepsilon X_1$, $\delta_\varepsilon(X_2) = \varepsilon X_2$ and $\delta_\varepsilon(X_3) = \varepsilon^2 X_3$.

Let d_1, d_2 be the CC distances on G_1, G_2 with respect to the left invariant distributions generated by X_1, X_2, transported on (a neighbourhood of 0 of) $\mathbb{R}^3$.

Suppose that we have the bracket relations:

$$[X_1, X_2]_i = X_3$$

$$[X_1, X_3]_i = A_i X_1 + B_i X_2$$

$$[X_2, X_3]_i = -B_i X_1 + D_i X_2$$

If $d_1(\delta_\varepsilon x, \delta_\varepsilon y) - d_1(\delta_\varepsilon x, \delta_\varepsilon y) = O(\varepsilon^2)$ uniformly with respect to x, y in a compact neighbourhood of 0, then the Lie brackets are identical.

The proof uses the Baker-Campbell-Hausdorff formula and the Ball Box theorem. The hypothesis implies that

$$d_1^2(\delta_\varepsilon x, \delta_\varepsilon y) - d_1^2(\delta_\varepsilon x, \delta_\varepsilon y) = O(\varepsilon^4)$$

Each distance d_i is left invariant. We shall note

$$\|u\|_i = d_i(0, u)$$

We know from the Ball Box theorem that $\|u\|^2$ is comparable with $\|u_1\|^2+ \mid u_2 \mid$, where $u = u_1 + u_2$ is the decomposition of u into the horizontal part $u_1 \in$ *span* $\{X_1, X_2\}$ and the vertical part $u_2 \in$ *span* $\{X_3\}$.

We shall denote by $\overset{1}{\cdot}$, $\overset{2}{\cdot}$ the operations in G_1, G_2 respectively. The hypothesis becomes:

$$\|\delta_\varepsilon(-x) \overset{1}{\cdot} \delta_\varepsilon y\|_1^2 - \|\delta_\varepsilon(-x) \overset{2}{\cdot} \delta_\varepsilon y\|_2^2 = 0(\varepsilon^4)$$

From the Baker-Campbell-Hausdorff formula and the bracket relations we see that we can approximate $\delta_\varepsilon(-x) \overset{1}{\cdot} \delta_\varepsilon y$ up to $o(\varepsilon^4)$ by using only terms in the Baker-Campbell-Hausdorff formula which contain at most two brackets. Same is true for the operation $\overset{2}{\cdot}$.

Moreover the norms $\| \cdot \|_i$ can be estimated from the Ball Box theorem. From here a careful computation resumes the proof.

References

[1] A. Bellaïche, The tangent space in sub-Riemannian geometry, in: *Sub-Riemannian Geometry*, A. Bellaïche, J.-J. Risler eds., Progress in Mathematics, **144**, Birkhäuser, (1996), 4 – 78

[2] P. Bieliavsky, E. Falbel, C. Gorodski, The classification of simply-connected contact sub-Riemannian symmetric spaces,*Pacific J. Math.* , **188** (1999), no. 1, 65–82

[3] M. Buliga, Sub-Riemannian geometry and Lie groups. Part I, arXiv:math.MG/0210189, (2002)

[4] M. Buliga, Tangent bundles to sub-Riemannian groups, arXiv:mathMG/0307342, (2003)

[5] D. Burago, Y. Burago, S. Ivanov, A Course in Metric Geometry, *Graduate Studies in Mathematics*, **33**, AMS Providence, Rhode Island, (2000)

[6] M. Gromov, Carnot-Caratheodory spaces seen from within, in: *Sub-Riemannian Geometry*, A. Bellaïche, J.-J. Risler eds., Progress in Mathematics, **144**, Birkhäuser, (1996), 79 – 323

[7] M. Gromov, Metric structures for Riemannian and non-Riemannian spaces, *Progress in Math.*, **152**, Birchäuser (1999), chapter 3.

[8] K. Hughen, The sub-Riemannian geometry of 3-manifolds, preprint DUK-M-95-13, (1995), available at

`http://www.math.duke.edu/preprints/1995.html`

[9] G.A. Margulis, G.D. Mostow, The differential of a quasi-conformal mapping of a Carnot-Carathéodory space, *Geom. Funct. Analysis*, **8** (1995), 2, 402–433

[10] G.A: Margulis, G.D. Mostow, Some remarks on the definition of tangent cones in a Carnot-Carathéodory space, *J. D'Analyse Math.*, **80** (2000), 299–317

[11] J. Mitchell, On Carnot-Carathéodory metrics, *J. Diff. Geometry*, **21** (1985), 35–45

[12] P. Pansu, Métriques de Carnot-Carathéodory et quasiisometries des espaces symetriques de rang un, *Ann. of Math.*, **129** (1989), 1–60

[13] S. K. Vodop'yanov, $\mathcal{P}$-Differentiability on Carnot Groups in Different Topologies and Related Topics, *Proc. on Analysis ans Geometry*, Novosibirsk: Sobolev Institute Press, (2000), 603 – 670

[14] S. K. Vodop'yanov, On differentiability of mappings of Sobolev classes on Carnot groups, *Matematicheskiy Sbornik* **194** (2003), 6, 67–86.

[15] S. K. Vodop'yanov, A. V. Greshnov, On differentiability of mappings of Carnot-Carathéodory spaces, *Dokl. R.A.S.*, t. **389**, 5, (2003)

[16] S. K. Vodop'yanov, A. V. Greshnov, Geometry of Carnot-Carathéodory spaces and differentiability of weakly contact mappings, *Izvestiya R.A.S., Matematicheskaya seriya*, to appear