Sur la construction des processus de Markov avec des méthodes de la théorie du potentiel et applications aux ED(P) stochastiques

Iulian Cîmpean

IMAR

Potentiel et probabilités

Conférence dédiée au 80ème anniversaire de Gheorghe Bucur

24, 25 Janvier 2019, Bucarest, Roumanie

Structure of the presentation:

1. Motivation: S(P)DEs with non-regular drifts.
2. A brief overview of potential theoretical tools for constructing Markov processes.
3. Back to S(P)DEs with non-regular drifts

Cauchy problem on $\mathbb{R}^d$: existence and uniqueness

Consider on $\mathbb{R}^d$, $d \geq 1$

$$\begin{cases} x'(t) = b(x(t)), & t > 0 \\ x(0) = x_0. \end{cases} \tag{0.1}$$

- b is continuous: local existence (Peano)
- b is locally Lipschitz: local existence and uniqueness
- b is locally Lipschitz (or monotone) + $|b(x)| \leq c(1 + |x|)$: global existence and uniqueness

! If $\mathbb{R}^d$ is replaced by an infinite dimensional Banach space, then Peano's local existence may fail (Jean Dieudonné):

- $C_0 := \{x := (x_n)_{n\geq 1} : x_n \underset{n}{\to} 0\}$, $|\cdot|_\infty$
- $b(x) = (\sqrt{|x(n)|} + \frac{1}{n+1})_{n\geq 1}$
- b is Holder continuous, but there is NO solution for (0.1)

Back to $\mathbb{R}^d$, $d = 1$

! If b is not locally Lipschitz then uniqueness is in general not guaranteed:

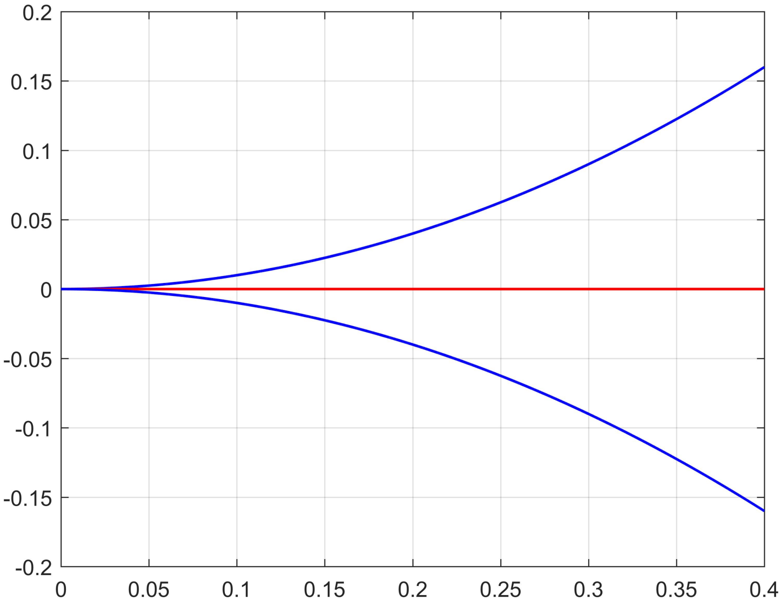
$$b(x) = 2sgn(x)\sqrt{|x|}$$

is Holder continuous and has sublinear growth, but

$$\begin{cases} x'(t) = 2sgn(x(t))\sqrt{|x(t)|}, & t > 0 \\ x(0) = 0. \end{cases} \tag{0.2}$$

has infinitely many (global) solutions:

- $x_0(t) = 0$, $x_+ = t^2$, $x_-(t) = -t^2$
- $x(t) = \begin{cases} 0, & 0 \leq t \leq t_0 \\ (t - t_0)^2, & t \geq t_0. \end{cases}$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SDEs: existence and uniqueness

Consider now an SDE (with additive noise) on $\mathbb{R}^d$, $d \geq 1$

$$\begin{cases} dX(t) = b(X(t))dt + \sigma dW(t), & t > 0 \\ X(0) = x \in \mathbb{R}^d, \end{cases} \tag{0.3}$$

where $(W(t))_{t\geq 0}$ is a d-Brownian motion on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$, and $\sigma \in \mathbb{R}^{d\times d}$. In Ito formulation

$$X(t) = x + \int_0^t b(X(s))ds + \sigma W(t), \quad t \geq 0 \;\; \mathbb{P}\text{-a.e.} \tag{0.4}$$

Different notions of solutions:

• X is a **strong** solution if: for any given B.m. W on $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$, X is $\mathcal{F}_t$-adapted, path-continuous and satisfies (0.4).

• **pathwise uniqueness** holds if: for any two solutions X, Y s.t. $X(0) = Y(0)$ $\mathbb{P}$-a.e., then $X(t) = Y(t), t \geq 0$ $\mathbb{P}$-a.e.

Well posedeness: if b is locally Lipschitz (or monotone) and of at most linear growth, then there exists a pathwise unique strong solution for (0.4); Krylov '99, Liu & Röckner '15...

Weak solutions and uniqueness in law

- X is a **weak** solution if: there exists a B.m. W on some $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$, X is $\mathcal{F}_t$-adapted, path-continuous and satisfies (0.4).
- **uniqueness in law** holds if: for any two weak solution $X, \mathbb{P}, W$ and $X', \mathbb{P}', W'$, we have for any $0 \leq t_1 \leq \dots t_n < \infty$

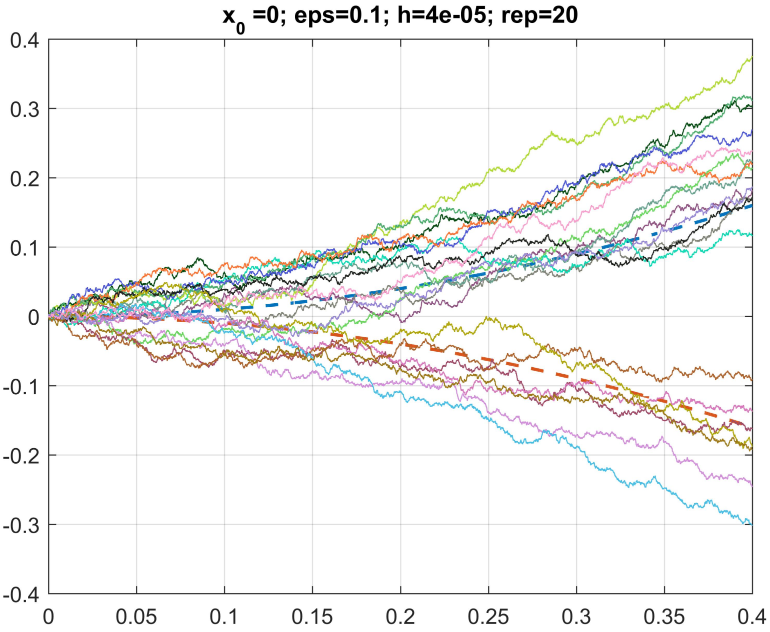
$$\mathbb{P} \circ (X_{t_1}, \cdots, X_{t_n})^{-1} = \mathbb{P}' \circ (X'_{t_1}, \cdots, X'_{t_n})^{-1}$$

Girsanov transform

If $|b(x)| \leq c(1 + |x|)$ then weak existence and uniqueness in law hold.
In fact, $W(t)_{0 \leq t \leq T}$ is a weak solution under

$$\mathbb{Q}_T := e^{\int_0^T \langle b(W(s)), W(s) \rangle - \frac{1}{2} \int_0^t |b|^2(W(s)) ds} \cdot \mathbb{P}.$$

! In particular, if $b(x) = 2sgn(x)\sqrt{|x|}$, then there exists a (unique in law) weak solution for (0.3)



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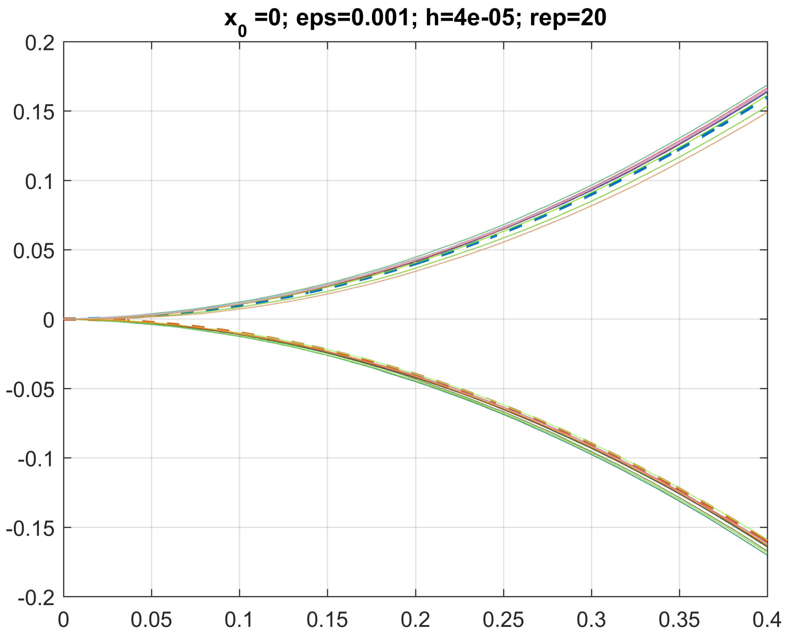
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Yamada-Watanabe and strong solutions

Theorem (Yamada-Watanabe '71)

Weak existence + pathwise uniqueness $\Rightarrow$ strong existence

Theorem (Veretennikov '81)

If $b \in L^\infty$ then there exist a pathwise unique strong solution.

Theorem (Krylov and Röckner '05)

If $b \in L^p, p > d$, then there exist a pathwise unique strong solution.

Martingale solutions (Strook & Varadhan)

If X is a (weak or strong) solution for (0.4) and $f \in C_b^2(\mathbb{R}^d)$, then by Ito's formula:

$$
\begin{aligned}
f(X(t)) & \\
&= f(X(0)) + \int_0^t \left[\langle \nabla f(X(s)), b(X(s)) \rangle + \frac{1}{2} tr(\sigma^T \sigma D^2 f)(X(s)) \right] ds \\
&\qquad + \int_0^t \langle Df(X(s)), dW(s) \rangle \\
&= f(x) + \int_0^t Lf(X(s)) ds + \text{ martingale },
\end{aligned}
$$

where

$$
Lf = \frac{1}{2} tr(\sigma^T \sigma D^2 f) + \langle \nabla f, b \rangle, \quad f \in C^2.
$$

- A process X is a **martingale** solution starting at $x \in \mathbb{R}^d$ if

$$
f(X(t)) - f(x) + \int_0^t Lf(X(s)) ds, \; t \geq 0 \quad \text{is a martingale}
$$

for a sufficiently large class of test functions f.

From martingale solutions to strong solutions

Strong solutions $\Rightarrow$ Weak solution $\Rightarrow$ Martingale solutions

Mart. sol. $\underbrace{\Longrightarrow}_{\text{often}}$ Weak sol. $\underbrace{\Longrightarrow}_{\text{path uniqueness}}$ Strong sol.

So, one strategy (see e.g. the recent work of Da Prato, Flandoli, Priola, Röckner, Veretennikov, Wang '02, '09, '16):

$(L, D(L)) \Longrightarrow (P_t)_{t\geq 0} \Longrightarrow$ Markov process $X \Longrightarrow$ mart. sol. $\Rightarrow$

$\Longrightarrow$ weak sol. $\underbrace{\Longrightarrow}_{\text{path uniqueness}}$ Strong solution.

Kolmogorov construction and Feller semigroups

Theorem (Kolmogorov)

Given a semigroup of Markov kernels $(P_t)_{t\geq 0}$ on $(E, \mathcal{B})$, there is always a canonical Markov process $(X_t)_{t\geq 0}$ s.t.

$$P_t f(x) = \mathbb{E}^x f(X_t) \quad \text{for all bounded } f$$

Such a "raw" Markov process is not enough, we need path-regularity.

Definition: If E is locally compact separable metric space, then $(P_t)_{t\geq 0}$ is called Feller if

1. $P_t(C_0) \subset C_0$
2. $\lim_{t\to 0} P_t f = f$ for all $f \in C_0$.

Theorem (classic)

If $(P_t)_{t\geq 0}$ is Feller, then there exists an associated normal strong Markov process with cadlag trajectories.

Pathwise-regular Markov processes by potential theory

- E be a Lusin topological space (e.g. a Polish space) with Borel σ-algebra $\mathcal{B}$. **For several slides we work only with $\mathcal{B}$!**
- $(P_t)_{t\geq 0}$ is a semigroup of Markov kernels on E, with resolvent

$$U_\alpha f(x) = \int_0^\infty P_t f(x) dt, \quad f : E \to \mathbb{R} \text{ bounded}$$

- $u : E \to [0, \infty]$ is called α**-excessive** if $P_t^\alpha u := e^{-\alpha t} P_t u \underset{t\to 0}{\nearrow} u$.
- $\mathcal{E}(\mathcal{U}_\alpha) := \{u : u \text{ is } \alpha\text{-excessive}\}$

Fact: $u \in \mathcal{E}(\mathcal{U}_\alpha) \Leftrightarrow \beta U_{\alpha+\beta} u \underset{\beta\to\infty}{\nearrow} u$

- A σ-finite measure ξ on E is called α-excessive if $\xi \circ (\beta U_{\alpha+\beta}) \leq \xi$ for all $\beta > 0$.

Fine and natural topologies on E

In order to have a "fine" potential theory for $\mathcal{U}$ we assume:

(H.1) $\sigma(\mathcal{E}(\mathcal{U}_\alpha)) = \mathcal{B}$, $1 \in \mathcal{E}(\mathcal{U}_\alpha)$ and $\mathcal{E}(\mathcal{U}_\alpha)$ is min-stable.

- **The fine topology**: is the one generated by $\mathcal{E}(\mathcal{U}_\alpha)$ for some $\alpha > 0$.

Remark: The fine topology is non-metrizable and hard to characterize! Therefore we introduce:

Definition: Any Lusin topology on E coarser than the fine topology, whose Borel σ-algebra is $\mathcal{B}$ is called a **natural** topology.

Definition (on brief): A process $(X_t)_{t\geq 0}$ on $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P}^x, x \in E)$ is called a right process with state space (E, τ), and transition function $(P_t)_{t\geq 0}$ if τ is natural and

- X is a normal strong Markov process with τ-r.c. trajectories under all $\mathbb{P}^x$
- $t \mapsto u(X_t)$ is $\mathbb{P}^x$-a.s. right continuous for all $u \in \mathcal{E}(\mathcal{U}_\alpha)$.
- $P_t f(x) = \mathbb{E}^x f(X_t)$ for all bounded f.

There is always a right process behind!

Remark: If there exists a right Markov process, then (H.1) is satisfied.

Question: Is (H.1) also sufficient? **Answer:** No, but...

Theorem (L. Beznea, N. Boboc, M. Röckner '04,'06)

Under (H.1), there is always a larger measurable space $(E, \mathcal{B}) \subset (E_1, \mathcal{B}_1)$ and a resolvent $\mathcal{U}^1$ on E_1 such that:

- $\mathcal{U}^1$ has a right process w.r.t. any natural topology τ_1 on E_1.
- $\mathcal{U}^1$ is an extension of $\mathcal{U}$:
  - $U^1_\alpha f|_E = U_\alpha(f|_E)$ on E for all bounded f and $\alpha > 0$
  - $U^1_\alpha(1_E) = 0$ on $E_1 \setminus E$.

Remark: 1. The extension $(E_1, \mathcal{U}^1)$ can be taken to be maximal and unique (up to an isomorphism)!
2. τ_1 can be chosen s.t. $\tau_1|_E$ is natural on E and s.t. it makes continuous any given countable $\mathcal{S}_0 \subset \mathcal{E}(\mathcal{U}_\alpha)$.

Polarity, solidity of potentials and right processes

Reduced and balayaged functions: If $u \in \mathcal{E}(\mathcal{U}_\alpha)$ and $A \in \mathcal{B}$

$$R_\alpha^A u := \inf\{v \in \mathcal{E}(\mathcal{U}_\alpha) : \ v \geq u \text{ on } A\}.$$
$$B_\alpha^A u = \lim_\beta \beta U_{\alpha+\beta}(R_\alpha^A u) \in \mathcal{E}(\mathcal{U}_\alpha).$$

Definition: $A \in \mathcal{B}$ is called **polar** if $B_\alpha^A 1 = 0$, $\alpha > 0$.

Theorem (Beznea, Boboc, Röckner '06), (Steffens '89)

The following assertions are equivalent, if (H.1) holds:

1. There exists a right process for $\mathcal{U}$ on E w.r.t. one (hence any) natural topology.
2. $E_1 \setminus E$ is polar w.r.t. $\mathcal{U}^1$.
3. If ξ is α-excessive s.t. $\xi \leq \mu \circ U_\alpha$ then $\xi = \nu \circ U_\alpha$.

Cadlag trajectories: tightness of capacity

Let λ be a reference measure on E: $\lambda(A) = 0 \Rightarrow U_\alpha 1_A \equiv 0$.
Let τ be a natural topology on E.
Choquet capacity:

$$c^1_\lambda(A) := \inf_{G=\overset{\circ}{G}} \{\lambda(R^G_1 1) : \ A \subset G\}, \ \ A \subset E.$$

Theorem (Lyons, Röckner '92) (Beznea, Boboc, Röckner '06)

Assume that X is a right process for $\mathcal{U}$. Then the following assertions are equivalent.

1. X has τ-cadlag trajectories.
2. The capacity is tight w.r.t. τ-compact sets.
3. There exists $v \in \mathcal{U}_\alpha$ with τ-compact sublevel sets, s.t. $v < \infty$ λ a.e.

Continuous paths: harmonic measure

Assume that X is a right process for $\mathcal{U}$ and τ is natural, generated by some metric d.

Theorem (Hunt):

$$B_1^A u(x) = \mathbb{E}^x\{e^{-T_A} u(X_{T_A}); T_A < \infty\},$$

where $T_A := \inf\{t > 0 : X_t \in A\}$.

If A is closed, then $u \mapsto B_1^A u(x)$ is a measure μ_x^A supported on A.

Theorem (essentially known), e.g. Fukushima, Stannat

The following assertions are equivalent:

1. X has continuous paths w.r.t. any natural topology.
2. If $B := B(z, r)$ is any ball in E, then $\mu_x^{B^c}$ is supported on ∂B.

Back to S(P)DEs with non-regular drifts

We are in the framework of Da Prato, Flandoli, Priola, Röckner, Veretennikov, Wang '02, '09, '16.
On $(H, \langle\cdot,\cdot\rangle)$:

$$\begin{cases} dX(t) = (AX(t) + F_0(X(t)) + B(X(t)))dt + \sigma dW(t) \\ X(0) = x \in H. \end{cases} \quad (0.5)$$

Hypothesis 1. (i) $A : D(A) \subset H \to H$ is a self-adjoint

$$\langle Ax, x\rangle \leq \omega|x|^2,$$

(ii) $F : D(F) \subset H \to 2^H$ is m-dissip.

$$F_0(x) := \underset{y \in F(x)}{arg\,min}\,|y|, \quad x \in D(F),$$

(iii) $\sigma \geq 0$ is sym. s.t. $\sigma^{-1} \in L(H)$ and for some $\alpha > 0$

$$\int_0^\infty (1 + t^{-\alpha})|T_t\sigma|_{\mathrm{HS}}^2\, dt < \infty$$

(iv) $B : H \to H$ is bounded and measurable.

The Kolmogorov operator for $B \equiv 0$

$$L_0\varphi = \frac{1}{2}\mathrm{Tr}[\sigma^2 D^2\varphi] + \langle x, AD\varphi\rangle + \langle F_0, D\varphi\rangle, \quad \varphi \in \mathcal{E}_A(H),$$

where $\mathcal{E}_A(H)$ is the linear space generated by the (real parts of) functions of type $\varphi(x) = \exp\{i\langle x, h\rangle\}$ with $h \in D(A)$.

Hypothesis 2. There exists a Borel probability measure μ on H such that

(i) $\int_{D(F)} (1 + |x|^2)(1 + |F_0(x)|)\, \nu(dx) < \infty$.

(ii) $L_0(\mathcal{E}_A(H)) \subset L^2(H, \nu)$ and

$$\int_H L_0\varphi\, d\nu = 0.$$

(iii) $\nu(D(F)) = 1$.

- $H_0 := supp(\nu)$.

The case $B \equiv 0$

Theorem (Da Prato and Röckner, '02)

If (H.1) and (H.2) hold, then

- $(L_0, \mathcal{E}_A(H))$ is closable on $L^{2(H_0,\nu)}$
- The closure $(L, D(L))$ is m-dissipative and there exits a semigroup of Markov kernels $(P_t)_{t \geq 0}$ on H_0 s.t. $P_t = e^{tA}$ on $L^2(\nu)$;
- P_t is Lip strong Feller and ν is invariant.

Theorem (Da Prato, Röckner, Wang '02, '09)

There exists a set $M \subset H_0$ s.t. $\nu(H_0 \setminus M) = 0$, and a $|\cdot|$-continuous right process $(X(t))_{t \geq 0}$ on M with transition function $(P_t)_{t \geq 0}$.
Moreover, for all "good" starting points $x \in M$

- X is a weak solution for (0.5).
- Pathwise uniqueness holds, so by Yamada-Watanabe, (0.5) admits also strong solutions.

General B

$$\begin{cases} dX = (AX + F_0(X) + B(X))dt + dW_t \\ X(0) = x \in H_0. \end{cases} \tag{0.6}$$

We fix W on $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$, and $(X_t^x)_{t\geq 0}$ to be the extended solution for $B \equiv 0$.

$$\rho_t^x := e^{\int_0^t \langle B(X_s^x) dW_s \rangle - \frac{1}{2}\int_0^t |B|^2(X_s^x) ds}$$

Proposition (Girsanov transform)

If $x \in M$ then $(X_t^x)_{t\in[0,T]}$ is a solution for (0.6) under $\rho_T^x \cdot \mathbb{P}$. Moreover, uniqueness in law holds.

!!! On M, if $F_0 = \nabla V$, pathwise uniqueness was obtained in [Da Prato, Flandoli, Priola, Röckner and Veretennikov, '16].

What about the "bad" starting points $H_0 \setminus M$?

Ongoing work, jointly with L. Beznea and M. Röckner:

Aim

Extend X_t^x to a Markov diffusion process on the entire H_0 and investigate in which sense the extended process is a solution to equation (0.6)!

- use potential theoretical tools in order to provide a general and natural way of extending Markov processes.

Thank you!