Ordered Groups: A Construction of Real Numbers of Professor Gh. Bucur

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Ordered Groups

Definition

An *ordered group* is, by definition, an *Abelian group* $(G;+)$ endowed with an *order relation* $\leq$ subject to the following *compatibility condition*: for any $x, y, z \in G$ we have

$$x \leq y \text{ if and only if } x + z \leq y + z.$$

In the following, we assume that the order relation is *total*, that is, for any $x, y \in G$ we have either $x \leq y$ or $y \leq x$.
As usually, for $x, y \in G$, we denote $x < y$ if $x \leq y$ and $x \neq y$. Also,

$$G_+ := \{x \in G \mid x \geq 0\}, \quad G_+^* := \{x \in G \mid x > 0\} = G_+ \setminus \{0\}.$$

Remark. If $p \in G_+$ and the set $\{np \mid n \in \mathbb{N}\}$ has supremum, then $p = 0$.

Definition

The totally ordered group $(G;+)$ is called *Archimedean* if for any $p \in G_+^*$, the set $\{np \mid n \in \mathbb{N}\}$ is not upper bounded.

Order Complete Groups

Definition

A totally ordered group $(G;+)$ is *order complete*, or simply *complete*, if any upper bounded and nonempty subset of G has supremum.

Proposition

If the totally ordered group $(G;+)$ is order complete then it is Archimedean.

Definition

A nontrivial totally ordered group $(G;+)$ is called *nondiscrete* (respectively, *discrete*) if $G_+^* = G_+ \setminus \{0\}$ does not have (respectively, has) a least element.

Remark. If G is discrete and we denote by e the least element of G_+^*, then

$$\{ne \mid n \in \mathbb{Z}\} = G,$$

if and only if G is Archimedean.

Nondiscrete Groups

Remarks. (Density) The group G is nondiscrete if and only if for any $x, y \in G$ with $x < y$ there exists $z \in G$ such that $x < z < y$.

(Decomposition Property) If G is nondiscrete and $g, g_1, g_2 \in G$ are such that $g < g_1 + g_2$ then there exist $g_1', g_2' \in G$ such that $g = g_1' + g_2'$, $g_1' < g_1$, and $g_2' < g_2$.

The Completion Theorem

Definition

Let $(G_i; +; \leq)$, $i = 1, 2$ be two totally ordered groups. A map $\varphi \colon G_1 \to G_2$ is an *isotonic morphism* if

- $\varphi(x + y) = \varphi(x) + \varphi(y)$ for all $x, y \in G_1$ (additive);
- $\varphi(x) < \varphi(y)$ for all $x, y \in G_1$ with $x < y$ (isotonic).

Remark. Any isotonic morphism of totally ordered groups is injective and hence an embedding.

Theorem (The Completion Theorem)

For any totally ordered group $(G; +; \leq)$ that is Archimedean and nondiscrete there exists a complete totally ordered (hence Archimedean) nondiscrete group $(\widetilde{G}; +; \leq)$ and an isotonic morphism $\varphi \colon G \to \widetilde{G}$.

The Completion Theorem — Idea of the Proof

A nonempty subset $S \subset G$ is called a *cut* of G if

- (Left Hereditary) Whenever $x \in S$ and $y \in G$ with $x \leq y$ it follows $x \in S$.
- (Upper Bounded) There exists $z \in G$ such that $x \leq z$ for all $x \in S$.
- (No Largest Element) For any $x \in S$ there exists $y \in S$ with $x < y$.

$\widetilde{G}$ is the collection of cuts, on which there are naturally defined addition $+$ and order $\leq$ such that $(\widetilde{G}; +; \leq)$ is a totally ordered group.

In addition, letting $\varphi \colon G \to \widetilde{G}$ be defined by

$$G \ni g \mapsto \varphi(g) = S_g := \{x \in G \mid x < g\},$$

then φ is an isotonic morphism.

The Completion Theorem — Density

Lemma (The Partition Lemma)

Let $(G; +; \leq)$ be a nondiscrete totally ordered group.

- *For any $p \in G_+^*$ and $n \in \mathbb{N}$ there exists $q_n \in G_+^*$ such that $nq_n \leq p$.*
- *Assuming that G is complete, for any $p \in G_+^*$ and $n \in \mathbb{N}$ there exists a unique $p_n \in G$ such that $np_n = p$. Also, $p_n \in G_+^*$.*

Definition

Given $(G; +; \leq)$ a totally ordered group, a subset M of G is *order dense*, or simply *dense*, in G if for any $x, y \in G$ with $x < y$ there exists $m \in M$ such that $x < m < y$.

Proposition

Let $(G; +; \leq)$ be an Archimedean totally ordered group and G_0 a nondiscrete subgroup of G. Then G_0 is dense in G.

Convergence in Totally Ordered Archimedean Nondiscrete Groups

In the following we let $(G; +; \leq)$ be a totally ordered Archimedean nondiscrete group.

For any $x \in G$ the *modulus* of x is

$$|x| := \begin{cases} x, & x > 0, \\ 0, & x = 0, \\ -x, & x < 0. \end{cases}$$

The Triangle Inequality holds: for any $x, y \in G$ we have

$$|x + y| \leq |x| + |y|, \quad ||x| - |y|| \leq |x - y|.$$

Also, if either $x, y \in G_+$ or $x, y \in -G_+$ then $|x + y| = |x| + |y|$.

Convergence in Totally Ordered Archimedean Nondiscrete Groups

Definition

A sequence $(x_n)_n$ in G is *convergent* if there exists $x \in G$ such that, for all $\epsilon \in G_+*$ there exists $N \in \mathbb{N}$ such that for all $n \in \mathbb{N}$ with $n \geq N$ we have $|x_n - x| < \epsilon$.

Definition

A sequence $(x_n)_n$ in G is *fundamental* if for any $\epsilon \in G_+^*$ there exists $N \in \mathbb{N}$ such that for all $m, n \in \mathbb{N}$ with $m, n \geq N$ we have $|x_m - x_n| < \epsilon$.

Theorem (Cauchy's Theorem)

If G is complete then a sequence in G is convergent if and only if it is fundamental.

Construction of the Real Field

One considers the ordered field of rational numbers $(\mathbb{Q}; +; \cdot; \leq)$. Clearly $(\mathbb{Q}; +; \leq)$ is a totally ordered Archimedean nondiscrete group and let $(\widetilde{\mathbb{Q}}; +; \leq)$ denote its completion to a complete totally ordered (hence Archimedean) nondiscrete group.

Let $\mathbb{R} := \widetilde{\mathbb{Q}}$. For any $x, y \in \mathbb{R}$ there exist sequences $(x_n)_n$ and $(y_n)_n$ in $\mathbb{Q}$ such that $x_n \to x$ and $y_n \to y$ when $n \to \infty$. Then,

$$xy := \lim_{n \to \infty} x_n y_n.$$

The definition is correct and $(\mathbb{R}; +; \cdot; \leq)$ is a complete ordered field that extends the ordered field $(\mathbb{Q}; +; \cdot; \leq)$.

Uniqueness of Complete Ordered Groups

Theorem (Uniqueness of Complete Ordered Groups)

Let $(G; +; \leq)$ be a complete nondiscrete totally ordered group (hence Archimedean) and let $e \in G_+^*$. Then:
(a) There exists $\Phi_e \colon \mathbb{R} \to G$ an isomorphism of ordered groups (additive, isotonic, and surjective) such that $\Phi_e(1) = e$.
(b) In addition, if $\Phi \colon \mathbb{R} \to G$ is additive, increasing (that is, $e < f$ implies $\Phi(e) \leq \Phi(f)$) and $\Phi(1) = e$ then $\Phi = \Phi_e$.

Corollary

Let $(G_i; +; \leq)$, $i = 1, 2$ be two complete nondiscrete totally ordered groups and $e_i \in G_{i,+}^*$. Then there exists a unique isomorphism of ordered groups $\Psi \colon G_1 \to G_2$ such that $\Psi(e_1) = e_2$.

Theorem (Uniqueness of the ordered field $\mathbb{R}$)

If $(K; +; \cdot; \leq)$ is a complete nondiscrete ordered field then there exists a unique isomorphism of ordered fields $\Phi \colon \mathbb{R} \to K$.

Transcendental Functions

Theorem (Existence and Uniqueness of the Exponential Function)

Let $a > 1$.

(a) There exists a bijective function $\exp_a :: \mathbb{R} \to \mathbb{R}^*_+$ *subject to the following properties:*

(i) $\exp_a(x+y) = \exp_a(x)\exp_a(y)$, *for all* $x, y \in \mathbb{R}$.

(ii) $\exp_a(1) = a$ *and* $\exp_a(y) < \exp_a(y)$ *for all* $x, y \in \mathbb{R}$ *with* $x < y$.

(b) Any function $g \colon \mathbb{R} \to \mathbb{R}^*_+$ *having the properties:*

(i) $g(x+y) = g(x)g(y)$ *for all* $x, y \in \mathbb{R}$;

(ii) $g(1) = a$ *and* $g(x) \leq g(y)$ *whenver* $x \leq y$;

coincides with $\exp_a$.

GH. BUCUR, *Analiză matematică*, Editura Universității din București 2006.