Modélisation stochastique des avalanches par des processus de fragmentation-branchement

Madalina Deaconu

Inria - Institut Élie Cartan de Lorraine, Nancy

avec L. Beznea et O. Lupașcu-Stamate

Conférence dédiée à Gheorghe Bucur
24-25 Janvier 2019, Potentiel et Probabilités, IMAR Bucarest

Plan

- **Model introduction**
  - Motivation: fragmentation examples
  - Fragmentation equation
- **Stochastic approach**
  - SDE with jumps
  - Branching processes and fragmentation
- **Application - modelisation**
  - Fragmentation, branching and avalanche
  - Numerical algorithm

- **Fragmenation models**
  - in astrophysics: stellar fragmentation, meteorits
  - in crystallography: crystals fragmentation
  - in nuclear physics: atoms fission
  - in geophysics: rupture phenomena like avalanches, earthquakes, etc.

- **Continuous mass model**
  - Consider the evolution of an infinite particle system
  - The particles are completely characterized by their mass (continuous)
  - Equation which describes the evolution of the concentration of particles $c(t,x)$ in the system
- **Fragmentation equation** (EF)

$$\begin{cases} \dfrac{\partial}{\partial t} c(t,x) &= \displaystyle\int_x^1 F(x, y-x) c(t,y)\, \mathrm{d}y \\ & \quad -\dfrac{1}{2} c(t,x) \displaystyle\int_0^x F(y, x-y)\, \mathrm{d}y, \forall t \geqslant 0, \forall x \in [0,1], \\ c(0,x) &= c_0(x), \forall x \in [0,1]. \end{cases} \qquad (EF)$$

- **Properties / Remarks**
  - Binary fragmentation: a particle of mass $x + y$ splits in two particles of masses x and y
  - $F : [0,1] \times [0,1] \rightarrow \mathbb{R}_+$ fragm. kernel, symmetric $F(x,y) = F(y,x)$
  - **Mass conservation**
  - Complex linear integro-differential —> difficult to solve

- **Key property**
  - Mass conservation

$$\int_0^1 xc(t,x)\mathrm{d}x = \int_0^1 xc_0(x)\mathrm{d}x = 1,\ \forall t \geq 0$$

  - Probability $Q_t(\mathrm{d}x) = xc(t,x)\mathrm{d}x,\ \forall t \geq 0$

- **Objectives and steps**
  - Construct a stochastic process with jumps $(X_t)_{t\geq 0}$ whose law is $Q_t(\mathrm{d}x)$
  - Weak form of the equation (EF) in order to introduce the stochastic approach and the infinitesimal generator
  - Construct a Markov process with jumps (stochastic differential equation)
  - Get properties for the solution of (EF) via this approach

- **Hypothesis on F**

(H) F is continuous from $[0,1]^2$ to $\mathbb{R}_+ \cup \{+\infty\}$.
The rate of loss of mass for the particle of mass x is:

$$\psi(x) = \begin{cases} \dfrac{1}{x}\displaystyle\int_0^x y(x-y)F(y,x-y)\mathrm{d}y & \text{if } x > 0, \\ 0 & \text{if } x = 0. \end{cases}$$

F is s.t. ψ is continuous on $[0,1]$.

- **Remark: the particle mass (size) $x \in [0,1]$** - connexion with the fragmentation introduced by Bertoin

- **Example:** $F(x,y) := x + y$

- **Definition** The family $(Q_t)_{t\geqslant 0}$ of probabilities on $[0,1]$ is a weak solution of (EF) with initial data Q_0 if:

$$\langle Q_t, \phi\rangle = \langle Q_0, \phi\rangle + \int_0^t \langle Q_s, \mathcal{F}\phi\rangle \mathrm{d}s,\ \forall \phi \in \mathcal{C}^1([0,1]),\ t\geqslant 0, \qquad \text{(EF-weak)}$$

where $\langle Q_t, \phi\rangle = \int_0^1 \phi(y) Q_t(\mathrm{d}y)$ and for all $x \in [0,1]$:

$$\boxed{\mathcal{F}\phi(x) = \int_0^x [\phi(x-y) - \phi(x)] \frac{x-y}{x} F(y, x-y)\mathrm{d}y.}$$

- **Aim:** Construct a process $(X_t)_{t\geqslant 0}$ whose law is $(Q_t)_{t\geqslant 0}$

- **Definition of the solution of the SDE** $(SDE - F)$

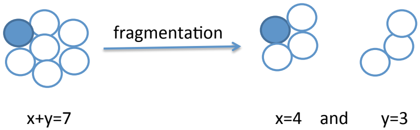
  Let $(\Omega, \mathcal{G}, (\mathcal{G}_t)_{t\geqslant 0}, \mathbb{P})$ be a probability space and Q_0 a probability on $[0,1]$.

  X is a solution of (SDE-F) if:

  - $X = (X_t)_{t\geqslant 0}$ is an adapted process on $(\Omega, \mathcal{G}, (\mathcal{G}_t)_{t\geqslant 0}, \mathbb{P})$ having trajectories in $\mathbb{D}([0,+\infty),[0,1])$.
  - The law of $X_0 = Q_0$.
  - There exists a Poisson measure $N(\mathrm{d}s, \mathrm{d}y, \mathrm{d}u)$ adapted to $(\mathcal{G}_t)_{t\geqslant 0}$ on $[0,+\infty)\times[0,1)\times[0,1)$ with intensity $\mathrm{d}s\,\mathrm{d}y\,\mathrm{d}u$ s.t.:

$$X_t = X_0 - \int_0^t \int_0^1 \int_0^1 y 1_{\{y\in(0,X_{s-})\}} 1_{\left\{u \leq \frac{X_{s-}-y}{X_{s-}} F(y, X_{s-}-y)\right\}} N(\mathrm{d}s, \mathrm{d}y, \mathrm{d}u)$$

- **Illustration**



- **Interpretation**

At random Poissonian times the particle splits into two particles with smaller masses

  - Mass the new particle is $X_{s-} - y$, $y \in (0, X_{s-})$
  - Time at a rate $F(y, X_{s-} - y)\frac{X_{s-} - y}{X_{s-}}$

Theorem

Under (H) there exists a solution $X = (X_t)_{t \geqslant 0}$ of the SDE of fragmentation $(SDE - F)$. Let Q_t be the law of (X_t), $t \geqslant 0$. Then, the family $\{Q_t\}_{t \geqslant 0}$ is a solution of $(EF - weak)$.

- **How to come back to the initial equation (EF) ?**

  Suppose that for all $t \geqslant 0$, Q_t is s.t. $Q_t(\mathrm{d}x) = q(t,x)\mathrm{d}x$. Then $c(t,x) := \frac{q(t,x)}{x}$ is a solution, in the weak sense, of (EF): for all $t \geq 0$ and all test function φ s.t. $\phi(x) = \frac{\varphi(x)}{x} \in \mathcal{C}^1([0,1])$ we have:

$$\int_0^1 \varphi(x) c(t,x) \mathrm{d}x = \int_0^1 \varphi(x) c_0(x) \mathrm{d}x$$

$$+ \frac{1}{2} \int_0^t \int_0^1 \int_0^x [\varphi(x-y) + \varphi(y) - \varphi(x)] F(y, x-y) c(s,x) \mathrm{d}y \mathrm{d}x \mathrm{d}s.$$

Probabilistic approach

Branching process

E a measurable space (for us $E = [0, 1]$)

$$\widehat{E} := \left\{ \mu \text{ non-negative measure on } E : \ \mu = \sum_{k=1}^{m} \delta_{x_k}, \ x_1, \dots, x_m \in E \right\} \cup \{\mathbf{0}\}.$$

Definition

A Markov process X on $\widehat{E}$ is a branching process iff for all measures $\mu_1, \mu_2 \in \widehat{E}$:

$$X^{\mu_1 + \mu_2} \stackrel{(d)}{=} X^{\mu_1} + X^{\mu_2}.$$

- **Probabilistic interpretation** At the initial time the particle starts in a point from E and evolves according with a basis process X, up to a random time, when it gives birth to a number m of independent particles having same law as the mother particle (same law as X).
- **Fragmentation** The particles split independently one to each other so we can associate a branching property

- **Steps**
  - Markov processes and fragmentation equation
  - Branching process associated to a fragmentation kernel
  - Fragmentation as a limit of the branching process

Theorem

Let F be a **bounded fragmentation kernel** *and $Q_0 = \delta_x$. Then:*

- $(EF - weak)$ *has a unique solution* $(Q_{t,x})_{t \geqslant 0}$.
- *The family of kernels* $(Q_t)_{t \geqslant 0}$ *on* $[0,1]$:

$$Q_t f(x) := \langle Q_{t,x}, f \rangle, \ \forall\ f \in p\mathcal{C}_b([0,1]), x \in [0,1],$$

  is also the transition function of the Markov process with jumps $X^0 = (\Omega, \mathcal{G}, \mathcal{G}_t, X_t^0, \mathbb{P}^x)$, *with values in* $[0,1]$ *(Ethier, Kurtz).*
- *For every function* $\phi \in \mathcal{C}([0,1])$ *and* $\nu \in \mathcal{P}([0,1])$, $(\phi(X_t^0) - \int_0^t \mathcal{F}\phi(X_s^0)\mathrm{d}s,\ t \geqslant 0)$ *is a* $(\mathcal{G}_t)_{t \geqslant 0}$*-martingale under* $\mathbb{P}^\nu$.
- *The SDE* $(SDE - F)$, *with initial condition* δ_x, *has a solution and this solution has same law as* $(X^0, \mathbb{P}^x)$.

Tools

- **Construct a sequence of thresholds for the particle size** fix a sequence $(d_n)_{n\geqslant 1} \subseteq (0,1)$, strictly non-increasing converging to 0.
  - **Truncated kernel (bounded)**

    Let F be a fragmentation kernel. For $n\geqslant 1$ define

    $$F_n(x,y) := \mathbb{1}_{(d_n,1]}(x \wedge y)F(x,y),\ x,y \in E = [0,1].$$

  - **Markov process for the kernel F_n** (size greater than d_n), by using the theorem.
- **Markov process truncated by the size d_n**
- **Result** Construct a branching process associated with. We need to:
  - Define the Markov process: $X = (\Omega, \mathcal{F}, \mathcal{F}_t, X_t, \theta_t, P^x)$, on E
  - A Markovian kernel: $B : bp\mathcal{B}(\widehat{E}) \longrightarrow bp\mathcal{B}(E)$

Theorem

(Beznea, Deaconu, Lupașcu, SPA, 2015)

There exists a branching process on $\widehat{E}$, induced by the process X with state space E, and by the Markovian kernel B.

- **Branching process induced by the solution of the fragmentation equation**

  By the Theorem, for all $n \geqslant 1$, there exists a branching process on $\widehat{E_n}$, induced by the process X^n on E_n, and by the kernel B^n, associated to the fragmentation kernel F.

- **Idea of proof**

  For $n \geqslant 1$ we note $(\widehat{P_t^n})_{t \geqslant 0}$ the transition function of the branching process on $\widehat{E_n}$, induced by the basis process X^n and by the Markovian kernel B^n.

  We can prove that there exists a projective limit of the sequence $(\widehat{P_t})_{t \geqslant 0}$ on $S^{\downarrow}$, which is a transition function

- $S^{\downarrow}$ link with the fragmentation model introduced by Bertoin

- A simplified physical model (fragmentation kernel) for the avalanche
- Stochastic differential equation of fragmentation
- Simulation

- **Describe the avalanche via the fragmentation model** with a fragmentation kernel which capture the physical properties.

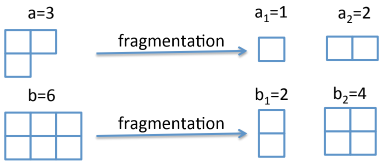
- **Fragmentation kernel for the avalanche**

$$x + y \to x, y$$

such that

$$\frac{\min(x,y)}{\max(x,y)} = cst.$$

- **Example** If $cst. = 1/2$



- **Fragmentation kernel for the avalanche model**

  There exists a function $\Phi : (0,\infty) \longrightarrow (0,\infty)$ such that

$$F(x,y) = \Phi\left(\frac{x}{y}\right), \ \forall\, x,y > 0, \qquad \text{and} \qquad \Phi(x) = \Phi\left(\frac{1}{x}\right), \ \forall\, x > 0.$$

- **Example:** For $r > 0$, define the function

$$\Phi^r(x) := \frac{1}{2}\delta_r(x) + \frac{1}{2}\delta_{1/r}(x), \quad \forall\, x > 0.$$

  In this situation $F^r(x,y) = \frac{1}{2}(\delta_r(\frac{x}{y}) + \delta_{1/r}(\frac{y}{x}))$.

- **Difficulty:** Φ^r is not continuos but we can construct a sequence of functions which approximates it and use the results before.

- **Branching kernel associated with F^r**:

$$N_x^{F^r} := \lambda_0(\beta x\delta_{\beta x} + (1-\beta)x\delta_{(1-\beta)x}),$$

where $\lambda_0 := \frac{\beta^2+(1-\beta)^2}{4}$ with $\beta := \frac{r}{1+r}$

- **Warning:** N^{F^r} is no more Markovian and has no density w.r.t. the Lebesgue measure.
Take $d_1 < \beta \leqslant 1/2$ and $d_{n+1}/d_n < \beta$ for all $n \geqslant 1$. For n fixed let $E_n = \bigcup_{k=1}^n E'_{k-1}$.
Define the kernel N_n^r on E_n as

$$N_n^r f := \sum_{k=1}^n \mathbb{1}_{E'_{k-1}} N^{F^r}(f\mathbb{1}_{E'_{k-1}}), \forall f \in bp\mathcal{B}(E_n).$$

- First order integral operator $\widetilde{N_n^r}$,

$$\mathcal{F}_n^r f(x) := \widetilde{N_n^r} f(x) = \int_{E_n} [f(y)-f(x)](N_n^r)_x(\mathrm{d}y), \forall f \in bp\mathcal{B}(E_n) \text{ and } x \in E_n.$$

$\mathcal{F}_n^r$ is the generator of a (continuous time) jump Markov process $X^{r,n} = (X_t^{r,n})_{t\geqslant 0}$. Its transition function is $P_t^{r,n} := e^{\mathcal{F}_n^r t}$, $t \geqslant 0$.

Probabilistic approach
Stochastic differential equation for the avalanche - discontinuous kernel

$$E_{\beta,x} := \{\, \beta^i(1-\beta)^j x : \; i,j \in \mathbb{N}\} \cup \{0\}, \; E_{\beta,x,n} := E_{\beta,x} \cap E_n$$

Theorem

(Beznea, Deaconu, Lupașcu-Stamate, MATCOM, 2018)
If $n \geqslant 1$ then E_n is absorbing w.r.t. the Markov process $X^{r,n}$ (in E_n) and

(i) *For every $\phi \in bp\mathcal{B}(E_n)$ and each probability ν on E_n, $(\phi(X_t^{r,n}) - \int_0^t \mathcal{F}_n^r \phi(X_s^{r,n})\mathrm{d}s, \; t \geqslant 0)$ is a martingale under $\mathbb{P}^\nu$, w.r.t. the natural filtration of $X^{r,n}$.*

(ii) *If $x \in E_n$, $n \geqslant 1$, then the following* SDE of fragmentation for avalanches, *with initial distribution δ_x, has a solution which is equal in law with $(X^{r,n}, \mathbb{P}^x)$:*

$$X_t = X_0 - \int_0^t \int_0^\infty p(\mathrm{d}\alpha, \mathrm{d}s) X_{\alpha-} \sum_{k=1}^{n} (1-\beta) \mathbb{1}_{[\frac{d_k}{\beta} \leqslant X_{\alpha-} < d_{k-1}, \; \frac{s}{\lambda_0 \beta} < X_{\alpha-}]}$$
$$+\beta \mathbb{1}_{[\frac{d_k}{1-\beta} \leqslant X_{\alpha-} < \frac{d_k}{\beta}, \; \frac{s}{\lambda_0(1-\beta)} < X_{\alpha-} \leqslant \frac{s}{\lambda_0 \beta}] \cup [\frac{d_k}{\beta} \leqslant X_{\alpha-} < d_{k-1}, \; \frac{s}{\lambda_0} < X_{\alpha-} \leqslant \frac{s}{\lambda_0 \beta}]},$$

where $p(\mathrm{d}\alpha, \mathrm{d}s)$ is a Poisson measure with intensity $q := \mathrm{d}\alpha\mathrm{d}s$.

(iii) *If $x \in E_n$ then $\mathbb{P}^x$-a.s. $X_t^{r,n} \in E_{\beta,x,n}$ for all $t \geqslant 0$.*

Probabilistic approach

Algorithm

- **Initialisation**: Sample the initial particle $X_0 \sim Q_0$
- **Step p**:
  - Sample a random variable $S_p \sim \mathrm{Exp}(\lambda_0)$
  - Let $T_p = T_{p-1} + S_p$
  - Set $X_t = X_{p-1}$ for all $t \in [T_{p-1}, T_p[$
  - Define

$$X_p = \begin{cases} \beta X_{p-1} & \text{with probability } \beta X_{p-1}, \\ (1-\beta) X_{p-1} & \text{with probability } (1-\beta) X_{p-1}, \\ X_{p-1} & \text{with probability } 1 - X_{p-1} \end{cases} \tag{0.1}$$

  - **Stop**: When $T_p > T$
  - **Outcome**: The approximation of the mass of the particle at time T, X_{p-1}

Numerical results

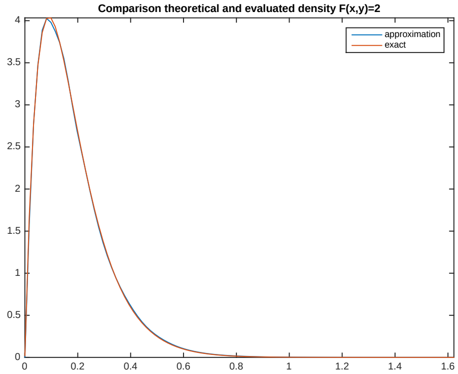
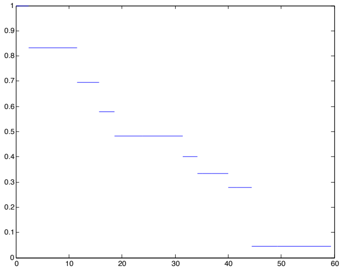
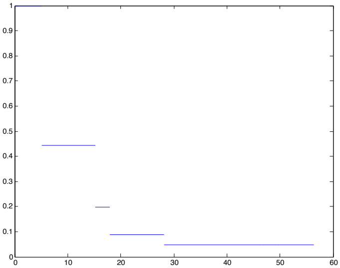


Figure : Comparison between the exact solution and the algorithm in the case $F(x, y) = 2$.

Numerical results



β=1/6, T=60



β=4/9, T=60

Figure : Path of the fragmentation process with the discontinuous kernel F^r and the size of the initial particle 1

Numerical results

Monte Carlo with 10^5 simulations

β	Mean $\widehat{I}_M$	Confidence interval
$\frac{1}{6}$	0.1566	0.0020
$\frac{1}{3}$	0.1356	0.0017
$\frac{4}{9}$	0.1350	0.0016

Numerical results

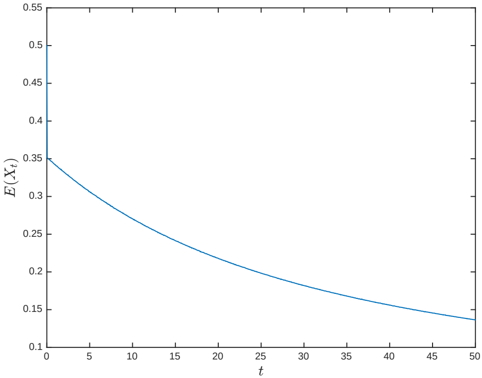


Figure : Time evolution of the empirical mean $t \mapsto \hat{I}_M(t)$ for $r = 0.2$, Monte Carlo parameter $M = 10^6$, and $t \in [0, 50]$.

Conclusion and perspectives

- **Go further**

  Construct a complex model of coagulation / fragmentation depending on the position and with physical kernels

  - Phase of snow accumulation (coagulation) before the avalanche begins
  - The rupture phase (fragmentation)
  - The phase of accumulation at the end of the avalanche (coagulation)

  - Evaluate and control the risk connected to the avalanche

- **Collaboration**

  - Geophysicians - Irstea Grenoble

- L. Beznea, M. Deaconu & O. Lupașcu-Stamate, *Numerical approach for stochastic differential equations of fragmentation; application to avalanches*, Accepted for publication, MATCOM, 2018.
- L. Beznea, M. Deaconu & O. Lupașcu, *Stochastic equation of fragmentation and branching processes related to avalanches*, Journal of Statistical Physics, 162 (2016), 824-841.
- L. Beznea, M. Deaconu & O. Lupașcu, *Branching processes for the fragmentation equation*, Stochastic Processes and Their Applications, 125 (2015), no. 5, 1861–1885.
- N. Fournier, J-S. Giet, *On small particles in coagulation-fragmentation equations*, J. Stat. Physics, 2003,
- M. Deaconu, N. Fournier, E. Tanré, *A pure jump Markov process associated with Smoluchowski's coagulation equation*, Ann. Probab., 2002.
- M. Deaconu, N. Fournier, *Probabilistic approach of discret and continuous coagulation equations with diffusion*, Stoch. Process. Appl., 2002.