Sur les hypothèses optimales dans deux résultats classiques d'analyse

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Then $u \geqslant 0$ in Ω.
Moreover, the following alternative holds:
(i) either $u \equiv 0$ in Ω;
(ii) or $u > 0$ in Ω.

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Let $a \in L^\infty(\Omega)$ be such that, for some $\alpha > 0$,

$$\int_\Omega (|\nabla u|^2 + a(x)u^2)dx \geqslant \alpha \|u\|^2_{H^1_0(\Omega)}, \quad \forall u \in H^1_0(\Omega).$$

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Assume that

$$\begin{cases} -\Delta u + a(x)u \geqslant 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

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Assume that

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Then either $u \equiv 0$ in Ω or $u > 0$ in Ω and $\dfrac{\partial u}{\partial \nu} < 0$ on $\partial\Omega$.

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Let $f : [0, \infty) \to \mathbb{R}$ be a continuous nondecreasing function such that $f(0) = 0$ (with $f > 0$ in $(0, \infty)$) and $\int_{0+} F(t)^{-1/2} dt = +\infty$, $F(t) := \int_0^t f(s) ds$.

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$$\begin{cases} -\Delta u + f(u) \geqslant 0 & \text{in } \Omega \\ u \geqslant 0 & \text{on } \partial\Omega. \end{cases}$$

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Then either $u \equiv 0$ in Ω or $u > 0$ in Ω.

Remark. The growth assumption $\int_{0+} F(t)^{-1/2} dt = +\infty$ implies that $f(t)$ must not be very large near t_0. This condition is satisfied if $f(t) \leqslant ct$ for a certain $c > 0$ and $0 < t < t_0$. But it is also satisfied by some f for which $f(t)/t$ is not bounded at 0, for instance $f(t) \leqslant t(\log t)^2$. This version of the maximum principle holds for **superlinear** nonlinearities like $f(t) = t^p$, $p > 1$.

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Let $\Omega \subset \mathbb{R}^N$ be a domain and consider the canonical divergence structure inequality

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The variational integral associated to (P) is

$$(V) \qquad I(u) = \int_\Omega \{\mathcal{G}(|\nabla u|) + F(u)\}dx, \quad F(u) := \int_0^u f(s)ds,$$

where $A(s) = \mathcal{G}'(s)/s$, $s > 0$.

Let

$$\Phi(s) := sA(s) \quad \text{for } s > 0; \quad \Phi(0) = 0$$

$$H(s) := s\Phi(s) - \int_0^s \Phi(t)dt, \quad s \geqslant 0.$$

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Examples. For the Laplace operator, the p-Laplace operator and the mean curvature operator, respectively, we have

$$H(s) = \frac{s^2}{2}, \quad H(s) = (p-1)\,\frac{s^p}{p}, \quad H(s) = 1 - \frac{1}{\sqrt{1+s^2}}\,.$$

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For the variational problem (V) we have

$$H(s) = s\mathcal{G}'(s) - \mathcal{G}(s).$$

By the *strong maximum principle* for (P) we mean the statement that if u is a non-negative classical solution of (P) with $u(x_0) = 0$ at some point $x_0 \in \Omega$, then $u \equiv 0$ in Ω.

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Strong Maximum Principle (P. Pucci and J. Serrin) *In order for the strong maximum principle to hold for problem (P) it is necessary and sufficient that either* $f \equiv 0$ *in* $[0, d]$ *with* $d > 0$, *or that* $f(s) > 0$ *for* $s \in (0, \delta)$ *and*

$$\int_{0+} \frac{ds}{H^{-1}(F(s))} = +\infty \, .$$

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(F2) $f(0) = 0, f > 0$ in $(0, \infty)$ and

$$\int_{0+} \frac{ds}{H^{-1}(F(s))} = +\infty\,.$$

Consider the problem

$$(P) \qquad -\mathrm{div}\,\{A(|\nabla u|)\nabla u\} + f(u) \geqslant 0 \quad \text{in } \Omega.$$

Theorem (P. Pucci, V.R., 2018) *Let $x_0 \in \partial\Omega$ and suppose that Ω satisfies the interior sphere condition at x_0.*
(i) If $u \in C^1(\overline{\Omega})$ satisfies $u(x_0) = 0$ and

$$\begin{cases} -\mathrm{div}\,\{A(|\nabla u|)\nabla u\} + f(u) \geqslant 0 & \text{in } \Omega \\ u > 0 & \text{in } \Omega \end{cases}$$

then $\frac{\partial u}{\partial \nu}(x_0) < 0$.

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then $\frac{\partial u}{\partial \nu}(x_0) < 0$.
(ii) Assume that $u \in C^1(\overline{\Omega})$ is a solution of

$$\begin{cases} -\mathrm{div}\,\{A(|\nabla u|)\nabla u\} + f(u) \geqslant 0 & \text{in } \Omega \\ u \geqslant 0 & \text{in } \Omega. \end{cases}$$

If u vanishes somewhere in Ω, then $u \equiv 0$ in Ω.

Sketch of the proof in a particular case. Let $B_\rho \subset \Omega$ be a ball such that ∂B_ρ is tangent to $\partial\Omega$ at x_0. Let $A_\rho = B_\rho \setminus B_{\rho/2}$ and consider the problem

$$(3) \qquad \begin{cases} -\Delta_p \phi + f(\phi) = 0 & \text{in} \quad A_\rho \\ \phi = 0 & \text{on} \quad \partial B_\rho \\ \phi = c & \text{on} \quad \partial B_{\rho/2}. \end{cases}$$

where $c = \min\{u(x); x \in \partial B_{\rho/2}\} > 0$.
Step 1. Problem (3) has a minimal solution $\phi \geqslant 0$ in A_ρ.

Step 2: ϕ is a radial function.
The function $\phi \circ R$ is still a nonnegative solution of (3), provided that R is any rotation of $\mathbb{R}^N$. By minimality of ϕ,

$$\phi(x) \leqslant \phi(R(x)) \qquad \text{for all} \qquad x \in A_\rho.$$

Applying this inequality at $y = R^{-1}(x)$ we deduce that ϕ is radial.
Thus, problem (3) reduces to:
(4)

$$\begin{cases} -(p-1)|\phi'|^{p-2}\phi'' - \frac{N-1}{r}|\phi'|^{p-2}\phi' + f(\phi) = 0 & \text{in} \quad (\frac{\rho}{2};\rho) \\ \phi(\frac{\rho}{2}) = c, \quad \phi(\rho) = 0. \end{cases}$$

Step 3: $\phi' < 0$. Since $\phi \geqslant 0$ in $(\rho/2, \rho)$ and $\phi(\rho) = 0$, then $\phi'(\rho) \leqslant 0$. We multiply by r^{N-1} in (4) and integrate on $[r, \rho]$, where $\rho/2 \leqslant r < \rho$. Then

$$-r^{N-1}|\phi'(r)|^{p-2}\phi'(r) = \int_r^{\rho} t^{N-1} f(\phi(t))dt \tag{5}$$

Since $f \geqslant 0$, relation (5) shows that the mapping

$$[\rho/2, \rho) \ni r \mapsto r^{N-1}|\phi'(r)|^{p-2}\phi'(r)$$

is nonpositive and nondecreasing in $[\rho/2, \rho)$. Moreover, since $f > 0$ on $(0, \infty)$ and $\phi(\rho/2) = c > 0$, relation (5) shows that $\phi'(\rho/2) < 0$.

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is nonpositive and nondecreasing in $[\rho/2, \rho)$. Moreover, since $f > 0$ on $(0, \infty)$ and $\phi(\rho/2) = c > 0$, relation (5) shows that $\phi'(\rho/2) < 0$. By contradiction, assume that $\exists\ r_0 \in (\rho/2, \rho]$ such that $\phi'(r_0) = 0$ and $\phi'(r) < 0$ for all $r \in (\rho/2, r_0]$. Applying (5) for $r = r_0$ we conclude that $\phi \equiv 0$ in $[r_0, \rho]$.

Next, we multiply (4) by ϕ' and integrate on $[r, r_0]$, where $\rho/2 < r < r_0$. It follows that

$$\text{(6)} \qquad \frac{p-1}{p}|\phi'(r)|^p - (N-1)\int_r^{r_0} \frac{|\phi'(t)|^p}{t}dt - F(\phi(r)) = 0$$

But

$$\begin{aligned} \int_r^{r_0} \frac{|\phi'(t)|^p}{t}dt &= \int_r^{r_0} t^{p-1-pN}|t^{N-1}\phi'(t)|^p dt \\ &\leqslant |r^{N-1}\phi'(r)|^p \int_r^{r_0} t^{p-1-pN}dt. \end{aligned}$$

Therefore

$$\lim_{r\to r_0^-} \frac{\int_r^{r_0} \frac{|\phi'(t)|^p}{t}dt}{|\phi'(r)|^p} = 0.$$

Returning to (6) we obtain

$$\frac{p-1}{p}|\phi'(r)|^p(1+o(1)) = F(\phi(r)) \qquad \text{as} \qquad r \to r_0^-. \tag{7}$$

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Fix $\epsilon > 0$. Then, by (7) and for all $r < r_0$ sufficiently close to r_0,

$$\left(\frac{p-1}{p}\right)^{1/p} \int_r^{r_0} \frac{-\phi'(t)}{(F(\phi(t)))^{\frac{1}{p}}} dt \leqslant (1+\epsilon)(r_0 - r).$$

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Since $\phi' < 0$ in $(\rho/2, r_0)$, the change of variable $s = \phi(t)$ yields

$$\left(\frac{p-1}{p}\right)^{1/p} \int_0^{\phi(r)} \frac{ds}{(F(s))^{1/p}} \leqslant (1+\epsilon)(r_0 - r) < +\infty,$$

contradiction. This implies that $\phi'(\rho) < 0$.

Step 4: Sign of the normal derivative.
By construction, $u \geqslant \phi$ in A_ρ. Therefore

$$-\frac{\partial u}{\partial \nu}(x_0) = \lim_{t \to 0^+} \frac{u((1-t)x_0)}{t} \geqslant \lim_{t \to 0^+} \frac{\phi((1-t)\rho)}{t} = -\phi^{'}(\rho) > 0.$$

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Step 5: Proof of Theorem concluded. We argue by contradiction and assume that u vanishes at some point in Ω but the set

$$\omega = \{x \in \Omega; u(x) > 0\} \neq \emptyset.$$

We consider a point $x_1 \in \omega$ which is closer to $\partial\omega$ than to $\partial\Omega$ and consider the largest ball $B \subset \omega$ centered at x_1. Then $u(x_0) = 0$ for some $x_0 \in \partial B$, while $u > 0$ in B. Since x_0 is an interior minimum point of ω, we have $\nabla u(x_0) = 0$. On the other hand, $\frac{\partial u}{\partial \nu}(x_0) < 0$, hence $\nabla u(x_0) \neq 0$. This contradicts the fact that x_0 is an interior minimum point of u.

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Consider the problem

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where $f \in C^1[0,\infty)$, f is non-decreasing in $(0,\infty)$, $f(0) = 0$ and $f(s) > 0$ for $s > 0$.

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Theorem (Keller and Osserman) *Under these hypotheses, problem (1) has a solution if and only if*

$$(KO) \qquad \int^{\infty} \frac{dt}{\sqrt{F(t)}} < \infty, \quad \text{where } F(t) = \int_0^t f(s)\, ds.$$

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Let

$$\Phi(\alpha) := \int_\alpha^\infty \frac{ds}{\sqrt{F(s) - F(\alpha)}} ,$$

where we let by convention $\Phi(\alpha) = +\infty$, whenever the integral is divergent or $F(s) = F(\alpha)$ on a set of positive measure.

We say that f satisfies the *weak Keller-Osserman condition* whenever

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Remark. In general, $\lim_{\alpha \to \infty} \Phi(\alpha)$ may not exist: for example, if $f(u) = u^2(1 + \cos u)$, then $\limsup_{\alpha \to \infty} \Phi(\alpha) = +\infty$. However, (sKO) still holds.

Theorem (Dumont, Dupaigne, Goubet, V.R.) *The following statements are equivalent:*

(a) f satisfies the Keller-Osserman condition (KO);

(b) f satisfies the weak Keller-Osserman condition (wKO);

(c) f satisfies the sharpened Keller-Osserman condition (sKO);

(d) there exists a ball $\Omega = B_R$ such that problem (1) admits at least one solution;

(e) given any smooth bounded domain Ω, problem (1) admits at least one solution.

Sketch of the proof. *Step 1: Mimimality Principle.* Let $f \in C(\mathbb{R})$ and $g \in C(\partial\Omega)$ and assume that there exist $\underline{u}, \overline{u} \in C(\overline{\Omega})$ such that $\underline{u} \leqslant \overline{u}$ and

$$(2) \qquad \begin{cases} \Delta \underline{u} \geqslant f(\underline{u}) \text{ in } \mathcal{D}'(\Omega), & (\text{resp. } \Delta \overline{u} \leqslant f(\overline{u}) \text{ in } \mathcal{D}'(\Omega)) \\ \underline{u} \leqslant g \text{ on } \partial\Omega, & (\text{resp. } \overline{u} \geqslant g \text{ on } \partial\Omega), \end{cases}$$

Consider the problem

$$(3) \qquad \begin{cases} \Delta u = f(u) \text{ in } \mathcal{D}'(\Omega), \\ u = g \text{ on } \partial\Omega. \end{cases}$$

Then there exists a unique solution $u \in C(\overline{\Omega})$ of (3) such that $\underline{u} \leqslant u$ and $u|_\omega \leqslant \overline{v}$ for any open subset ω of Ω and any function $\overline{v} \in C(\bar{\omega})$ satisfying

$$\begin{cases} \Delta \overline{v} \leqslant f(\overline{v}) \text{ in } \mathcal{D}'(\omega), \\ \overline{v} \geqslant \underline{u} \text{ in } \omega, \\ \overline{v} \geqslant u \text{ on } \partial\omega. \end{cases}$$

Step 2: Minimality Principle for blow-up solutions. Let $f \in C(\mathbb{R})$. Assume that there exist $\underline{u} \in C(\overline{\Omega})$ such that $\Delta \underline{u} \geqslant f(\underline{u})$ in $\mathcal{D}'(\Omega)$ and $v \in C(\Omega)$ such that $\Delta v \leqslant f(v)$ in $\mathcal{D}'(\Omega)$, $\lim_{x\to x_0} v(x) = +\infty$ for all $x_0 \in \partial\Omega$ and $v \geqslant \underline{u}$.
Then there exists a unique solution $u \in C(\Omega)$ of problem (1) such that $\underline{u} \leqslant u$ and $u|_\omega \leqslant \overline{v}$ for any open subset $\omega \subset \Omega$ and any $\overline{v} \in C(\omega)$ satisfying

$$\begin{cases} \Delta \overline{v} \leqslant f(\overline{v}) & \text{in } \mathcal{D}'(\omega), \\ \overline{v} \geqslant \underline{u} & \text{in } \omega, \\ \lim\limits_{x\to x_0} \overline{v}(x) = +\infty & \text{for all } x_0 \in \partial\omega. \end{cases}$$

We call u the minimal solution of (1) relative to $\underline{u}$.

Step 3: (wKO) $\Rightarrow$ existence of a blow-up boundary solution on some ball

Lemma

Let $\phi \in C^2(0,R)$ *be a nondecreasing function solving*

$$\phi'' + \frac{N-1}{r}\phi' = f(\phi) \quad \textit{in } (0,R).$$

Then, given $0 < r_1 < r_2 < R$*,*

$$\frac{1}{\sqrt{2}}\int_{\phi(r_1)}^{\phi(r_2)} \frac{1}{\sqrt{F(s) - F(\phi(r_1))}}\, ds \geqslant \frac{1}{N-2} r_1 \left(1 - \left(\frac{r_1}{r_2}\right)^{D-2}\right),$$

if $N \geqslant 3$ *and*

$$\frac{1}{\sqrt{2}}\int_{\phi(r_1)}^{\phi(r_2)} \frac{1}{\sqrt{F(s) - F(\phi(r_1))}}\, ds \geqslant r_1 \ln \frac{r_2}{r_1},$$

if $N = 2$.

Let B be a ball of radius R and assume that $\underline{u} \in C(\overline{B})$ is such that $\Delta \underline{u} \geqslant f(\underline{u})$ in B. Assume (wKO) holds for some $\alpha \geqslant \sup_B \underline{u}$ and let $\overline{u} = \alpha$. Let u be the minimal solution relative to $\underline{u}$ of

$$\begin{cases} \Delta u = f(u) \text{ in } B, \\ \quad u = \alpha \text{ on } \partial B. \end{cases}$$

Repeating the proof of the previous step, we conclude that u can be extended to a radially symmetric boundary blow-up solution on some ball $\tilde{B}$ of radius $\tilde{R} > R$, satisfying $u \geqslant \underline{u}$ in B.

Step 4: (sKO) $\Rightarrow$ existence of solutions on small balls

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Step 6: Conclusion

ABC concave-convex problem

A. Ambrosetti, H. Brezis, G. Cerami, Combined effects of concave and convex nonlinearities in some elliptic problems, *J. Funct. Anal.* **122** (1994), 519-543.

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Consider the concave-convex problem

$$\begin{cases} -\Delta u = \lambda u^q + u^p & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $0 < q < 1 < p$ and $\lambda > 0$.

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Authors' remark. "One can substitute u^q with any concave function that behaves like u^q near $u \equiv 0$. When $p < (N+2)/(N-2)$, u^p can be substituted by any superlinear function that behaves like u^p near $u = 0$ and near $u = +\infty$."

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Open problem 1. Do the results of this paper remain valid if $u^q \to f(u)$, $u^p \to g(u)$, f is concave, g is convex but they are **not** monotone?

Open Problems

The sublinear Brezis-Kamin problem

H. Brezis, S. Kamin, Sublinear elliptic equations in $\mathbb{R}^n$, *Manuscripta Math.* **74** (1992), 87-106.

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Consider the problem

$$(BK) \qquad -\Delta u = \rho(x) u^{\alpha} \quad \text{in } \mathbb{R}^n,\ n \geqslant 3,$$

with $0 < \alpha < 1$, $\rho \in L^{\infty}_{loc}(\mathbb{R}^n)$, $\rho \geqslant 0$, $\rho \not\equiv 0$.

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Theorem. *Problem (BK) has a nonnegative (positive) bounded solution if and only if the linear problem*

$$-\Delta U = \rho \quad \textit{in } \mathbb{R}^n$$

has a bounded solution.

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Open problem 2. Does this result remain true if u^{α} is replaced by a sublinear function with no monotonicity assumption?

Open Problems

The Pucci-Serrin dead core phenomenon

P. Pucci, J. Serrin, Dead cores and bursts for quasilinear singular elliptic equations, *SIAM J. Math. Anal.* **38** (2006), 259-278.

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Open problem 3. Does this result remain true for general non-monotone sublinear terms?

Joyeux Anniversaire,
Monsieur le Professeur Bucur!