# The generalized Hermite-Hadamard Inequality on Simplices

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# The Hermite-Hadamard Inequality

Ch. Hermite, 1883:

$$f : [a, b] \to \mathbb{R} \text{ convex} \Longrightarrow f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x \le \frac{f(a)+f(b)}{2}.$$

J. Hadamard, 1893: Left inequality.

Generalization for a Borel probability measure $\mu$ on $[a, b]$:

$$f(b_\mu) \le \int_a^b f(x)\,\mathrm{d}\mu(x) \le \frac{b-b_\mu}{b-a}f(a) + \frac{b_\mu - a}{b-a}f(b)\,,$$

where $b_\mu := \int_a^b x\,\mathrm{d}\mu(x)$.

Generalization to several dimensions:

G. Choquet, in the 1960s:

Let $E$ be a locally convex real Hausdorff space, $K \subset E$ a compact convex subset, $\mu$ a Borel probability measure on $K$. Then, for every continuous convex function $f : K \to \mathbb{R}$,

$$f(b_\mu) \le \int_K f(x)\,\mathrm{d}\mu(x)\,,$$

where $b_\mu \in K$, the barycenter of $\mu$, is the unique point such that $l(b_\mu) = \int_K l(x)\,\mathrm{d}\mu(x)$ for every continuous linear functional $l : E \to \mathbb{R}$.

G. Choquet, E. Bishop, K. de Leeuw, 1959 ( $K$ and $\mu$ as above, $K$ metrizable):

Let $\mathrm{Ext}(K)$ denote the set of all extreme points of $K$. Then, there exists a probability measure $\lambda$ on $K$ which has the same barycenter as $\mu$ and is concentrated on $\mathrm{Ext}(K)$, such that for every continuous convex function $f : K \to \mathbb{R}$,

$$\int_K f(x)\,\mathrm{d}\mu(x) \leq \int_{\mathrm{Ext}(K)} f(x)\,\mathrm{d}\lambda(x)\,.$$

**Corollary** in the case of simplices:

Let $\Delta \subset \mathbb{R}^n$ be a simplex with vertices $P_1, \ldots, P_{n+1}$, $\mu$ a nonzero Borel measure on $\Delta$ with barycenter $b_\mu := \frac{1}{\mu(\Delta)} \int_\Delta x\,\mathrm{d}\mu(x)$. Let $\lambda_1, \ldots, \lambda_{n+1}$ be nonnegative numbers such that

$$b_\mu = \sum_{j=1}^{n+1} \lambda_j P_j \quad \text{and} \quad \sum_{j=1}^{n+1} \lambda_j = 1\,.$$

If $f : \Delta \to \mathbb{R}$ is continuous and convex, then

$$f(b_\mu) \leq \frac{1}{\mu(\Delta)} \int_\Delta f(x)\,\mathrm{d}\mu(x) \leq \sum_{j=1}^{n+1} \lambda_j f(P_j)\,.$$

Proof: $f$ convex $\Longrightarrow$ $f$ has a supporting hyperplane at $b_\mu$, that is, there is a linear functional $h$ such that

$$f(x) \geq f(b_\mu) + h(x - b_\mu)$$

for every $x \in \Delta$. Therefore,

$$\frac{1}{\mu(\Delta)} \int_\Delta f(x)\,\mathrm{d}\mu(x) \geq f(b_\mu) + \frac{1}{\mu(\Delta)} \int_\Delta h(x - b_\mu)\,\mathrm{d}\mu(x)$$

$$= f(b_\mu) + h\left(\frac{1}{\mu(\Delta)} \int_\Delta (x - b_\mu)\,\mathrm{d}\mu(x)\right) = f(b_\mu) + h(0) = f(b_\mu).$$

For the right inequality let $x = \sum_{j=1}^{n+1} \alpha_j(x) P_j$ with continuous $\alpha_1, \ldots, \alpha_{n+1}$ such that $\sum_{j=1}^{n+1} \alpha_j(x) = 1$. Then,

$$b_\mu = \frac{1}{\mu(\Delta)} \int_\Delta x\,\mathrm{d}\mu(x) = \frac{1}{\mu(\Delta)} \sum_{j=1}^{n+1} \int_\Delta \alpha_j(x)\,\mathrm{d}\mu(x) \cdot P_j,$$

so $\lambda_j = \frac{1}{\mu(\Delta)} \int_\Delta \alpha_j(x)\,\mathrm{d}\mu(x)$ for $1 \leq j \leq n+1$. The Jensen inequality now supplies:

$$\frac{1}{\mu(\Delta)} \int_\Delta f(x)\,\mathrm{d}\mu(x) = \frac{1}{\mu(\Delta)} \int_\Delta f\left(\sum_{j=1}^{n+1} \alpha_j(x) P_j\right) \mathrm{d}\mu(x)$$

$$\leq \frac{1}{\mu(\Delta)} \int_\Delta \left(\sum_{j=1}^{n+1} \alpha_j(x) f(P_j)\right) \mathrm{d}\mu(x) = \sum_{j=1}^{n+1} \lambda_j f(P_j). \qquad \square$$

We will refer to these inequalities as (LHH) and (RHH).

## The Converse

**Theorem (2012)**

Let $D \subseteq \mathbb{R}^n$ be nonempty, open, convex, $\mu$ a Borel measure on $D$ such that $\mathrm{d}\mu(x) = p(x)\mathrm{d}x$, where $p : D \to [0, \infty)$ is continuous and $p^{-1}(\{0\})$ does not contain any nontrivial segment. Let $f : D \to \mathbb{R}$ be continuous.

1. If $f$ satisfies (LHH) for all simplices $\Delta \subset D$, then $f$ is convex.
2. If $f$ satisfies (RHH) for all simplices $\Delta \subset D$, then $f$ is convex.

For a proof see [F.-C. Mitroi & E. Symeonidis, The converse of the Hermite-Hadamard inequality on simplices, *Expo. Math. 30 (2012), 389-396*].

For $p(x) \equiv 1$ this was proved in 2008 [T. Trif, Characterizations of convex functions of a vector variable via Hermite-Hadamard's inequality, *J. Math. Inequal. 2 (1) (2008), 37-44*].

**Theorem (Generalization)**

The previous theorem remains valid, when the condition

> $p^{-1}(\{0\})$ *does not contain any nontrivial segment*

is replaced by the (weaker) condition

> *There is a dense subset* $S$ *of* $D$*, such that for every* $a, b \in S$*,* $p^{-1}(\{0\}) \cap [a,b]$ *(*$[a,b]$ *stands for the segment of endpoints* $a$ *and* $b$*) is a Lebesgue null set in* $[a,b]$*.*

Proof: Let $f$ satisfy (LHH) or (RHH), the same for all simplices in $D$. By continuity, it suffices to prove that $f$ is convex on every segment $[a,b]$, where $a, b \in S$.

By reductio ad absurdum let $a, b \in S$ and $\varepsilon \in (0,1)$ such that

$$f((1-\varepsilon)a + \varepsilon b) > (1-\varepsilon)f(a) + \varepsilon f(b)\,.$$

It follows that there is a subsegment $[c,d]$ of $[a,b]$, such that $f|_{[c,d]}$ is strictly concave.

Let $v_1, \ldots, v_{n-1} \in \mathbb{R}^n$ be such that $v_1, \ldots, v_{n-1}, d-c$ is a basis of $\mathbb{R}^n$. We assume that $v_1, \ldots, v_{n-1}$ are small enough, so that $c + v_i \in D$ for $1 \le i \le n-1$. Let

$$P_{i,m} := c + \frac{1}{m} v_i$$

for $m \in \mathbb{N}$ and $1 \le i \le n-1$, let $\Delta_m$ be the simplex of vertices $c, P_{1,m}, \ldots, P_{n-1,m}, d$.

For a continuous function $h : \Delta_m \to \mathbb{R}$ it then holds:

$$\frac{1}{\mu(\Delta_m)} \int_{\Delta_m} h(x)\, \mathrm{d}\mu(x) = \frac{\int_{\Delta_m} h(x) p(x) \mathrm{d}x}{\int_{\Delta_m} p(x) \mathrm{d}x}$$

$$\overset{m\to\infty}{\longrightarrow} \;(\ldots)\; \int_0^1 h(c + t(d-c)) \cdot \frac{p(c + t(d-c))(1-t)^{n-1} \mathrm{d}t}{\int_0^1 p(c + \tau(d-c))(1-\tau)^{n-1} \mathrm{d}\tau}$$

$$= \frac{1}{\nu([c,d])} \int_{[c,d]} h(x)\, \mathrm{d}\nu(x)\,,$$

where $\nu$ is the push-forward (image) by $s \mapsto c + s(d-c)$ of the measure

$$p(c + s(d-c))(1-s)^{n-1} \mathrm{d}s$$

on $[0,1]$. In particular, the barycenter $b_{\mu,m}$ of $\mu$ on $\Delta_m$ converges to the barycenter $b_\nu$ of $\nu$.

— Case 1, $f$ satisfies (LHH).

Letting $m \to \infty$ in

$$f(b_{\mu,m}) \le \frac{1}{\mu(\Delta_m)} \int_{\Delta_m} f(x) \, \mathrm{d}\mu(x)$$

we obtain

$$f(b_\nu) \le \frac{1}{\nu([c,d])} \int_{[c,d]} f(x) \, \mathrm{d}\nu(x) \, .$$

Since $f|_{[c,d]}$ is strictly concave, there exists a linear functional $h$ such that

$$f(x) \le f(b_\nu) + h(x - b_\nu)$$

for $x \in [c,d]$ and strictly on an open subset of $[c,d]$. This leads to a contradiction.

— Case 2, $f$ satisfies (RHH).

We start with the convex combinations

$$b_{\mu,m} = \left( \sum_{j=1}^{n-1} \lambda_j^{(m)} P_{j,m} \right) + \lambda_n^{(m)} c + \lambda_{n+1}^{(m)} d$$

for $m \in \mathbb{N}$, assume without restriction that all coefficients converge, and set

$$\lambda_j^\infty = \lim_{m\to\infty} \lambda_j^m \, , \qquad \text{for} \quad 1 \le j \le n+1 \, .$$

Then,

$$b_\nu = \left( \sum_{j=1}^{n-1} \lambda_j^\infty c \right) + \lambda_n^\infty c + \lambda_{n+1}^\infty d = (1 - \lambda_{n+1}^\infty) c + \lambda_{n+1}^\infty d \, .$$

By assumption,

$$\frac{1}{\mu(\Delta_m)} \int_{\Delta_m} f(x)\,\mathrm{d}\mu(x) \le \left( \sum_{j=1}^{n-1} \lambda_j^{(m)} f(P_{j,m}) \right) + \lambda_n^{(m)} f(c) + \lambda_{n+1}^{(m)} f(d)\,,$$

so in the limit,

$$\frac{1}{\nu([c,d])} \int_{[c,d]} f(x)\,\mathrm{d}\nu(x) \le \left( \sum_{j=1}^{n-1} \lambda_j^{\infty} f(c) \right) + \lambda_n^{\infty} f(c) + \lambda_{n+1}^{\infty} f(d)$$

$$= (1-\lambda_{n+1}^{\infty}) f(c) + \lambda_{n+1}^{\infty} f(d)\,.$$

On the other hand,

$$(1-\lambda_{n+1}^{\infty})c + \lambda_{n+1}^{\infty} d = b_\nu = \frac{1}{\nu([c,d])} \int_{[c,d]} x\,\mathrm{d}\nu(x)$$

$$= \int_0^1 [(1-s)c + sd] \cdot \frac{p((1-s)c+sd)(1-s)^{n-1}}{\int_0^1 p((1-\tau)c+\tau d)(1-\tau)^{n-1}\mathrm{d}\tau}\,\mathrm{d}s,$$

so

$$\lambda_{n+1}^{\infty} = \int_0^1 \frac{s \cdot p((1-s)c+sd)(1-s)^{n-1}}{\int_0^1 p((1-\tau)c+\tau d)(1-\tau)^{n-1}\mathrm{d}\tau}\,\mathrm{d}s\,.$$

From the strict concavity of $f$ and the conditions on $p$ it therefore follows that

$$\frac{1}{\nu([c,d])}\int_{[c,d]} f(x)\,\mathrm{d}\nu(x)$$
$$= \int_0^1 f((1-s)c+sd)\cdot\frac{p((1-s)c+sd)(1-s)^{n-1}}{\int_0^1 p((1-\tau)c+\tau d)(1-\tau)^{n-1}\mathrm{d}\tau}\,\mathrm{d}s$$

$$> \int_0^1 [(1-s)f(c)+sf(d)]\cdot\frac{p((1-s)c+sd)(1-s)^{n-1}}{\int_0^1 p((1-\tau)c+\tau d)(1-\tau)^{n-1}\mathrm{d}\tau}\,\mathrm{d}s$$

$$= (1-\lambda_{n+1}^{\infty})f(c)+\lambda_{n+1}^{\infty}f(d)\,,$$

which leads to a contradiction. □