Existence et approximation pour des modèles variationnels de rupture fragile

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(avec S. Conti (Bonn), G. Francfort (Paris-Nord), F. Iurlano (Paris 6), V. Crismale (CMAP))

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Outline

- Griffith's theory and the corresponding energy;
- Francfort-Marigo's variational approach;
- Weak formulation. The space *GSBD*. A Poincaré-Korn inequality;
- Existence of (weak/strong) minimizers;
- Other results.

Idea behind Griffith' theory

The idea is that the amount of energy dissipated for opening a crack of a given length ($d = 2$) or surface ($d = 3$), that is, the amount of elastic energy released by this opening, should be proportional to the length or surface.

In the classical theory of Griffith ($\sim 1920's$), the crack path is fixed and K grows along this path with a rule which accounts for this energy balance.

Idea of Francfort-Marigo (1998) is to turn this into a global variational formulation.

Griffith energy

$$\mathcal{E}(u,K) = \int_{\Omega\setminus K} \mathbb{C}e(u) : e(u)dx + \gamma\mathcal{H}^{d-1}(K)$$

with $\Omega \subset \mathbb{R}^d$ ($d = 2, 3$) is the reference configuration of an elastic object, $u : \Omega \to \mathbb{R}^d$ an infinitesimal displacement (*linearized elasticity*) and K a possible fracture set.

- γ = *Toughness*; (*ténacité*);
- $e(u)$ = symmetrized gradient $(Du + (Du)^T)/2$ is the *infinitesimal strain*; (*déformation* infinitésimale);
- $\mathbb{C}$ = "Hooke's [constitutive] law" which expresses the stress in term of the strain $e(u)$, as a linear function. Typically:

  $$\sigma = \mathbb{C}e(u) = \lambda(\mathrm{Tr}e(u))I + 2\mu e(u) = \lambda(\mathrm{div}\, u)I + 2\mu e(u)$$

(linear elastic isotropic material), λ, μ are the "Lamé constants".

Francfort-Marigo's variational approach

We assume that the domain Ω has a Dirichlet boundary Γ^D where an increasing load will be applied: $u^0(t) = tU^0$. Then, assuming at $t = t_k = k\delta t$, the displacement u^k and the fracture K^k have been found, one looks for (u^{k+1}, K^{k+1}) which minimize

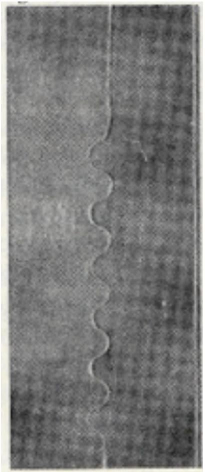
$$\min \left\{ \mathcal{E}(u, K) : u = t_{k+1} U^0 \text{ on } \Gamma^D, K \supseteq K^k \right\}.$$

(Inspired by Mumford-Shah's approach to image segmentation.)

Examples

Numerical experiments available at:
https://www.math.lsu.edu/~bourdin

Example: Oscillating crack (induced by a displacement due to a gradient of temperature) [Computed by Blaise Bourdin, LSU]



How is this justified?

Some results

- Dal Maso-Toader (2002): $\delta t \to 0$, scalar setting, $2D$, **connected** K; Then, if $K(t)$ is smooth, one finds back the motion originally defined by Griffith (1920).
- A.C. (2003) same in planar elasticity ($2D$, vectorial u);
- Francfort-Larsen: any dimension, scalar setting, "weak formulation": $u \in BV(\Omega)$, K replaced with J_u the "jump set" of u;
- Dal Maso-Francfort-Toader (2005): extension of the previous result to nonlinear elasticity;
- many subsequent directions (plasticity, damage...);
- Existence for weak minimizers of $\min_{u,K} \mathcal{E}(u, K)$ in $2D$: Friedrich-Solombrino 2017; weak minimizers are strong in $2D$: Conti-Focardi-Iurlano 2017 (to appear);
- Existence of evolutions ($\delta t \to 0$): Friedrich-Solombrino 2017 in $2D$.

More results

- A Poincaré-Korn inequality for displacements with small jump set (w. S. Conti, Bonn, G. Francfort, Paris);
- Phase-field approximation of problems with non-interpenetration (w. S. Conti, G. Francfort);
- Approximation of "GSBD" functions with regular functions $\rightarrow$ phase-field approximations (w. V. Crismale, Palaiseau);
- Weak minimizers are strong in any dimension (w. S. Conti, F. Iurlano, Paris);
- Existence of weak minimizers (w. S. Crismale).

Weak formulation

Minimizers of $\mathcal{E}(u, K)$ are usually intended in the following weak sense:

- $u \in BD(\Omega) = \{u \in L^1(\Omega; \mathbb{R}^d) : Eu := (Du + (Du)^\perp)/2 \in \mathcal{M}^1(\Omega; \mathbb{R}^{d\times d}))\}$;
- $u \in SBD(\Omega) = \{u \in BD(\Omega) : Eu = e(u)dx + (u^+ - u^-) \odot \nu_u \mathcal{H}^{d-1} \llcorner Ju\}$;
- $u \in SBD_2(\Omega) = \{u \in SBD(\Omega) : e(u) \in L^2, \mathcal{H}^{d-1}(J_u) < +\infty\}$;
- $\mathcal{E}(u, K)$ is replaced with

$$\mathcal{E}(u) := \int_\Omega \mathbb{C}e(u) : e(u)dx + \gamma \mathcal{H}^{d-1}(J_u).$$

- General (weak) minimizers should be in "$GSBD_2(\Omega)$" (Dal Maso, 2011)

Weak formulation (SBD)

That is, instead of a compact $(d-1)$-dimensional set K, the fracture is replaced with the jump set J_u of a displacement u with "bounded deformation", which is "only" a $\mathcal{H}^{d-1}$-*countably rectifiable* set.

The *jump set* J_u is defined for BD fields as the sets of points x where the blowups $u_\rho(y) := u(x+\rho y)$ converge, as $\rho \to 0$, to a function taking two different values $u^\pm$ in two half-spaces $\{\pm y \cdot \nu_u \geq 0\}$.

Weak formulation ($GSBD$)

Fundamental point: in a given direction ξ, $e(u)$ controls how much $u \cdot \xi$ varies ($(e(u)\xi) \cdot \xi = \xi \cdot \nabla(u \cdot \xi)$).

Definition [Dal Maso, 2011] $u \in GBD(\Omega)$: $u : \Omega \to \mathbb{R}^d$ and there exists λ nonnegative bounded measure with, for every $\xi \in \mathbb{S}^{d-1}$:

- for every smooth 1-Lipschitz truncation T, $|D_\xi T(u \cdot \xi)| \leq \lambda$,
- or equivalently, for $\mathcal{H}^{d-1} - a.e.\ y \in \xi^\perp$, $u_y^\xi : s \mapsto u(y + s\xi) \cdot \xi$ is in BV_{loc}

$$\int_{\xi^\perp} \left(\int_{\mathbb{R}} |D^a u_y^\xi| + \sum_{J_{u_y^\xi}} (|[u_y^\xi]| \wedge 1) \right) \leq \lambda$$

On can then define $e(u)$, Cu, J_u and the jump of u appropriately.

Weak formulation ($GSBD$)

Definitions $u \in GSBD(\Omega)$: $u \in GBD$ and in the previous definition, $u_y^\xi \in SBV_{loc}$. $u \in GSBD_p(\Omega)$ if in addition $e(u) \in L^p$, $\mathcal{H}^{d-1}(J_u) < \infty$.

Important points:

- Generalization of Ambrosio's lower semicontinuity theorem for SBD [*i.e.*, semi-continuity of **Griffith's energy**];
- $GSBD$ is defined by the properties of $1D$ slices. "Essentially", if $[x, y] \cap J_u = \emptyset$, then

$$(u(y)-u(x))\cdot(y-x) = \int_0^1 (e(u)(x+s(y-x))(y-x))\cdot(y-x)ds.$$

A Poincaré-Korn inequality...

Theorem [A.C., S. Conti, G. Francfort (IUMJ 2016)]: Let $\delta > 0$ $\theta > 0$, $Q = (-\delta, \delta)^d$, $Q' = (1+\theta)Q$, $Q'' = (1+2\theta)Q$, $p \in (1, \infty)$, $u \in GSBD_p(Q'')$. There exists $c(\theta, p, d) > 0$ such that
1. There exists $\omega \subset Q'$ and an affine function $a : \mathbb{R}^d \to \mathbb{R}^d$ with $e(a) = 0$ ($a = Ax + b$, $A + A^T = 0$) such that:

$$|\omega| \leq c\delta\mathcal{H}^{d-1}(J_u)$$

$$\|u - a\|_{L^{dp/(d-1)}(Q'\setminus\omega)} \leq c\delta^{1-1/d}\|e(u)\|_{L^p(Q'')}.$$

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2. Letting $v = u\chi_{Q' \setminus \omega} + a\chi_\omega$, and for ϕ a smooth symmetric mollifier with support in $B(0, \theta/2)$,

$$\int_Q |e(v * \phi_\delta) - e(u) * \phi_\delta|^p dx \leq c \left(\frac{\mathcal{H}^{d-1}(J_u)}{\delta^{d-1}} \right)^s \int_{Q''} |e(u)|^p dx$$

for some exponent $s = s(p, d) > 0$.

... for functions with small jump set

The point here is that when

$$\mathcal{H}^{d-1}(J_u) << \delta^{d-1}$$

then ω is small, v is close to u (but in ω), and $e(v * \phi_\delta)$ is close to $e(u) * \phi_\delta$.

Recent results of M. Friedrich (16/17) improve [C-Conti-Francfort 2016] (in particular the perimeter of ω can be controlled).

Existence of weak mimimizers

The problem has the form

$$\min_{u=u^0 \partial^D \Omega} \mathcal{E}(u) := \int_\Omega \mathbb{C}e(u) : e(u)dx + \gamma \mathcal{H}^{d-1}(J_u).$$

Main issues: very little control on u itself. Compactness?

Compactness in $GSBD$ (C.-Crismale 2018)

Theorem Let (u_n) with $\sup_n \mathcal{E}(u_n) < \infty$. Then up to a subsequence, $u_n \to u$ a.e. in $\Omega \setminus A$ for some set A and letting $u = 0$ on A, one has $u \in GSBD(\Omega)$ and

$$\mathcal{E}(u) \leq \liminf_n \mathcal{E}(u_n).$$

The proof is essentially the same as for Rellich's thm... The set A is where the function u_n go to ∞.

Proof

Step. 1: discretize. Choose a scale δ, cover the domain with cubes of size $\delta > 0$. In each cube pick $a_{Q,n}$ a rigid motion with

$$\int_{Q\setminus\omega_{Q,n}} |u_n - a_{Q,n}|^2 dx \leq C\delta^2 \int_Q |e(u_n)|^2$$

for some $|\omega_{Q,n}| \leq c\delta\mathcal{H}^{d-1}(J_{u_n} \cap Q)$.
Observe that if $\omega_n = \bigcup_Q \omega_{Q,n}$, then $|\omega_n| \leq c\delta\mathcal{H}^{d-1}(J_{u_n})$.

Step 2. Use the compactness of the discrete approximation. Indeed let $a_n = \sum_Q a_{Q,n}\chi_Q$. Then it lives in a finite dimensional space: hence we can extract a subsequence such that a_{n_k} converges. Or almost.

Proof

We can either compactify the space or, which is natural here, consider $\tanh(e \cdot a_{n_k})$ for a finite set of directions e.
Observe moreover that

$$\int |\tanh e \cdot u_n - \tanh e \cdot a_n|^2 dx \leq C\delta^2 \int |e(u_n)|^2 + C\delta \mathcal{H}^{d-1}(J_{u_n}).$$

Hence by a diagonal argument, we get convergence for $\tanh e \cdot u_n$.
It is easy to see that it is of the form $\tanh e \cdot u$ for some function u with components in $[-\infty, +\infty]$. The set where it is infinite is A.

Proof

Step 3. The rest is standard and based on slicing along lines (and essentially contained in [Dal Maso 2011], following older results on SBD and SBV functions [Ambrosio 89, Bellettini Coscia Dal Maso 98]).

Weak minimizers are strong (A.C, S. Conti, F. Iurlano 2017)

Theorem If u is a weak minimizer, then $\mathcal{H}^{d-1}(\overline{J_u} \setminus J_u \cap \Omega) = 0$.

- Extends a previous result in 2D of Conti-Focardi-Iurlano. Based on a previous result of De Giorgi, Carriero, Leaci for the Mumford-Shah functional (1989).
- Extension up to the boundary is in preparation with V. Crismale.

Other results

The rigidity theorem has allowed to prove many approximation results, with as consequences

- The Γ-convergence of phase-field approximations of the Griffith energy *with a constraint of non-interpenetration* (in 2D, with S. Conti, G. Francfort);
- The Γ-convergence and compactness for phase-field approximations of the Griffith energy without any artificial assumption (L^∞ bounds, coercivity), with V. Crismale.

Approximation (C.-Crismale 2017)

Theorem Let $u \in GSBD_p(\Omega)$. Then, there exists $u_n \to u$ a.e. such that

1. $e(u_n) \to e(u)$ in $L^p(\Omega)$, $\mathcal{H}^{d-1}(J_{u_n}) \to \mathcal{H}^{d-1}(J_u)$;
2. J_{u_n} is contained in a finite union of C^1 pieces, u_n is smooth up to its jump.

Slight improvement upon recent results of Iurlano ($p = 2$), Conti, Focardi, Iurlano (2017): we do **not** assume $u \in L^p$.

Remark

I must admit that most of the results (but the compactness) I have mentioned are based on almost the same construction, based on point **2.** of the rigidity theorem, which allows to smooth a function in cubes with little jump.

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