Distances between homotopy classes of $W^{s,p}(\mathbb{S}^N, \mathbb{S}^N)$

Itai Shafrir

Technion—I.I.T.

Based on:
Rubinstein-Sh 07, Levi-Sh 14, Brezis-Mironescu-Sh 16, Sh 18

Transitions de phase et équations non locales
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  The first is

$$\mathrm{dist}_{W^{s,p}}(\mathcal{E}_{d_1}, \mathcal{E}_{d_2}) = \inf_{u \in \mathcal{E}_{d_1}} \inf_{v \in \mathcal{E}_{d_2}} \|u - v\|_{W^{s,p}(S^N, S^N)},$$

  where $\|\cdot\|_{W^{s,p}}$ is a semi-norm.

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Theorem (Rubinstein-Sh 07)

$\mathrm{dist}_{W^{1,p}}(\mathcal{E}_{d_1}, \mathcal{E}_{d_2}) = \left(\frac{2}{\pi}\right) \Sigma_{W^{1,p}}(d_1 - d_2) = \frac{2^{1+1/p}}{\pi^{1-1/p}} |d_2 - d_1|$.

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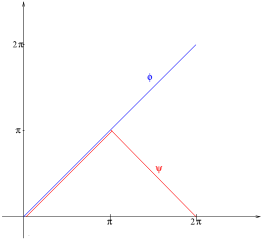

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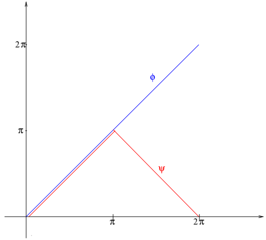

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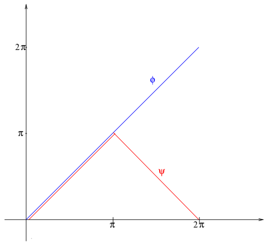

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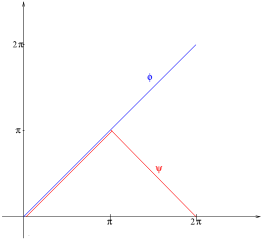

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Corollary

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Theorem (BMS 16)

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Proof uses again the fact that $|(u - v)(x)|$ does

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- $\exists s \in S^N$ s.t. $v(s) = -u(s) = \mathbf{N}$ (W.l.o.g.)
- Since $d_2 \neq 0$, $\exists t \in S^N$ s.t. $v(t) = \mathbf{S}$.
- Define $f : S^N \to \mathbb{R}$ by $f(x) = \{(v-u)(x)\}_N$.
- $f(s) = 2$, $f(t) \leq 0$, hence
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Distance between classes of $W^{1,p}(S^N, S^N)$ when $N \geq 2$

Theorem (Levi-Sh 14)

(i) For $p = N$: $\mathrm{dist}_{W^{1,N}}(\mathcal{E}_{d_1}, \mathcal{E}_{d_2}) = 0, \ \forall d_1, d_2$.
(ii) For $p > N$: $\mathrm{dist}_{W^{1,p}}(\mathcal{E}_{d_1}, \mathcal{E}_{d_2}) = c_{p,N}$ *for all* $d_1 \neq d_2$,
where $c_{p,N} > 0$ *is a universal (explicit) constant.*

(i) Follows similarly to the case $W^{1/p/p}(S^1, S^1)$.

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  $A_{p,N}$ is the sharp constant in the Sobolev-type inequality:

  $$\max_{S^N} g - \min_{S^N} g \leq A_{p,N} \|\nabla g\|_{L^p(S^N)} \text{ (Talenti, Cianchi).}$$

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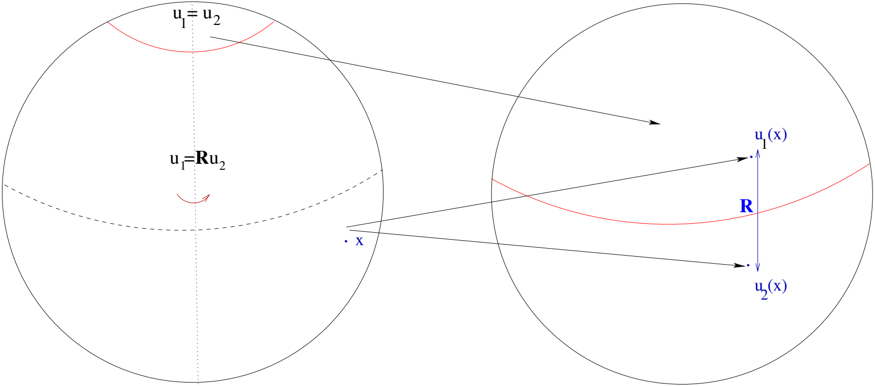
Degree-difference is due to rotations around the z-axis



An application to $W^{1,2}(\Omega, S^2)$ in dimension 3

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Answer(BMS 19?):

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(ii) If $[1 < p < \infty$ *and* $s > N/p]$ *or* $[p = 1$ *and* $s \geq N]$ *then*

$$C' \leq \text{dist}_{W^{s,p}}(\mathcal{E}_{d_1}, \mathcal{E}_{d_2}) \leq C \,,$$

For $C, C' > 0$ *depending on* s, p *and* N *(*$N \geq 2$ *is essential!).*

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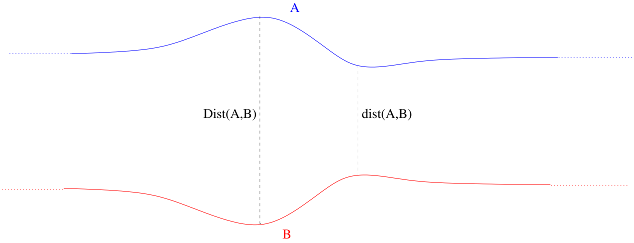

General results on $\text{Dist}_{W^{s,p}}$ in $W^{s,p}(S^N, S^N)$

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The answer is known only for $N = 1$, see below.

$\text{Dist}_{W^{1,1}(S^1,S^1)}$ and $\text{Dist}_{W^{1/p,p}(S^1,S^1)}$

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Since $\Sigma_{W^{1/p,p}}(d) \geq C_p|d|^{1/p}$ (Bourgain-Brezis-Mironescu), we get:

$\mathrm{Dist}_{W^{1,1}(S^1,S^1)}$ and $\mathrm{Dist}_{W^{1/p,p}(S^1,S^1)}$

Theorem (BMS16)

$\mathrm{Dist}_{W^{1,1}(S^1,S^1)}(\mathcal{E}_{d_1},\mathcal{E}_{d_2}) = 2\pi|d_1 - d_2|$ $(= \Sigma_{W^{1,1}}(d_2 - d_1))$

(recall $\mathrm{dist}_{W^{1,1}(S^1,S^1)}(\mathcal{E}_{d_1},\mathcal{E}_{d_2}) = 4|d_1 - d_2|$*)*

Theorem (Sh18)

For $p > 1$, $\mathrm{Dist}_{W^{1/p,p}(S^1,S^1)}(\mathcal{E}_{d_1},\mathcal{E}_{d_2}) = \Sigma_{W^{1/p,p}}(d_2 - d_1)$

(recall $\mathrm{dist}_{W^{1/p,p}(S^1,S^1)}(\mathcal{E}_{d_1},\mathcal{E}_{d_2}) = 0$*)*

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$$C_p|d_2 - d_1|^{1/p} \leq \mathrm{Dist}_{W^{1/p,p}(S^1,S^1)}(\mathcal{E}_{d_1},\mathcal{E}_{d_2}) \leq C'_p|d_2 - d_1|^{1/p}.$$

The case $W^{1/2,2}(S^1, S^1)$

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Corollary

$\mathrm{Dist}_{W^{1/2,2}(S^1,S^1)}(\mathcal{E}_{d_1}, \mathcal{E}_{d_2}) = 2\pi|d_2 - d_1|^{1/2}.$

The lower bound for $\mathrm{Dist}_{W^{1,1}}$: preventing large bubbles

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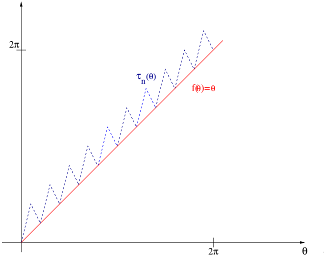

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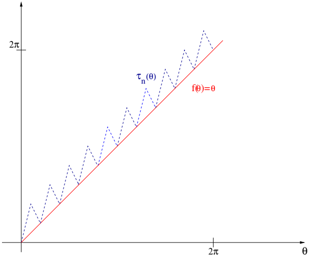

Proposition (BMS 18)

$\forall d_1 \neq d_2$, $\forall u \in \mathcal{E}_{d_1}$ *let* $u_n = T_n \circ u \in \mathcal{E}_{d_1}$. *Then*

$\lim_{n\to\infty} \inf_{v \in \mathcal{E}_{d_2}} \int_{S^1} |(u_n - v)'| = 2\pi|d_2 - d_1|$

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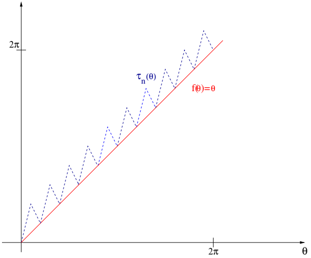

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$(\Longrightarrow \mathrm{Dist}_{W^{1,1}}(\mathcal{E}_{d_1}, \mathcal{E}_{d_2}) \geq 2\pi|d_2 - d_1|)$

Thank you for your attention!