Deux problèmes variationnels liés aux opérateurs en forme divergence avec symbole à croissance rapide

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Definition of rapidly growing operators in divergence form

• $\Phi : \mathbb{R} \to \mathbb{R}^+$ is called N-function if Φ is even, convex; $\Phi(t) = 0$ iff $t = 0$; $\lim_{t\to 0} t^{-1}\Phi(t) = 0$ and $\lim_{t\to\infty} t^{-1}\Phi(t) = \infty$;

• assume Φ is C^1 and let $\varphi := \Phi'$ and define

$$\varphi^- := \inf_{s>0} \frac{s\varphi(s)}{\Phi(s)} \quad \text{and} \quad \varphi^+ := \sup_{s>0} \frac{s\varphi(s)}{\Phi(s)};$$

If

$$1 < \varphi^- \quad \text{and} \quad \varphi^+ = \infty\,,$$

we call the operator

$$\operatorname{div}\left(\frac{\varphi(|\nabla u|)}{|\nabla u|}\nabla u\right),$$

a rapidly growing operator in divergence form.

Examples: For each $p \in (1, \infty)$ rapidly growing operators in divergence form can be built with the aid of the following N-functions

$$\Phi(t) = \exp(|t|^p) - 1;$$

$$\Phi(t) = \sinh(|t|^p) = \frac{\exp(|t|^p) - \exp(|t|^{-p})}{2};$$

$$\Phi(t) = \cosh(|t|^p) - 1 = \frac{\exp(|t|^p) + \exp(|t|^{-p})}{2} - 1.$$

For

$$\Phi(t) = \exp(|t|^p) - 1\,,$$

we have

$$\varphi(t) = p\exp(|t|^p)|t|^{p-2}t\,.$$

A solution u of the equation

$$\operatorname{div}\left(\frac{\varphi(|\nabla u|)}{|\nabla u|}\nabla u\right) = 0\,,$$

will be called an exponentially harmonic function.

Outline:

- I. Exponentially harmonic maps and related problems;

- II. Minimization problems for inhomogeneous Rayleigh quotients.

In the rest of the presentation let $\Omega \subset \mathbb{R}^D$ $(D \geq 1)$

be an open, bounded, convex domain with smooth

boundary, $\partial\Omega$.

I. Exponentially harmonic maps and related problems

Let $g \in C^1(\Omega) \cap C(\overline{\Omega})$ be given. Consider the problem

$$\begin{cases} -\Delta u - 2\Delta_\infty u = 0 & \text{in} \quad \Omega \\ u = g & \text{on} \quad \partial\Omega. \end{cases} \tag{1}$$

Theorem. (**Lieberman**, 1992)

Define

$$W_g^{1,1}(\Omega) := \{w \in W^{1,1}(\Omega) : \ w = g \text{ on } \partial\Omega\} .$$

If there exists $v \in W_g^{1,1}(\Omega)$ s.t.

$$\int_\Omega e^{|\nabla v|^2} \, dx \leq \int_\Omega e^{|\nabla w|^2} \, dx, \quad \forall \, w \in W_g^{1,1}(\Omega) ,$$

then v is a classical solution of problem (1).

Note that the problem (1) is equivalent to

$$\begin{cases} -\text{div}(e^{|\nabla u|^2}\nabla u) = 0 & \text{in} \quad \Omega \\ u = g & \text{on} \quad \partial\Omega. \end{cases} \tag{2}$$

The integral from the above **Theorem**, i.e.,

$$u \longrightarrow \int_\Omega e^{|\nabla u|^2}\, dx,$$

plays the role of the Euler-Lagrange functional for problem (2).

Lieberman refers to **Duc & Eells** (1991) for alternative methods showing that a minimizer of the above integral is a weak solution of (2). Here we propose a different approach that is more natural from the point of view of the calculus of variations.

A suitable function space for analyzing problem (1) from a variational point of view is an Orlicz-Sobolev space, denoted by

$$W^{1,\Phi}(\Omega)\,,$$

corresponding to the N-function

$$\Phi:\mathbb{R}\to\mathbb{R},\ \ \Phi(t):=e^{t^2}-1\,,$$

whose derivative

$$\varphi(t):=2te^{t^2}\,,$$

is an odd, increasing homeomorphism from $\mathbb{R}$ onto $\mathbb{R}$.

- $W^{1,\Phi}(\Omega)$ is not reflexive but it is a weakly* closed subset of the dual of a separable space;

- $W^{1,\Phi}(\Omega)\subset W^{1,q}(\Omega)$, for all $q\in(1,\infty)$.

The natural Euler-Lagrange functional corresponding to problem (2) is

$I : W^{1,\Phi}(\Omega) \to \mathbb{R}$ defined by

$$I(u) := \int_\Omega \Phi(|\nabla u(x)|)\, dx = \int_\Omega \left(e^{|\nabla u(x)|^2} - 1 \right)\, dx\,,$$

or its equivalent form,

$$J(u) := \int_\Omega e^{|\nabla u(x)|^2}\, dx\,.$$

To account for the boundary condition in (2), we need to consider the restriction of I to a proper subset of the Orlicz-Sobolev space $W^{1,\Phi}(\Omega)$, namely

$$X := W^{1,\Phi}(\Omega) \cap \left(\cap_{q>1} W_g^{1,q}(\Omega) \right),$$

where, for each $q \in (1, \infty)$,

$$W_g^{1,q}(\Omega) := \{ u \in W^{1,q}(\Omega) \mid u = g \text{ on } \partial\Omega \},$$

is a closed and convex subset of the Sobolev space $W^{1,q}(\Omega)$.

- X is NOT a linear subspace of $W^{1,\Phi}(\Omega)$ (except when $g = 0$)

- X is a convex and closed subset of $W^{1,\Phi}(\Omega)$.

- I is convex and weakly* lower semicontinuous, but $I \notin C^1(W^{1,\Phi}(\Omega), \mathbb{R})$.

Our main result is the following theorem.

Theorem 1. *(Bocea & M, 2018) The functional I admits a unique minimizer in X, and this is the unique classical solution of the problem (1).*

Proof of Theorem 1 (existence of a minimizer)

For each integer $n \geq 2$, consider the family of problems

$$\begin{cases} -\sum_{k=1}^{n} \frac{1}{(k-1)!} \Delta_{2k} u = 0 & \text{in } \Omega \\ u = g & \text{on } \partial\Omega, \end{cases} \tag{3}$$

where $\Delta_{2k} u := \text{div}(|\nabla u|^{2k-2} \nabla u)$ stands for the $2k$-Laplace operator.

A weak solution of the problem (3) is a function $u \in W_g^{1,2n}(\Omega)$ such that

$$\sum_{k=1}^{n} \frac{1}{(k-1)!} \int_\Omega |\nabla u|^{2k-2} \nabla u \nabla \varphi \, dx = 0 \quad \forall \ \varphi \in W_0^{1,2n}(\Omega) \,. \tag{4}$$

Lemma 1. *For each integer* $n \geq 2$, *the problem (3) admits a unique weak solution.*

Lemma 2. *For each integer* $n \geq 2$, *let* $u_n \in W_g^{1,2n}(\Omega)$ *be a weak solution of the problem (3). Then there exists a subsequence of* $\{u_n\}$ *that converges uniformly in* Ω *to some function* $u_\infty \in C(\overline{\Omega})$.

A Γ-convergence result

Lemma 3. *Define* $J_n : L^1(\Omega) \to \overline{\mathbb{R}}$ $(n \geq 2)$ *and* $J_\infty : L^1(\Omega) \to \overline{\mathbb{R}}$ *by*

$$J_n(u) := \begin{cases} \displaystyle\sum_{k=1}^{n} \frac{1}{k!} \int_\Omega |\nabla u|^{2k}\, dx & \text{if } u \in W_g^{1,2n}(\Omega) \\ +\infty & \text{if } u \in L^1(\Omega) \setminus W_g^{1,2n}(\Omega)\,, \end{cases}$$

and

$$J_\infty(u) := \begin{cases} \displaystyle\int_\Omega \left(e^{|\nabla u|^2} - 1 \right) dx & \text{if } u \in X \\ +\infty & \text{if } u \in L^1(\Omega) \setminus X, \end{cases}$$

respectively.

$$\Gamma(L^1(\Omega)) - \lim_{n \to \infty} J_n = J_\infty\,.$$

Proposition 1. *Let Y be a topological space satisfying the first axiom of countability, and assume that the sequence $\{F_n\}$ of functionals $F_n : Y \to \overline{\mathbb{R}}$ $\Gamma - converge$ to $F : Y \to \overline{\mathbb{R}}$. Let z_n be a minimizer for F_n. If $z_n \to z$ in Y, then z is a minimizer of F, and $F(z) = \liminf_{n\to\infty} F_n(z_n)$.*

Taking into account Lemmas 1, 3, and 2, we can now apply Proposition 1 with $Y = L^1(\Omega)$, $F_n = J_n$, $F = J_\infty$, $z_n = u_n$ to deduce the existence of a minimizer for J_∞ and consequently a minimizer for I on X.

Connections with infinity harmonic and harmonic functions

For each real parameter $\varepsilon > 0$, consider the problem

$$\begin{cases} -\varepsilon\Delta u - 2\Delta_\infty u = 0 & \text{in} \quad \Omega \\ u = g & \text{on} \quad \partial\Omega. \end{cases} \tag{5}$$

This problem is equivalent to

$$\begin{cases} -\text{div}\left(e^{\varepsilon^{-1}|\nabla u|^2}\nabla u\right) = 0 & \text{in} \quad \Omega \\ u = g & \text{on} \quad \partial\Omega, \end{cases} \tag{6}$$

which can be analyzed using similar arguments to those used for problem (2). The Euler-Lagrange functional for problem (6) is $J_\varepsilon : W^{1,\Phi}(\Omega) \to \mathbb{R}$ given by

$$J_\varepsilon(u) := \int_\Omega e^{\varepsilon^{-1}|\nabla u|^2}\, dx.$$

Arguing as for problem (2), it can be shown that $u_\varepsilon \in X$ is the unique minimizer of J_ε in X.

Thus, for each $\varepsilon > 0$ problem (5) has a unique classical solution $u_\varepsilon \in X$.

The limit of u_ε as $\varepsilon \to 0^+$

Let u be the (unique; by **Jensen**, 1993) viscosity solution of the problem

$$\begin{cases} -\Delta_\infty u = 0 & \text{in} \quad \Omega \\ u = g & \text{on} \quad \partial\Omega. \end{cases} \tag{7}$$

Theorem (**Evans & Yu**, 2007) *For every integer $m > D$, the sequence $\{u_\varepsilon\}$ is bounded in $W^{1,2m}(\Omega)$. Moreover, u_ε converges uniformly to u in $\overline{\Omega}$ as $\varepsilon \to 0^+$.*

The limit of u_ε as $\varepsilon \to \infty$

Let v be the unique solution of the problem

$$\begin{cases} -\Delta v = 0 & \text{in} \quad \Omega \\ v = g & \text{on} \quad \partial\Omega. \end{cases} \tag{8}$$

Theorem 2. *(Bocea & M, 2018) The sequence $\{u_\varepsilon\}$ converges to v in $W^{1,4}(\Omega)$ as $\varepsilon \to \infty$. Consequently, if $D \leq 3$ then u_ε converges to v uniformly in $\overline{\Omega}$ as $\varepsilon \to \infty$.*

II. Minimization problems for inhomogeneous Rayleigh quotients

Let $p \in (1, \infty)$.

Our goal is to analyze the minimization problem

$$\inf_{u \in X_0 \setminus \{0\}} \frac{\int_\Omega (\exp(|\nabla u|^p) - 1)\, dx}{\int_\Omega (\exp(|u|^p) - 1)\, dx},$$

where $X_0 = W^{1,\infty}(\Omega) \cap (\cap_{q>1} W_0^{1,q}(\Omega))$ and to discuss some related problems.

Motivation

Given an N-function $\Phi : \mathbb{R} \to \mathbb{R}^+$ (i.e., Φ is even, convex; $\Phi(t) = 0$ iff $t = 0$; $\lim_{t\to 0} t^{-1}\Phi(t) = 0$ and $\lim_{t\to\infty} t^{-1}\Phi(t) = \infty$), consider the minimization problem

$$\lambda_\Phi := \inf_{u \in W_0^{1,\Phi}(\Omega)\setminus\{0\}} \frac{\int_\Omega \Phi(|\nabla u|)\, dx}{\int_\Omega \Phi(|u|)\, dx}, \tag{9}$$

where $W_0^{1,\Phi}(\Omega)$ is a function space (an Orlicz-Sobolev type space) built with the aid of the N-function Φ.

A particular case

• If $\Phi(t) = \Phi_p(t) := |t|^p$ with $p \in (1, \infty)$, (9) has been investigated in connection with the *eigenvalue problem for the p-Laplacian*

$$\begin{cases} -\mathrm{div}(|\nabla u|^{p-2}\nabla u) = \lambda |u|^{p-2}u & \text{for } \ x \in \Omega \\ u = 0 & \text{for } \ x \in \partial\Omega. \end{cases} \tag{10}$$

In this case, λ_{Φ_p} is denoted by $\lambda_1(p)$ (or $\lambda_1(p;\Omega)$).

More precisely, in this case

$$\lambda_1(p) = \inf_{u \in W_0^{1,p}(\Omega)\setminus\{0\}} \frac{\displaystyle\int_\Omega |\nabla u|^p \, dx}{\displaystyle\int_\Omega |u|^p \, dx}.$$

Properties of $\lambda_1(p;\Omega)$

Fix $p \in (1,\infty)$**.**

- If $\Omega_1 \subset \Omega_2$ then $\lambda_1(p;\Omega_1) \geq \lambda_1(p;\Omega_2)$.

- $\lambda_1(p;B_R) = R^{-p}\lambda_1(p;B_1)$, for each $R > 0$, $p \in (1,\infty)$ and B_R ball of radius R.

Fix $\Omega \subset \mathbb{R}^D$**.**

- The map $\lambda_1(\cdot;\Omega) : (1,\infty) \to (0,\infty)$ is continuous (**Lindqvist** 1993, **Huang** 1997).

• If

$$\Lambda_\infty(\Omega) := \inf_{\varphi \in X_0 \setminus \{0\}} \frac{\|\nabla \varphi\|_{L^\infty(\Omega)}}{\|\varphi\|_{L^\infty(\Omega)}} \tag{11}$$

it is known that $\Lambda_\infty(\Omega) = \|\delta\|^{-1}_{L^\infty(\Omega)}$ (**Juutinen, Lindqvist, & Manfredi**, 1999; **Fukagai, Ito & Narukawa** 1999) and

$$\lim_{p\to\infty} \sqrt[p]{\lambda_1(p;\Omega)} = \Lambda_\infty(\Omega) = \|\delta\|^{-1}_{L^\infty(\Omega)},$$

here $\delta : \Omega \to [0,\infty)$ stands for the distance function to $\partial\Omega$, namely

$$\delta(x) := \mathrm{dist}(x, \partial\Omega) = \inf_{y\in\partial\Omega} |x-y|, \quad \forall\, x \in \Omega\,.$$

It is also well-known that $\Lambda_\infty(\Omega) = R_\Omega^{-1}$, where R_Ω stands for the radius of the largest ball contained in Ω.

• The map $\lambda_1(\cdot;\Omega) : (1,\infty) \to (0,\infty)$ is **not monotone** for general domains Ω (**Bobkov & Tanaka**, 2015).

Nevertheless, recall if $\partial\Omega$ is sufficiently smooth, then (**Lindqvist** 1993) function

$$(1,\infty) \ni p \longrightarrow p\sqrt[p]{\lambda_1(p;\Omega)}\,,$$

is increasing, i.e.

$$t\sqrt[t]{\lambda_1(t;\Omega)} \le s\sqrt[s]{\lambda_1(s;\Omega)}, \quad \forall\, 1 < t < s < \infty\,.$$

• The following estimate holds true (**Kajikia** 2015)

$$\frac{p-1}{R_\Omega^p}\left(\frac{\pi/p}{\sin(\pi/p)}\right)^p \leq \lambda_1(p;\Omega) \leq \lambda_1(p;B_{R_\Omega}) \,.$$

This result is sharp in the sense that we know that the left hand side represents the infimum of $\lambda_1(p;\Omega)$ over all convex, bounded domains that have the radius of the largest inscribed ball equal to R_Ω.

More precisely, if we define, for $T>0$, $\Omega(T) := (-R_\Omega, R_\Omega) \times B_T^{D-1}(0)$, where $B_T^{D-1}(0)$ stands for the ball in $\mathbb{R}^{D-1}$ centered at the origin and with radius T, then

$$\lim_{T\to\infty} \lambda_1(p;\Omega(T)) = \frac{p-1}{R_\Omega^p}\left(\frac{\pi/p}{\sin(\pi/p)}\right)^p.$$

Back to the general case

• Denote $\varphi(t) := \Phi'(t)$. If

$$1 \leq \varphi^- := \inf_{s>0} \frac{s\varphi(s)}{\Phi(s)} \leq \frac{t\varphi(t)}{\Phi(t)} \leq \sup_{s>0} \frac{s\varphi(s)}{\Phi(s)} =: \varphi^+ < \infty \quad \forall\, t > 0\,,$$

then it can be showed that $\lambda_\Phi > 0$ (**Gossez**, 1974; **Bocea & M**, 2012).

• If $1 < \varphi^- \leq \varphi^+ \leq \infty$, it was established the fact that for each $r > 0$ the minimization problem

$$\Lambda_r := \inf \left\{ \int_\Omega \Phi(|\nabla u|)\, dx :\; u \in W_0^{1,\Phi}(\Omega),\; \int_\Omega \Phi(|u|)\, dx = r \right\},$$

has a solution u_r (**Mustonen & Tienari**, 1999). Here, $W_0^{1,\Phi}(\Omega)$ stands for the corresponding Orlicz-Sobolev space associated to the N-function Φ. Moreover, u_r is a weak solution for the corresponding Euler-Lagrange equation, i.e.

$$\begin{cases} -\mathrm{div}\left(\frac{\varphi(|\nabla u|)}{|\nabla u|}\nabla u\right) = \lambda_r \frac{\varphi(|u|)}{|u|} u & \text{for} \quad x \in \Omega \\ u = 0 & \text{for} \quad x \in \partial\Omega, \end{cases}$$

for some $\lambda_r > 0$. Note that the case $\varphi^+ = +\infty$ is allowed.

When $\varphi^{+} = +\infty$, **Garcia-Huidobro, Le, Manasevich, & Schmitt** (1999) observed that we can have $\inf_{r>0} \Lambda_r \geq 0$, and thus $\lambda_\Phi = 0$ may, in fact, occur.

Back to our problem

Consider the minimization problem (9) when

$$\Phi(t) = \exp(|t|^p) - 1\,,$$

with $p \in (1,\infty)$, and when $W_0^{1,\Phi}(\Omega)$ is replaced by

$$X_0 = W^{1,\infty}(\Omega) \cap (\cap_{q>1} W_0^{1,q}(\Omega))\,,$$

i.e.,

$$\Lambda_1(p) := \inf_{u \in X_0 \setminus \{0\}} \frac{\displaystyle\int_\Omega (\exp(|\nabla u|^p) - 1)\, dx}{\displaystyle\int_\Omega (\exp(|u|^p) - 1)\, dx}\,. \tag{12}$$

The distance function to $\partial\Omega$, namely $\delta : \Omega \to [0,\infty)$ defined by $\delta(x) := \text{dist}(x, \partial\Omega) = \inf_{y \in \partial\Omega} |x - y|$, plays a key role in establishing the positivity of $\Lambda_1(p)$.

Our results

First, we note that the following Hardy-type inequality related to the study of (12) can be established.

Theorem 3. *(***Bocea & M***, 2018) For each* $p \in (1, \infty)$ *we have*

$$\inf_{u \in C_0^\infty(\Omega) \setminus \{0\}} \frac{\int_\Omega (\exp(|\nabla u|^p) - 1)\, dx}{\int_\Omega \left(\exp\left(\left|\frac{u}{\delta}\right|^p\right) - 1\right)\, dx} = \left(\frac{p-1}{p}\right)^p. \tag{13}$$

Remark.

By Theorem 3 we get that if $\|\delta\|_{L^\infty(\Omega)} \leq 1$ then

$$\inf_{u \in C_0^\infty(\Omega) \setminus \{0\}} \frac{\int_\Omega (\exp(|\nabla u|^p) - 1)\, dx}{\int_\Omega (\exp(|u|^p) - 1)\, dx} > 0\,.$$

Indeed, if $\|\delta\|_{L^\infty(\Omega)} \leq 1$ then

$$\exp\left(\left|\frac{v(x)}{\delta(x)}\right|^p\right) - 1 \leq \exp(|v(x)|^p) - 1, \quad \text{a.e. } x \in \Omega\ \forall\, v \in C_0^\infty(\Omega),\ \forall\, p > 1\,.$$

Thus, for each $v \in C_0^\infty(\Omega) \setminus \{0\}$ we have

$$\left(\frac{p-1}{p}\right)^p \leq \frac{\int_\Omega (\exp(|\nabla v|^p) - 1)\, dx}{\int_\Omega (\exp(|\delta|^{-p}|v|^p) - 1)\, dx} \leq \frac{\int_\Omega (\exp(|\nabla v|^p) - 1)\, dx}{\int_\Omega (\exp(|v|^p) - 1)\, dx}\,.$$

Next, we point out the main result on problem (12).

Theorem 4. *(***Bocea & M***, 2018)*

- *if* $\|\delta\|_{L^\infty(\Omega)} > 1$ *then* $\Lambda_1(p) = 0$*;*

- *if* $\|\delta\|_{L^\infty(\Omega)} \in (0,1]$ *then* $\Lambda_1(p) > 0$*;*

- *if* $\|\delta\|_{L^\infty(\Omega)} \in (0,1]$ *then* $\inf\{\lambda_1(kp) : \ k \in \mathbb{N}^\star\} \le \Lambda_1(p) \le \lambda_1(p)$, **for all** $p \in (1,\infty)$*;*

- *if* $\|\delta\|_{L^\infty(\Omega)} \in (0,1)$ *then* $\Lambda_1(p) = \lambda_1(p)$ **for all** $p > 1$ **sufficiently large**.

The proof of the case where $\|\delta\|_{L^\infty(\Omega)} > 1$

Let $\epsilon_0 > 0$ and $\omega \subset \Omega$ with $|\omega| > 0$ s.t.

$$\delta(x) \geq 1 + \epsilon_0, \quad \forall\, x \in \omega\,.$$

Since $\delta \in X_0 \setminus \{0\}$,

$$\Lambda_1(p) \leq \frac{\displaystyle\int_\Omega (\exp(|\nabla(n\delta)|^p) - 1)\, dx}{\displaystyle\int_\Omega (\exp((n\delta)^p) - 1)\, dx} \quad \forall\, n \geq 1. \tag{14}$$

Taking into account that $|\nabla\delta(x)| = 1$ for a.e. $x \in \Omega$, we have

$$\frac{\displaystyle\int_\Omega (\exp(|\nabla(n\delta)|^p) - 1)}{\displaystyle\int_\Omega (\exp((n\delta)^p) - 1)} = \frac{|\Omega|(\exp(n^p) - 1)}{\displaystyle\int_\Omega (\exp(n^p\delta(x)^p) - 1)} \leq \frac{|\Omega|(\exp(n^p) - 1)}{\displaystyle\int_\omega (\exp(n^p\delta(x)^p) - 1)}$$
$$\leq \frac{|\Omega|(\exp(n^p) - 1)}{|\omega|(\exp(n^p(1+\epsilon_0)^p) - 1)},$$

for every integer $n \geq 1$. The fact that $\Lambda_1(p) = 0$ now follows by letting $n \to \infty$ in (14).

We identify some cases where $\Lambda_1(p) = \lambda_1(p)$ for all $p \in (1, \infty)$.

The simplest such situation occurs in the **one-dimensional case**, when $\lambda_1(p)$ can be explicitly computed for every $p \in (1, \infty)$ (see, e.g. Lindqvist 1993) and thus, we can show that the function $(1, \infty) \ni p \to \lambda_1(p) \in (0, \infty)$ **is increasing** provided that $\|\delta\|_{L^\infty(\Omega)} \in (0, 1]$.

More precisely, the following result holds true.
Theorem 5. *(***Bocea & M***, 2018)*

- *If* $\Omega = (a, b)$ *and* $\|\delta\|_{L^\infty(\Omega)} \in (0, 1]$ *then*

$$\Lambda_1(p) = \lambda_1(p), \ \forall \, p \in (1, \infty) \, .$$

- *In the D-dimensional case,* $D \geq 2$*, if* $\|\delta\|_{L^\infty(\Omega)} \in (0, 1]$ *and* $\Omega := (-\|\delta\|_{L^\infty(\Omega)}, \|\delta\|_{L^\infty(\Omega)}) \times B_T^{D-1}(0)$*, where* $B_T^{D-1}(0)$ **stands for the ball centered at the origin of radius** T **in** $\mathbb{R}^{D-1}$ *then, for sufficiently large* $T > 0$*, we have*

$$\Lambda_1(p) = \lambda_1(p), \ \forall \, p \in (1, \infty) \, .$$

Unfortunately, in dimensions two or higher, the **monotonicity** of the map $(1,\infty) \ni p \to \lambda_1(p) \in (0,\infty)$ when $\|\delta\|_{L^\infty(\Omega)} \in (0,1]$ **is not known in general**. Nevertheless, using as our main argument the monotonicity property

$$t\sqrt[t]{\lambda_1(t;\Omega)} \leq s\sqrt[s]{\lambda_1(s;\Omega)}, \quad \forall\, 1 < t < s < \infty\,.$$

it can be shown that the above property remains valid in higher dimensions provided that the volume of the domain where we investigate the problem is sufficiently small.

Theorem 6. *(**Bocea & M**, 2018)*

• *Assume that $D \geq 2$ and $\Omega \subset \mathbb{R}^D$ is an open bounded domain that satisfies $|\Omega| < v_D \exp(-D)$, where v_D denotes the volume of the unit ball in $\mathbb{R}^D$. Then*

$$\Lambda_1(p) = \lambda_1(p),\ \forall\, p \in (1,\infty)\,.$$

We would like to point out that a well-known property of the principal frequency of the p-Laplacian fails to hold for Λ_1.

Recall that,

$$\lambda_1(p; B_R) = R^{-p}\lambda_1(p; B_1),\ \forall\, p \in (1, \infty),\ \text{and } R > 0\,.$$

It follows that

$$\lim_{R \searrow 1} \lambda_1(p; B_R) = \lambda_1(p; B_1)\,.$$

It is clear that this property fails if we replace λ_1 by Λ_1, since in this case we have

$$\Lambda_1(p; B_R) = 0,\ \forall\, p \in (1, \infty),\ \ R > 1\,,$$

while

$$\Lambda_1(p; B_1) > 0,\ \forall\, p \in (1, \infty)\,.$$

Thank you for your attention!