Hölder Topology for the Heisenberg group

Armin Schikorra

Bucharest, April, 26 2018.

Homotopy groups and analysis of Sobolev-spaces

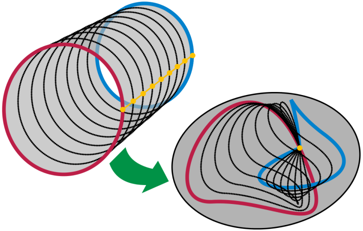
Definition

k-th homotopy group $\pi_k(\mathcal{M})$ of a metric space $(\mathcal{M}, d)$:

Set of all maps $f : \mathbb{S}^k \to \mathcal{M}$ modulo homotopies:

$f \sim g \Leftrightarrow f$ and g homotopic $\Leftrightarrow$

$$\exists H : [0,1] \times \mathbb{S}^k \to \mathcal{M} \text{ continuous}, \quad H(0,\cdot) = f, \quad H(1,\cdot) = g.$$

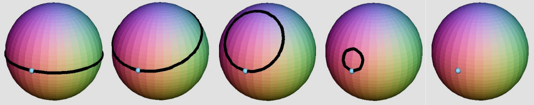


Homotopy groups and analysis of Sobolev-spaces

$f : \mathbb{S}^k \to \mathcal{M} \sim 0 \Leftrightarrow$

f can be contracted in $\mathcal{M}$ to a point $\Leftrightarrow$

$$\exists \text{ continuous extension } F : \mathbb{B}^{k+1} \to \mathcal{M} \quad F\Big|_{\mathbb{S}^k = \partial\mathbb{B}^{k+1}} = f$$

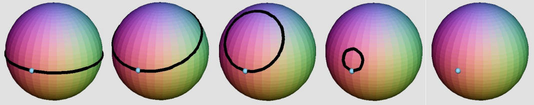


Homotopy groups and analysis of Sobolev-spaces

$f : \mathbb{S}^k \to \mathcal{M} \sim 0 \Leftrightarrow$

f can be contracted in $\mathcal{M}$ to a point $\Leftrightarrow$

$$\exists \text{ continuous extension } F : \mathbb{B}^{k+1} \to \mathcal{M} \quad F\Big|_{\mathbb{S}^k = \partial\mathbb{B}^{k+1}} = f$$



Trivial homotopy group, $\pi_k(\mathcal{M}) = \{0\}$, $\Leftrightarrow$

All maps $f : \mathbb{S}^k \to \mathcal{M}$ can be contracted to a point

nontrivial homotopy groups $\Leftrightarrow$ density of smooth maps in Sobolev spaces

Theorem (Bethuel-Zheng, Hang-Lin)

$\mathcal{M} \subset \mathbb{R}^N$ be a smooth compact manifold, $1 \leq p < n$. Smooth maps $C^\infty(\mathbb{B}^n, \mathcal{M})$ are dense in $W^{1,p}(\mathbb{B}^n, \mathcal{M}) \Leftrightarrow$ $\pi_{\lfloor p \rfloor}(\mathcal{M}) = \{0\}$.

nontrivial homotopy groups $\Leftrightarrow$ density of smooth maps in Sobolev spaces

Theorem (Bethuel-Zheng, Hang-Lin)

$\mathcal{M} \subset \mathbb{R}^N$ be a smooth compact manifold, $1 \leq p < n$. Smooth maps $C^\infty(\mathbb{B}^n, \mathcal{M})$ are dense in $W^{1,p}(\mathbb{B}^n, \mathcal{M}) \Leftrightarrow \pi_{\lfloor p \rfloor}(\mathcal{M}) = \{0\}$.

- for $p \geq n$, density always true (Schoen-Uhlenbeck).

nontrivial homotopy groups $\Leftrightarrow$ density of smooth maps in Sobolev spaces

Theorem (Bethuel-Zheng, Hang-Lin)

$\mathcal{M} \subset \mathbb{R}^N$ be a smooth compact manifold, $1 \leq p < n$. Smooth maps $C^\infty(\mathbb{B}^n, \mathcal{M})$ are dense in $W^{1,p}(\mathbb{B}^n, \mathcal{M}) \Leftrightarrow$ $\pi_{\lfloor p \rfloor}(\mathcal{M}) = \{0\}$.

- for $p \geq n$, density always true (Schoen-Uhlenbeck).
- if instead of $\mathbb{B}^n$ more general domain manifold, more complicated condition (Hang-Lin).

Theorem (Bethuel-Zheng, Hang-Lin)

For $1 \leq p < n$, smooth maps $C^\infty(\mathbb{B}^n, \mathcal{M})$ are dense in $W^{1,p}(\mathbb{B}^n, \mathcal{M})$ if and only if $\pi_{\lfloor p \rfloor}(\mathcal{M}) = \{0\}$.

Theorem (Bethuel-Zheng, Hang-Lin)

For $1 \leq p < n$*, smooth maps* $C^\infty(\mathbb{B}^n, \mathcal{M})$ *are dense in* $W^{1,p}(\mathbb{B}^n, \mathcal{M})$ *if and only if* $\pi_{\lfloor p \rfloor}(\mathcal{M}) = \{0\}$*.*

Idea $(n - 1 < p < n)$

- Assume $f : \mathbb{S}^{n-1} \to \mathcal{M} \in \pi_{n-1}(\mathcal{M}) \setminus \{0\}$

Theorem (Bethuel-Zheng, Hang-Lin)

For $1 \leq p < n$, *smooth maps* $C^\infty(\mathbb{B}^n, \mathcal{M})$ *are dense in* $W^{1,p}(\mathbb{B}^n, \mathcal{M})$ *if and only if* $\pi_{\lfloor p \rfloor}(\mathcal{M}) = \{0\}$.

Idea $(n - 1 < p < n)$

- Assume $f : \mathbb{S}^{n-1} \to \mathcal{M} \in \pi_{n-1}(\mathcal{M}) \backslash \{0\}$
- Set $F(x) := f(x/|x|) \in W^{1,p}(\mathbb{B}^n, \mathcal{M})$

Theorem (Bethuel-Zheng, Hang-Lin)

For $1 \leq p < n$*, smooth maps* $C^\infty(\mathbb{B}^n, \mathcal{M})$ *are dense in* $W^{1,p}(\mathbb{B}^n, \mathcal{M})$ *if and only if* $\pi_{\lfloor p \rfloor}(\mathcal{M}) = \{0\}$.

Idea $(n - 1 < p < n)$

- Assume $f : \mathbb{S}^{n-1} \to \mathcal{M} \in \pi_{n-1}(\mathcal{M})\backslash\{0\}$
- Set $F(x) := f(x/|x|) \in W^{1,p}(\mathbb{B}^n, \mathcal{M})$
- If there was a smooth approximation $G_k \in C^\infty(\mathbb{B}^n, \mathcal{M})$ for F in $W^{1,p}(\mathbb{B}^n, \mathcal{M})$

Theorem (Bethuel-Zheng, Hang-Lin)

For $1 \leq p < n$, smooth maps $C^\infty(\mathbb{B}^n, \mathcal{M})$ are dense in $W^{1,p}(\mathbb{B}^n, \mathcal{M})$ if and only if $\pi_{\lfloor p \rfloor}(\mathcal{M}) = \{0\}$.

Idea $(n-1 < p < n)$

- Assume $f : \mathbb{S}^{n-1} \to \mathcal{M} \in \pi_{n-1}(\mathcal{M}) \backslash \{0\}$
- Set $F(x) := f(x/|x|) \in W^{1,p}(\mathbb{B}^n, \mathcal{M})$
- If there was a smooth approximation $G_k \in C^\infty(\mathbb{B}^n, \mathcal{M})$ for F in $W^{1,p}(\mathbb{B}^n, \mathcal{M})$
- then (Fubini!) for some radius $r > 0$ we have $G_k\big|_{\partial \mathbb{B}_r} \to F\big|_{\partial \mathbb{B}_r}$ in $W^{1,p}(\partial \mathbb{B}_r, \mathcal{M})$

Theorem (Bethuel-Zheng, Hang-Lin)

For $1 \leq p < n$, smooth maps $C^\infty(\mathbb{B}^n, \mathcal{M})$ are dense in $W^{1,p}(\mathbb{B}^n, \mathcal{M})$ if and only if $\pi_{\lfloor p \rfloor}(\mathcal{M}) = \{0\}$.

Idea $(n-1 < p < n)$

- Assume $f : \mathbb{S}^{n-1} \to \mathcal{M} \in \pi_{n-1}(\mathcal{M}) \setminus \{0\}$
- Set $F(x) := f(x/|x|) \in W^{1,p}(\mathbb{B}^n, \mathcal{M})$
- If there was a smooth approximation $G_k \in C^\infty(\mathbb{B}^n, \mathcal{M})$ for F in $W^{1,p}(\mathbb{B}^n, \mathcal{M})$
- then (Fubini!) for some radius $r > 0$ we have $G_k\big|_{\partial \mathbb{B}_r} \to F\big|_{\partial \mathbb{B}_r}$ in $W^{1,p}(\partial \mathbb{B}_r, \mathcal{M})$
- Since $p > n-1 = \dim \partial \mathbb{B}_r$ by Sobolev embedding $G_k\big|_{\partial \mathbb{B}_r} \to F\big|_{\partial \mathbb{B}_r}$ in C^σ for $\sigma = 1 - \frac{n-1}{p} > 0$

Theorem (Bethuel-Zheng, Hang-Lin)

For $1 \leq p < n$, smooth maps $C^\infty(\mathbb{B}^n, \mathcal{M})$ are dense in $W^{1,p}(\mathbb{B}^n, \mathcal{M})$ if and only if $\pi_{\lfloor p \rfloor}(\mathcal{M}) = \{0\}$.

Idea $(n-1 < p < n)$

- Assume $f : \mathbb{S}^{n-1} \to \mathcal{M} \in \pi_{n-1}(\mathcal{M}) \setminus \{0\}$
- Set $F(x) := f(x/|x|) \in W^{1,p}(\mathbb{B}^n, \mathcal{M})$
- If there was a smooth approximation $G_k \in C^\infty(\mathbb{B}^n, \mathcal{M})$ for F in $W^{1,p}(\mathbb{B}^n, \mathcal{M})$
- then (Fubini!) for some radius $r > 0$ we have $G_k\big|_{\partial \mathbb{B}_r} \to F\big|_{\partial \mathbb{B}_r}$ in $W^{1,p}(\partial \mathbb{B}_r, \mathcal{M})$
- Since $p > n-1 = \dim \partial \mathbb{B}_r$ by Sobolev embedding $G_k\big|_{\partial \mathbb{B}_r} \to F\big|_{\partial \mathbb{B}_r}$ in C^σ for $\sigma = 1 - \frac{n-1}{p} > 0$
- $G_k\big|_{\partial \mathbb{B}_r} \sim F\big|_{\partial \mathbb{B}_r}$ for k large

Theorem (Bethuel-Zheng, Hang-Lin)

For $1 \leq p < n$, smooth maps $C^\infty(\mathbb{B}^n, \mathcal{M})$ are dense in $W^{1,p}(\mathbb{B}^n, \mathcal{M})$ if and only if $\pi_{\lfloor p \rfloor}(\mathcal{M}) = \{0\}$.

Idea $(n-1 < p < n)$

- Assume $f : \mathbb{S}^{n-1} \to \mathcal{M} \in \pi_{n-1}(\mathcal{M}) \backslash \{0\}$
- Set $F(x) := f(x/|x|) \in W^{1,p}(\mathbb{B}^n, \mathcal{M})$
- If there was a smooth approximation $G_k \in C^\infty(\mathbb{B}^n, \mathcal{M})$ for F in $W^{1,p}(\mathbb{B}^n, \mathcal{M})$
- then (Fubini!) for some radius $r > 0$ we have $G_k\big|_{\partial \mathbb{B}_r} \to F\big|_{\partial \mathbb{B}_r}$ in $W^{1,p}(\partial \mathbb{B}_r, \mathcal{M})$
- Since $p > n-1 = \dim \partial \mathbb{B}_r$ by Sobolev embedding $G_k\big|_{\partial \mathbb{B}_r} \to F\big|_{\partial \mathbb{B}_r}$ in C^σ for $\sigma = 1 - \frac{n-1}{p} > 0$
- $G_k\big|_{\partial \mathbb{B}_r} \sim F\big|_{\partial \mathbb{B}_r}$ for k large
- $G_k\big|_{\partial \mathbb{B}_r}$ is zero-homotopic (i.e. $G_k\big|_{\mathbb{B}_r}$ can be extended to the ball

Theorem (Bethuel-Zheng, Hang-Lin)

For $1 \leq p < n$, smooth maps $C^\infty(\mathbb{B}^n, \mathcal{M})$ are dense in $W^{1,p}(\mathbb{B}^n, \mathcal{M})$ if and only if $\pi_{\lfloor p \rfloor}(\mathcal{M}) = \{0\}$.

Idea $(n-1 < p < n)$

- Assume $f : \mathbb{S}^{n-1} \to \mathcal{M} \in \pi_{n-1}(\mathcal{M}) \backslash \{0\}$
- Set $F(x) := f(x/|x|) \in W^{1,p}(\mathbb{B}^n, \mathcal{M})$
- If there was a smooth approximation $G_k \in C^\infty(\mathbb{B}^n, \mathcal{M})$ for F in $W^{1,p}(\mathbb{B}^n, \mathcal{M})$
- then (Fubini!) for some radius $r > 0$ we have $G_k\big|_{\partial \mathbb{B}_r} \to F\big|_{\partial \mathbb{B}_r}$ in $W^{1,p}(\partial \mathbb{B}_r, \mathcal{M})$
- Since $p > n-1 = \dim \partial \mathbb{B}_r$ by Sobolev embedding $G_k\big|_{\partial \mathbb{B}_r} \to F\big|_{\partial \mathbb{B}_r}$ in C^σ for $\sigma = 1 - \frac{n-1}{p} > 0$
- $G_k\big|_{\partial \mathbb{B}_r} \sim F\big|_{\partial \mathbb{B}_r}$ for k large
- $G_k\big|_{\partial \mathbb{B}_r}$ is zero-homotopic (i.e. $G_k\big|_{\mathbb{B}_r}$ can be extended to the ball
- $F\big|_{\partial \mathbb{B}_r}$ is not zero-homotopic.

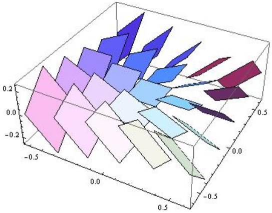
- Question: What do we do for non-Riemannian manifolds?
- Sub-Riemannian manifolds: "distance between points given via curves that are horizontal for a distribution $H\mathcal{M} \subsetneq T\mathcal{M}$.
- Special case Heisenberg-group $\mathbb{H}_n$.

Heisenberg-group (w. Hajłasz et al)

Heisenberg-group $\mathbb{H}_1 = (\mathbb{R}^3, d_{cc})$:

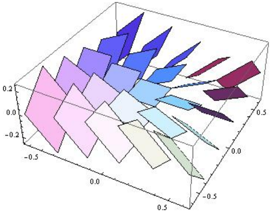
Heisenberg-group (w. Hajłasz et al)

Heisenberg-group $\mathbb{H}_1 = (\mathbb{R}^3, d_{cc})$:
metric d_{cc}: minimal lenght of horizontal curves, i.e. along a plane distribution $H_p\mathbb{H}_1$



Heisenberg-group (w. Hajłasz et al)

Heisenberg-group $\mathbb{H}_1 = (\mathbb{R}^3, d_{cc})$:
metric d_{cc}: minimal lenght of horizontal curves, i.e. along a plane distribution $H_p\mathbb{H}_1$



Given by $H_p\mathbb{H}_1 = \ker \alpha$

$$\alpha := dp^3 + 2(p^2\, dp^1 - p^1\, dp^2).$$

Horizontal curve $\nu : [0,1] \to \mathbb{H}_1$:

$$0 = \nu^*(\alpha) = \dot{\nu}^3(t) + 2(\nu^2(t)\, \dot{\nu}^1(t) - \nu^1(t)\, \dot{\nu}^2(t)).$$

Heisenberg-group $\mathbb{H}_1 = (\mathbb{R}^3, d_{cc})$:

Heisenberg-group $\mathbb{H}_1 = (\mathbb{R}^3, d_{cc})$:

- $\mathbb{H}_1$ homeomorphic to $\mathbb{R}^3$
- $\mathbb{H}_1$ $C^{1/2}$-homeomorphic to $\mathbb{R}^3$
- $\mathbb{H}_1$ not C^σ-homeomorphic to $\mathbb{R}^3$, $\sigma > \frac{1}{2}$.

Heisenberg-group $\mathbb{H}_1 = (\mathbb{R}^3, d_{cc})$:

- $\mathbb{H}_1$ homeomorphic to $\mathbb{R}^3$
- $\mathbb{H}_1$ $C^{1/2}$-homeomorphic to $\mathbb{R}^3$
- $\mathbb{H}_1$ not C^σ-homeomorphic to $\mathbb{R}^3$, $\sigma > \frac{1}{2}$.

Questions

- Lipschitz-approximations/extensions of maps in the Heisenberg group (Sobolev, Hölder) – what are topological restrictions?
- Which topological properties can we measure?

Heisenberg-group (w. Hajłasz et al)

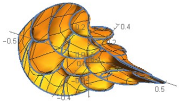
Conjecture (Gromov)

There is no C^σ-embedding $\varphi : \mathbb{B}^2 \subset \mathbb{R}^2 \to \mathbb{H}_1$ *for* $\sigma > \frac{1}{2}$.

Heisenberg-group (w. Hajłasz et al)

Conjecture (Gromov)

There is no C^σ-embedding $\varphi : \mathbb{B}^2 \subset \mathbb{R}^2 \to \mathbb{H}_1$ for $\sigma > \frac{1}{2}$.



numerical counterexample? (Hajłasz-Mirra-S.)

Heisenberg-group (w. Hajłasz et al)

Theorem (Gromov)

There is no C^σ-embedding $\varphi : \mathbb{B}^2 \subset \mathbb{R}^2 \to \mathbb{H}_1$ for $\sigma > \frac{2}{3}$.

Heisenberg-group (w. Hajłasz et al)

Theorem (Gromov)

There is no C^σ-embedding $\varphi : \mathbb{B}^2 \subset \mathbb{R}^2 \to \mathbb{H}_1$ for $\sigma > \frac{2}{3}$.

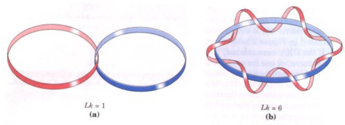
Proof: (Hajłasz-Mirra-S.): Linking number for $\varphi\Big|_{\partial\mathbb{B}^2}$!

Heisenberg-group (w. Hajłasz et al)

Theorem (Gromov)

There is no C^σ-embedding $\varphi : \mathbb{B}^2 \subset \mathbb{R}^2 \to \mathbb{H}_1$ for $\sigma > \frac{2}{3}$.

Proof: (Hajłasz-Mirra-S.): Linking number for $\varphi\Big|_{\partial \mathbb{B}^2}$!

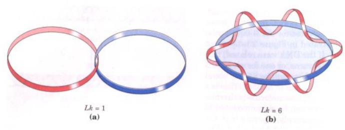


Heisenberg-group (w. Hajłasz et al)

Theorem (Gromov)

There is no C^σ-embedding $\varphi : \mathbb{B}^2 \subset \mathbb{R}^2 \to \mathbb{H}_1$ for $\sigma > \frac{2}{3}$.

Proof: (Hajłasz-Mirra-S.): Linking number for $\varphi\Big|_{\partial\mathbb{B}^2}$!



Idea (Phase 1): Algebraic Topology:

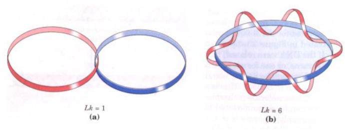
- If φ is embedding, then $\varphi(\partial\mathbb{B}^2)$ closed curve $\mathbb{R}^3$.

Heisenberg-group (w. Hajłasz et al)

Theorem (Gromov)

There is no C^σ-embedding $\varphi : \mathbb{B}^2 \subset \mathbb{R}^2 \to \mathbb{H}_1$ for $\sigma > \frac{2}{3}$.

Proof: (Hajłasz-Mirra-S.): Linking number for $\varphi\Big|_{\partial\mathbb{B}^2}$!



Idea (Phase 1): Algebraic Topology:

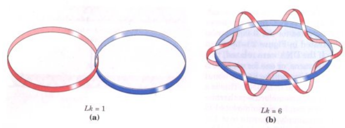
- If φ is embedding, then $\varphi(\partial\mathbb{B}^2)$ closed curve $\mathbb{R}^3$.
- Homology: $H_1(\mathbb{R}^3 \backslash \varphi(\partial\mathbb{B}^2)) \neq \{0\}$
  $\Leftrightarrow$ There exists another closed curve ω, which links with $\varphi(\partial\mathbb{B}^2)$.

Heisenberg-group (w. Hajłasz et al)

Theorem (Gromov)

There is no C^σ-embedding $\varphi : \mathbb{B}^2 \subset \mathbb{R}^2 \to \mathbb{H}_1$ for $\sigma > \frac{2}{3}$.

Proof: (Hajłasz-Mirra-S.): Linking number for $\varphi\Big|_{\partial\mathbb{B}^2}$!



Idea (Phase 1): Algebraic Topology:

- If φ is embedding, then $\varphi(\partial\mathbb{B}^2)$ closed curve $\mathbb{R}^3$.
- Homology: $H_1(\mathbb{R}^3 \backslash \varphi(\partial\mathbb{B}^2)) \neq \{0\}$
  $\Leftrightarrow$ There exists another closed curve ω, which links with $\varphi(\partial\mathbb{B}^2)$.
- Try to formulate this analytically and find for C^σ a contradiction

Heisenberg-group (w. Hajłasz et al)

Theorem (Gromov)

There is no C^σ-embedding $\varphi : \mathbb{B}^2 \subset \mathbb{R}^2 \to \mathbb{H}_1$ *for* $\sigma > \frac{2}{3}$.

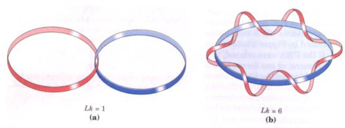
Proof (Hajłasz-Mirra-S.): Linking number for $\varphi\Big|_{\partial \mathbb{B}^2}$!

Heisenberg-group (w. Hajłasz et al)

Theorem (Gromov)

There is no C^σ-embedding $\varphi : \mathbb{B}^2 \subset \mathbb{R}^2 \to \mathbb{H}_1$ for $\sigma > \frac{2}{3}$.

Proof (Hajłasz-Mirra-S.): Linking number for $\varphi\Big|_{\partial \mathbb{B}^2}$!



Idee (Phase 2): Analytic Algebraic Topologie:

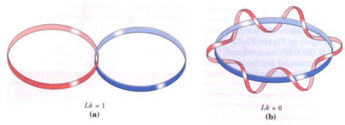
- φ a Lipschitz Embedding

Heisenberg-group (w. Hajłasz et al)

Theorem (Gromov)

There is no C^σ-embedding $\varphi : \mathbb{B}^2 \subset \mathbb{R}^2 \to \mathbb{H}_1$ for $\sigma > \frac{2}{3}$.

Proof (Hajłasz-Mirra-S.): Linking number for $\varphi\big|_{\partial\mathbb{B}^2}$!



Idee (Phase 2): Analytic Algebraic Topologie:

- φ a Lipschitz Embedding
- Isomorphism of cohomology $H^2(\mathbb{R}^3 \backslash \varphi(\partial\mathbb{B}^2))$

$$d\omega \mapsto \int_{\partial\mathbb{B}^2} \varphi^*(\omega)$$

Heisenberg-group (w. Hajłasz et al)

Theorem (Gromov)

There is no C^σ-embedding $\varphi : \mathbb{B}^2 \subset \mathbb{R}^2 \to \mathbb{H}_1$ for $\sigma > \frac{2}{3}$.

Proof. (Hajłasz-Mirra-S.): Analytic Linking number for $\varphi\Big|_{\partial\mathbb{B}^2}$!

Theorem (Hajłasz-Mirra-S.)

If φ is a $C^{0,1}$-embedding in $\mathbb{R}^3$, there exists a 1-form ω s.t.

$$\int_{\partial\mathbb{B}^2} \varphi^*(\omega) \neq 0$$

Heisenberg-group (w. Hajłasz et al)

Theorem (Gromov)

There is no C^σ-embedding $\varphi : \mathbb{B}^2 \subset \mathbb{R}^2 \to \mathbb{H}_1$ for $\sigma > \frac{2}{3}$.

Proof. (Hajłasz-Mirra-S.): Analytic Linking number for $\varphi\Big|_{\partial\mathbb{B}^2}$!

Theorem (Hajłasz-Mirra-S.)

If φ is a $C^{0,1}$-embedding in $\mathbb{R}^3$, there exists a 1-form ω s.t.

$$\int_{\partial\mathbb{B}^2} \varphi^*(\omega) \neq 0$$

but

Heisenberg-group (w. Hajłasz et al)

Theorem (Gromov)

There is no C^σ-embedding $\varphi : \mathbb{B}^2 \subset \mathbb{R}^2 \to \mathbb{H}_1$ for $\sigma > \frac{2}{3}$.

Proof. (Hajłasz-Mirra-S.): Analytic Linking number for $\varphi\Big|_{\partial\mathbb{B}^2}$!

Theorem (Hajłasz-Mirra-S.)

If φ is a $C^{0,1}$-embedding in $\mathbb{R}^3$, there exists a 1-form ω s.t.

$$\int_{\partial\mathbb{B}^2} \varphi^*(\omega) \neq 0$$

but

Theorem (Hajłasz-Mirra-S.)

If φ is a $C^{0,1}$-map $\varphi : \mathbb{B}^2 \to \mathbb{H}_1$, for every 1-form ω we have

$$\int_{\partial\mathbb{B}^2} \varphi^*(\omega) = 0$$

Theorem (Hajłasz-Mirra-S.)

If φ *is a* $C^{0,1}$*-map* $\varphi : \mathbb{B}^2 \to \mathbb{H}_1$, *for every* 1*-form* ω *we have*

$$\int_{\partial \mathbb{B}^2} \varphi^*(\omega) = 0$$

Theorem (Hajłasz-Mirra-S.)

If φ *is a* $C^{0,1}$*-map* $\varphi : \mathbb{B}^2 \to \mathbb{H}_1$*, for every* 1*-form* ω *we have*

$$\int_{\partial \mathbb{B}^2} \varphi^*(\omega) = 0$$

Proof: is φ Lipschitz, then for

$$\alpha := dp^3 + 2(p^2\, dp^1 - p^1\, dp^2)$$

we have

$$\varphi^*(\alpha) = 0$$

Theorem (Hajłasz-Mirra-S.)

If φ *is a* $C^{0,1}$*-map* $\varphi : \mathbb{B}^2 \to \mathbb{H}_1$*, for every* 1*-form* ω *we have*

$$\int_{\partial \mathbb{B}^2} \varphi^*(\omega) = 0$$

Proof: is φ Lipschitz, then for

$$\alpha := dp^3 + 2(p^2\, dp^1 - p^1\, dp^2)$$

we have

$$\varphi^*(\alpha) = 0, \quad \varphi^*(d\alpha) = 0$$

Theorem (Hajłasz-Mirra-S.)

If φ *is a* $C^{0,1}$*-map* $\varphi : \mathbb{B}^2 \to \mathbb{H}_1$, *for every* 1*-form* ω *we have*

$$\int_{\partial \mathbb{B}^2} \varphi^*(\omega) = 0$$

Proof: is φ Lipschitz, then for

$$\alpha := dp^3 + 2(p^2\, dp^1 - p^1\, dp^2)$$

we have

$$\varphi^*(\alpha) = 0, \quad \varphi^*(d\alpha) = 0$$

(Lefschetz-Lemma) Any two-form η can be written as (β one-form, c function).

$$\eta = \alpha \wedge \beta + c\, d\alpha$$

Theorem (Hajłasz-Mirra-S.)

If φ *is a* $C^{0,1}$*-map* $\varphi : \mathbb{B}^2 \to \mathbb{H}_1$*, for every 1-form* ω *we have*

$$\int_{\partial \mathbb{B}^2} \varphi^*(\omega) = 0$$

Proof: is φ Lipschitz, then for

$$\alpha := d p^3 + 2(p^2\, dp^1 - p^1\, dp^2)$$

we have

$$\varphi^*(\alpha) = 0, \quad \varphi^*(d\alpha) = 0$$

(Lefschetz-Lemma) Any two-form η can be written as (β one-form, c function).

$$\eta = \alpha \wedge \beta + c\, d\alpha$$

With Stokes' theorem

$$\int_{\partial \mathbb{B}^2} \varphi^*(\omega) = \int_{\mathbb{B}^2} \varphi^*(d\omega) = 0. \quad \square$$

Heisenberg-group (w. Hajłasz et al)

Theorem (Gromov)

There is no C^σ-embedding $\varphi : \mathbb{B}^2 \subset \mathbb{R}^2 \to \mathbb{H}_1$ for $\sigma > \frac{2}{3}$.

Proof. (Hajłasz-Mirra-S.): Analytic Linking number for $\varphi\Big|_{\partial\mathbb{B}^2}$!

Theorem (Hajłasz-Mirra-S.)

If φ is a $C^{0,1}$-embedding in $\mathbb{R}^3$, there exists a 1-form ω s.t.

$$\int_{\partial\mathbb{B}^2} \varphi^*(\omega) \neq 0$$

Heisenberg-group (w. Hajłasz et al)

Theorem (Gromov)

There is no C^σ*-embedding* $\varphi : \mathbb{B}^2 \subset \mathbb{R}^2 \to \mathbb{H}_1$ *for* $\sigma > \frac{2}{3}$*.*

Proof. (Hajłasz-Mirra-S.): Analytic Linking number for $\varphi\Big|_{\partial\mathbb{B}^2}$!

Theorem (Hajłasz-Mirra-S.)

If φ *is a* $C^{0,1}$*-embedding in* $\mathbb{R}^3$*, there exists a* 1*-form* ω *s.t.*

$$\int_{\partial\mathbb{B}^2} \varphi^*(\omega) \neq 0$$

but

Heisenberg-group (w. Hajłasz et al)

Theorem (Gromov)

There is no C^σ-embedding $\varphi : \mathbb{B}^2 \subset \mathbb{R}^2 \to \mathbb{H}_1$ for $\sigma > \frac{2}{3}$.

Proof. (Hajłasz-Mirra-S.): Analytic Linking number for $\varphi\Big|_{\partial\mathbb{B}^2}$!

Theorem (Hajłasz-Mirra-S.)

If φ is a $C^{0,1}$-embedding in $\mathbb{R}^3$, there exists a 1-form ω s.t.

$$\int_{\partial\mathbb{B}^2} \varphi^*(\omega) \neq 0$$

but

Theorem (Hajłasz-Mirra-S.)

If φ is a $C^{0,1}$-map $\varphi : \mathbb{B}^2 \to \mathbb{H}_1$, for every 1-form ω we have

$$\int_{\partial\mathbb{B}^2} \varphi^*(\omega) = 0$$

Heisenberg-group (w. Hajłasz et al)

Theorem (Gromov)

There is no C^σ-embedding $\varphi : \mathbb{B}^2 \subset \mathbb{R}^2 \to \mathbb{H}_1$ for $\sigma > \frac{2}{3}$.

Proof. (Hajłasz-Mirra-S.): Analytic Linking number for $\varphi\Big|_{\partial\mathbb{B}^2}$!

Theorem (Hajłasz-Mirra-S.)

If φ is a $C^{1/2+}$-embedding in $\mathbb{R}^3$, there exists a 1-form ω s.t.

$$\int_{\partial\mathbb{B}^2} \varphi^*(\omega) \neq 0$$

but

Theorem (Hajłasz-Mirra-S.)

If φ is a $C^{2/3+}$-map $\varphi : \mathbb{B}^2 \to \mathbb{H}_1$, for every 1-form ω we have

$$\int_{\partial\mathbb{B}^2} \varphi^*(\omega) = 0$$

(Phase 3): Harmonic Analysis:

- Problem:

$$\int_{\mathbb{S}^1} \varphi^*(\omega)$$

makes no sense for non-differentiable φ? 

(Phase 3): Harmonic Analysis:

- Problem:

$$\int_{\mathbb{S}^1} \varphi^*(\omega)$$

  makes no sense for non-differentiable φ? 

-  Yes it does: Commutator estimates

(Phase 3): Harmonic Analysis:

- Problem:

$$\int_{\mathbb{S}^1} \varphi^*(\omega)$$

  makes no sense for non-differentiable φ? 

-  Yes it does: Commutator estimates

Theorem (Coifman-Lions-Meyer-Semmes etc.)

$v \in C_c^\infty(\mathbb{R}^n)$ and $u = (u^1, \ldots, u^n) \in C_c^\infty(\mathbb{R}^n, \mathbb{R}^n)$. Dann

$$\int_{\mathbb{R}^n} v \, \det(\nabla u) \lesssim [v]_{BMO} \, \|\nabla u\|^n_{L^n(\mathbb{R}^n)}.$$

(Phase 3): Harmonic Analysis:

- Problem:

$$\int_{\mathbb{S}^1} \varphi^*(\omega)$$

makes no sense for non-differentiable φ? 

-  Yes it does: Commutator estimates

Theorem (Coifman-Lions-Meyer-Semmes etc.)

$v \in C_c^\infty(\mathbb{R}^n)$ and $u = (u^1, \ldots, u^n) \in C_c^\infty(\mathbb{R}^n, \mathbb{R}^n)$. Dann

$$\int_{\mathbb{R}^n} v\ \det(\nabla u) \lesssim [v]_{BMO}\ \|\nabla u\|^n_{L^n(\mathbb{R}^n)}.$$

$$\int_{\mathbb{R}^n} v\ \det(\nabla u) \lesssim [v]_{W^{s_0,p_0}}\ [u^1]_{W^{s_1,p_1}}\ \ldots\ [u^n]_{W^{s_n,p_n}}.$$

$$\sum_{i=0}^n s_i = n, \quad \sum_{i=0}^n \frac{1}{p_i} = 1$$

Corollary

- $\det(\nabla_n \psi)$ *is (distributional) well-defined if* $\psi \in C^{\frac{n}{n+1}+}(\mathbb{S}^n, \mathbb{R}^n)$

Corollary

- $\det(\nabla_n \psi)$ *is (distributional) well-defined if* $\psi \in C^{\frac{n}{n+1}+}(\mathbb{S}^n, \mathbb{R}^n)$
- φ *a* $C^{1/2+}$*-map, then* $\int_{\partial \mathbb{B}^2} \varphi^*(\omega)$ *well-defined*

Corollary

- $\det(\nabla_n \psi)$ *is (distributional) well-defined if* $\psi \in C^{\frac{n}{n+1}+}(\mathbb{S}^n, \mathbb{R}^n)$
- φ *a* $C^{1/2+}$*-map, then* $\int_{\partial \mathbb{B}^2} \varphi^*(\omega)$ *well-defined*
- φ *a* $C^{2/3+}$*-map, then* $\int_{\mathbb{B}^2} \varphi^*(d\omega)$ *well-defined*

Corollary

- $\det(\nabla_n \psi)$ *is (distributional) well-defined if* $\psi \in C^{\frac{n}{n+1}+}(\mathbb{S}^n, \mathbb{R}^n)$
- φ *a* $C^{1/2+}$*-map, then* $\int_{\partial \mathbb{B}^2} \varphi^*(\omega)$ *well-defined*
- φ *a* $C^{2/3+}$*-map, then* $\int_{\mathbb{B}^2} \varphi^*(d\omega)$ *well-defined*

Theorem (Hajłasz-Mirra-S.)

- *is* φ *a* $C^{1/2+}$*-embedding in* $\mathbb{R}^3$*, there exists a 1-form* ω *s.t.*

$$\int_{\partial \mathbb{B}^2} \varphi^*(\omega) \neq 0$$

- *is* φ *a* $C^{2/3+}$*-map* $\varphi : \mathbb{B}^2 \to \mathbb{H}_1$ *for any 1-form* ω

$$\int_{\partial \mathbb{B}^2} \varphi^*(\omega) = 0$$

Theorem (Hajlasz-Mirra-S. '2018)

Sei $k \geq n + 1$, $\gamma \in (\frac{1}{2}, 1]$, $\theta > 0$ *und*

$$2\gamma + \theta(k-1) - k > 0.$$

Then for arbitrary open set $\Omega \subset \mathbb{R}^k$ *with smooth boundary there is no* $f : \Omega \to \mathbb{H}_n$

- *boundary curve injective*
- *which is* C^γ *w.r.t.* $\mathbb{H}_n$*-metric and*
- *which is* C^θ *w.r.t Euclidean metric.*

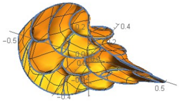
Theorem (Hajlasz-Mirra-S. '2018)

Sei $k \geq n + 1$, $\gamma \in (\frac{1}{2}, 1]$, $\theta > 0$ *und*

$$2\gamma + \theta(k - 1) - k > 0.$$

Then for arbitrary open set $\Omega \subset \mathbb{R}^k$ *with smooth boundary there is no* $f : \Omega \to \mathbb{H}_n$

- *boundary curve injective*
- *which is* C^γ *w.r.t.* $\mathbb{H}_n$*-metric and*
- *which is* C^θ *w.r.t Euclidean metric.*



Maybe sharp result? (Hajłasz-Mirra-S.)

With a combination of

- Algebraic Topology
- Analytic Algebraic Topology
- Harmonic Analysis

we also get homotopy groups results:

Theorem (Hajlasz-S-Tyson, Hajlasz-Mirra-S)

1. $\pi_n^\gamma(\mathbb{H}_n) \neq \{0\}$ *when* $\frac{n+1}{n+2} < \gamma \leq 1$.
2. $\pi_{4n-1}^\gamma(\mathbb{H}_{2n}) \neq \{0\}$ *when* $\frac{4n+1}{4n+2} < \gamma \leq 1$.

Rang-essential homotopy groups

Definition (Rang-essential (Hajłasz-S.-Tyson))

A homotopy group $\pi_k(\mathbb{S}^n) \neq \{0\}$ is rang-essential, if there is a map $f \in C^\infty(\mathbb{S}^k, \mathbb{S}^n)$ such that:
Any Lipschitz extension $F : \mathbb{B}^{k+1} \to \mathbb{R}^{n+1}$ with $F\big|_{\partial \mathbb{B}^{k+1}} = f$, satisfies

$$\operatorname{Rank} \nabla F = n+1$$

in a set of positive measure.

Rang-essential homotopy groups

Definition (Rang-essential (Hajłasz-S.-Tyson))

A homotopy group $\pi_k(\mathbb{S}^n) \neq \{0\}$ is rang-essential, if there is a map $f \in C^\infty(\mathbb{S}^k, \mathbb{S}^n)$ such that:
Any Lipschitz extension $F : \mathbb{B}^{k+1} \to \mathbb{R}^{n+1}$ with $F\big|_{\partial \mathbb{B}^{k+1}} = f$, satisfies

$$\operatorname{Rank} \nabla F = n + 1$$

in a set of positive measure.

Theorem (Hajłasz-S.-Tyson)

if $\pi_k(\mathbb{S}^n)$ *rang-essential, then* $\pi_k^{\text{Lip}}(\mathbb{H}_n) \neq \{0\}$.

- $f \in C^\infty(\mathbb{S}^k, \mathbb{S}^n)$ as above
- If $g := \phi \circ f \in \text{Lip}(\mathbb{S}^k, \mathbb{H}_n)$ trivial in homotopy group
- $G : \mathbb{B}^{k+1} \to \mathbb{H}_n$ extension of g
- $\operatorname{Rang} \nabla G \leq n$
- Invert: ϕ^{-1} - so we found and extension of f with rank $\leq n$ everywhere

Rang-essential homotopy groups

Theorem (Hajłasz-S.-Tyson)

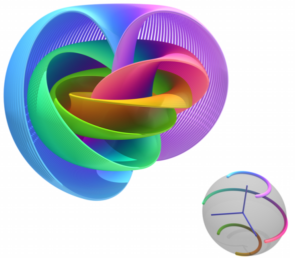
- $\pi_n(\mathbb{S}^n)$ *is Rang-essential*
- $\pi_{4n-1}(\mathbb{S}^{2n})$ *is Rang-essential*

$\pi_{4n-1}(\mathbb{S}^{2n})$ is Rang-essential

Topology: $\pi_{4n-1}(\mathbb{S}^{2n}) \neq \{0\}$:

- Hopf Map: $f \in \pi_{4n-1}(\mathbb{S}^{2n}) \backslash \{0\}$:

,



- for $p \neq q \in \mathbb{S}^{2n}$, consider Linking Number of $f^{-1}(p)$ and $f^{-1}(q)$
- Linking Number is Homotopy-invariant.

Analytic presentation of Hopf Invariant:

$$f : \mathbb{S}^{4n-1} \to \mathbb{S}^{2n}, \qquad \text{e.g. Hopf Fibration}$$

- Let $\sigma_{\mathbb{S}^{2n}} \in C^\infty(\bigwedge^{2n} \mathbb{S}^{2n})$ volume form of $\mathbb{S}^{2n}$

Analytic presentation of Hopf Invariant:

$$f : \mathbb{S}^{4n-1} \to \mathbb{S}^{2n}, \quad \text{e.g. Hopf Fibration}$$

- Let $\sigma_{\mathbb{S}^{2n}} \in C^\infty(\bigwedge^{2n} \mathbb{S}^{2n})$ volume form of $\mathbb{S}^{2n}$
- $f^*\sigma_{\mathbb{S}^{2n}} \in C^\infty(\bigwedge^{2n} \mathbb{S}^{4n-1})$

Analytic presentation of Hopf Invariant:

$$f : \mathbb{S}^{4n-1} \to \mathbb{S}^{2n}, \qquad \text{e.g. Hopf Fibration}$$

- Let $\sigma_{\mathbb{S}^{2n}} \in C^\infty(\bigwedge^{2n} \mathbb{S}^{2n})$ volume form of $\mathbb{S}^{2n}$
- $f^* \sigma_{\mathbb{S}^{2n}} \in C^\infty(\bigwedge^{2n} \mathbb{S}^{4n-1})$
- $d(f^* \sigma_{\mathbb{S}^{2n}}) = f^* d\sigma_{\mathbb{S}^{2n}} = 0$
  since $\sigma_{\mathbb{S}^{2n}}$ volume form of $\mathbb{S}^{2n}$

Analytic presentation of Hopf Invariant:

$$f : \mathbb{S}^{4n-1} \to \mathbb{S}^{2n}, \quad \text{e.g. Hopf Fibration}$$

- Let $\sigma_{\mathbb{S}^{2n}} \in C^\infty(\bigwedge^{2n} \mathbb{S}^{2n})$ volume form of $\mathbb{S}^{2n}$
- $f^*\sigma_{\mathbb{S}^{2n}} \in C^\infty(\bigwedge^{2n} \mathbb{S}^{4n-1})$
- $d(f^*\sigma_{\mathbb{S}^{2n}}) = f^* d\sigma_{\mathbb{S}^{2n}} = 0$
  since $\sigma_{\mathbb{S}^{2n}}$ volume form of $\mathbb{S}^{2n}$
- $f^*\sigma_{\mathbb{S}^{2n}} = d\omega$ for $\omega \in C^\infty(\bigwedge^{2n-1} \mathbb{S}^{4n-1})$, since $H^{2n}_{DR}(\mathbb{S}^{4n-1}) = \{0\}$

Analytic presentation of Hopf Invariant:

$$f : \mathbb{S}^{4n-1} \to \mathbb{S}^{2n}, \qquad \text{e.g. Hopf Fibration}$$

- Let $\sigma_{\mathbb{S}^{2n}} \in C^\infty(\bigwedge^{2n} \mathbb{S}^{2n})$ volume form of $\mathbb{S}^{2n}$
- $f^* \sigma_{\mathbb{S}^{2n}} \in C^\infty(\bigwedge^{2n} \mathbb{S}^{4n-1})$
- $d(f^* \sigma_{\mathbb{S}^{2n}}) = f^* d\sigma_{\mathbb{S}^{2n}} = 0$
  since $\sigma_{\mathbb{S}^{2n}}$ volume form of $\mathbb{S}^{2n}$
- $f^* \sigma_{\mathbb{S}^{2n}} = d\omega$ for $\omega \in C^\infty(\bigwedge^{2n-1} \mathbb{S}^{4n-1})$, since $H^{2n}_{DR}(\mathbb{S}^{4n-1}) = \{0\}$
- Hopf-invariant:

$$\mathcal{H}(f) := \int_{\mathbb{S}^{4n-1}} \omega \wedge d\omega$$

Analytic presentation of Hopf Invariant:

$$f : \mathbb{S}^{4n-1} \to \mathbb{S}^{2n}, \qquad \text{e.g. Hopf Fibration}$$

- Let $\sigma_{\mathbb{S}^{2n}} \in C^\infty(\bigwedge^{2n} \mathbb{S}^{2n})$ volume form of $\mathbb{S}^{2n}$
- $f^*\sigma_{\mathbb{S}^{2n}} \in C^\infty(\bigwedge^{2n} \mathbb{S}^{4n-1})$
- $d(f^*\sigma_{\mathbb{S}^{2n}}) = f^* d\sigma_{\mathbb{S}^{2n}} = 0$
  since $\sigma_{\mathbb{S}^{2n}}$ volume form of $\mathbb{S}^{2n}$
- $f^*\sigma_{\mathbb{S}^{2n}} = d\omega$ for $\omega \in C^\infty(\bigwedge^{2n-1} \mathbb{S}^{4n-1})$, since $H^{2n}_{DR}(\mathbb{S}^{4n-1}) = \{0\}$
- Hopf-invariant:
$$\mathcal{H}(f) := \int_{\mathbb{S}^{4n-1}} \omega \wedge d\omega$$
- $\mathcal{H}(f) \neq 0$ for Hopf map. $\mathcal{H}$ is homotopy invariant.

Analytic presentation of Hopf Invariant:

$$f : \mathbb{S}^{4n-1} \to \mathbb{R}^{2n+1}, \quad \text{Rank } \nabla f \leq 2n$$

- Let $\sigma_{\mathbb{S}^{2n}} \in C^\infty(\bigwedge^{2n} \mathbb{S}^{2n})$ volume form of $\mathbb{S}^{2n}$
- $f^*\sigma_{\mathbb{S}^{2n}} \in C^\infty(\bigwedge^{2n} \mathbb{S}^{4n-1})$
- $d(f^*\sigma_{\mathbb{S}^{2n}}) = f^* d\sigma_{\mathbb{S}^{2n}} = 0$
  since $\sigma_{\mathbb{S}^{2n}}$ $2n$-form and f has rank $\leq 2n$
- $f^*\sigma_{\mathbb{S}^{2n}} = d\omega$ for $\omega \in C^\infty(\bigwedge^{2n-1} \mathbb{S}^{4n-1})$, since $H^{2n}_{DR}(\mathbb{S}^{4n-1}) = \{0\}$
- Hopf-invariant:
  $$\mathcal{H}(f) := \int_{\mathbb{S}^{4n-1}} \omega \wedge d\omega$$
- $\mathcal{H}(f) \neq 0$ for Hopf map. $\mathcal{H}$ is homotopy invariant.

Analytic presentation of Hopf Invariant:

$$f : \mathbb{S}^{4n-1} \to \mathbb{R}^{2n+1}, \quad \text{Rank } \nabla f \leq 2n$$

- Let $\sigma_{\mathbb{S}^{2n}} \in C^\infty(\bigwedge^{2n} \mathbb{S}^{2n})$ volume form of $\mathbb{S}^{2n}$
- $f^*\sigma_{\mathbb{S}^{2n}} \in C^\infty(\bigwedge^{2n} \mathbb{S}^{4n-1})$
- $d(f^*\sigma_{\mathbb{S}^{2n}}) = f^* d\sigma_{\mathbb{S}^{2n}} = 0$
  since $\sigma_{\mathbb{S}^{2n}}$ $2n$-form and f has rank $\leq 2n$
- $f^*\sigma_{\mathbb{S}^{2n}} = d\omega$ for $\omega \in C^\infty(\bigwedge^{2n-1} \mathbb{S}^{4n-1})$, since $H^{2n}_{DR}(\mathbb{S}^{4n-1}) = \{0\}$
- Hopf-invariant:
$$\mathcal{H}(f) := \int_{\mathbb{S}^{4n-1}} \omega \wedge d\omega$$
- $\mathcal{H}(f) \neq 0$ for Hopf map. $\mathcal{H}$ is homotopy invariant.

Also is $\pi_{4n-1}(\mathbb{S}^{2n})$ Rank-essential! □

Hölder-version:

- for $f \in C^{1/2+}(\mathbb{S}^{2n}, \mathbb{H}_{2n})$ holds in distributional sense

$$f^*(\alpha) = 0$$

Hölder-version:

- for $f \in C^{1/2+}(\mathbb{S}^{2n}, \mathbb{H}_{2n})$ holds in distributional sense

$$f^*(\alpha) = 0$$

- for $f \in C^{\frac{4n+1}{4n+2}+}(\mathbb{S}^{2n}, \mathbb{H}_{2n})$ in distributional sense

$$f^*(\kappa) = 0$$

for any smooth k-form κ s.t. $k \geq 2n+1$.

Hölder-version:

- for $f \in C^{1/2+}(\mathbb{S}^{2n}, \mathbb{H}_{2n})$ holds in distributional sense

$$f^*(\alpha) = 0$$

- for $f \in C^{\frac{4n+1}{4n+2}+}(\mathbb{S}^{2n}, \mathbb{H}_{2n})$ in distributional sense

$$f^*(\kappa) = 0$$

  for any smooth k-form κ s.t. $k \geq 2n+1$.
- This is a weak rank condition..