Le spectre d'un opérateur de Schrödinger dans un domaine de type-fil avec un potentiel purement imaginaire dégénéré (d'après Almog-Helffer)

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Abstract

Consider a two-dimensional domain shaped like a wire, not necessarily of uniform cross section. Let V denote an electric potential driven by a voltage drop between the conducting surfaces of the wire. We consider the operator $\mathcal{A}_h = -h^2\Delta + iV$ in the semi-classical limit $h \to 0$.
We obtain both the asymptotic behaviour of the left margin of the spectrum, as well as resolvent estimates on the left side of this margin.
We extend here previous results by Henry, Almog-Henry, Helffer-Grebenkov,... obtained for potentials for which the set where the current (or ∇V) is normal to the boundary is discrete, in contrast with the present case where V is constant along the conducting surfaces.
This is a work in collaboration with Y. Almog (Louisiana State University).

Main assumptions

We consider the operator

$$\mathcal{A}_h = -h^2 \Delta + i\, V\,, \tag{1a}$$

defined on

$$D(\mathcal{A}_h) = \{\, u \in H^2(\Omega, \mathbb{C}) \,|\, u|_{\partial\Omega_D} = 0\,;\, \partial u/\partial\nu|_{\partial\Omega_N} = 0 \,\}\,. \tag{1b}$$

In the above, $\Omega \subset \mathbb{R}^2$ denotes a bounded, simply connected domain which has the same characteristics as in Almog-Helffer-Pan [8, 5]. In particular its boundary $\partial\Omega$ contains two disjoint open subsets $\partial\Omega_D$ and $\partial\Omega_N$ such that

$$\overline{\partial\Omega_D} \cup \overline{\partial\Omega_N} = \partial\Omega\,,$$

where $\partial\Omega_D$ is a union of two disjoint smooth interfaces on which we prescribe a Dirichlet boundary condition, and $\partial\Omega_N$ is a union of two disjoint smooth interfaces on which we prescribe a Neumann boundary condition. Hence $\overline{\partial\Omega_D} \cap \overline{\partial\Omega_N}$ consists of four points which will be called corners.

In the context of superconductivity $\partial\Omega_D$ and $\partial\Omega_N$, are respectively adjacent either to a normal metal or to an insulator. We denote each connected component of $\partial\Omega_\#$ ($\# \in \{D, N\}$) by a superscript $i \in \{1, 2\}$, i.e.,

$$\partial\Omega_\# = \partial\Omega_\#^1 \cup \partial\Omega_\#^2 \,, \quad \# \in \{D, N\}\,,\ i \in \{1, 2\}\,.$$

We say that $\partial\Omega$ is of class $C^{n,+}$ for some $n \in \mathbb{N}$, if there exists $\check{\beta} > 0$ such that $\partial\Omega$ is of class $C^{n,\check{\beta}}$. As in [3, 8, 5] we make the following assumptions on $\partial\Omega$

$$(R1) \begin{cases} (a)\ \overline{\partial\Omega_\#} \text{ is of class } C^{n,+} \text{ for } \# \in \{D, N\}\,; \\ (b) \text{Near each corner, } \overline{\partial\Omega_D} \text{ and } \overline{\partial\Omega_N} \text{ meet with an angle of } \frac{\pi}{2}\,. \end{cases} \tag{2}$$

The needed regularity n depends on the various results ($n = 2, 3, 4$). In addition, we assume
(R2) Near the corners, we assume in addition that there exists a conformal map, mapping the vicinity of the corner onto a vicinity of rectangular corner.

We consider potentials $V \in H^2(\Omega)$ satisfying

$$\begin{cases} \Delta V = 0 & \text{in } \Omega\,, \\ V = C_i & \text{on } \partial\Omega_D^i \text{ for } i = 1, 2\,, \\ \frac{\partial V}{\partial \nu} = 0 & \text{on } \partial\Omega_N\,, \end{cases} \tag{3}$$

describing a potential drop along a wire.
We assume further, as in [6], that V satisfies

$$|\nabla V(x)| \neq 0\,,\ \forall x \in \overline{\Omega}\,. \tag{4}$$

This implies that

$$C_1 \neq C_2\,.$$

The mathematical analysis of Equation (3) has a very long record in the literature. We refer to Polya-Szegö. In the case of the square

$$V(x_1, x_2) = V_0 + J x_2\,.$$

Typical sample with properties (R1) and (R2)

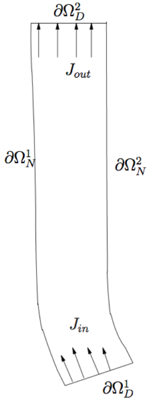

Figure 1: A typical wire-like domain. The arrows denote the direction of the potential gradient (or the current flow: J_{in} for the inlet, and J_{out} for the outlet).

The current flows into the sample from of $\partial\Omega_D^1$, and exits from $\partial\Omega_D^2$.

We distinguish in the sequel between two types of potentials satisfying (3).

V1 Potentials for which all points where $\inf_{x\in\overline{\partial\Omega_D}} |\partial V/\partial\nu|$ is attained, lie in $\partial\Omega_D$.

V2 Potentials for which all points where $\inf_{x\in\overline{\partial\Omega_D}} |\partial V/\partial\nu|$ is attained are corners.

Examples of potentials satisfying (3)

We now derive some simple examples of potentials satisfying (3), demonstrating that both types of potentials may exist. The basic idea is again that the problem introduced in (3) exhibits some invariance to conformal mapping (see Polya-Szegö). Hence starting form a problem defined on the square □, where the solution of (3) is a linear function, we can get from family of conformal maps a corresponding family of potentials satisfying (3) in various domains, together with (4), (R1) and (R2).

Let $\square = (0,1)\times(0,1) \subset \mathbb{C}$ and $\Omega = f(\square)$ where, for $w = u + iv$, f is the conformal map

$$f(w) = w + \delta\Big(\frac{1}{2}w^2 + \frac{\gamma}{3}w^3\Big)\,,$$

in which $\delta > 0$ and $\gamma \in \mathbb{R}$. Let further $f(w) = z = x + iy \in \Omega$, and set $g = f^{-1} : \Omega \to \square$, for sufficiently small δ. We may now set

$$\partial\Omega_D^1 = \{f(u)\,,\; u \in (0,1)\} \quad ; \quad \partial\Omega_D^2 = \{f(u+i)\,,\; u \in (0,1)\}\,,$$

and

$$\partial\Omega_N^1 = \{f(iv)\,,\; v \in (0,1)\} \quad ; \quad \partial\Omega_N^2 = \{f(1+iv)\,,\; v \in (0,1)\}\,.$$

Hence, depending on the value of γ, we can either find δ and a pair (V,Ω) for which $(V1)$ is satisfied or find δ and a pair (V,Ω) for which $(V2)$ is satisfied.

The spectral analysis of a Schrödinger operator with a purely imaginary potential has several applications in mathematical physics, among them are

- the Orr-Sommerfeld equations in fluid dynamics (Shkalikov) [22],
- the Ginzburg-Landau equation in the presence of electric current (when magnetic field effects are neglected) (Almog 2008) [3, 20],
- the null controllability of Kolmogorov type equations (Beauchard-Helffer-Henry-Robbiano (2015)) [12],
- the diffusion nuclear magnetic resonance (Grebenkov and references therein) [24, 25, 14].

Here we focus on the Ginzburg-Landau model, in the absence of magnetic field, and choose an electric potential satisfying (3).
Such a potential was considered in Almog (2008) where the asymptotics of a lower bound of $\inf \mathrm{Re}\, \sigma(\mathcal{A}_h)$ have been obtained as $h \to 0$.
Assuming a smooth domain, a similar result has been established in Henry (2014), using a more constructive technique, which provides also resolvent estimates.

In Almog-Helffer-Henry [AGH] [6], improving previous results from [10] (Almog-Henry (2016)), we obtained in collaboration with D. Grebenkov, the asymptotic behaviour of an upper bound for $\inf \mathrm{Re}\,\sigma(-h^2\Delta + iV)$ on smooth bounded domains (Dirichlet realization) in $\mathbb{R}^d$.

To characterize the potentials addressed in [AGH] we first define (for $d = 2$, which is the case considered in this work)

$$\partial\Omega_\perp = \{x \in \partial\Omega \,|\, \det(\nabla V(x), \vec{\nu}(x)) = 0\}\,,$$

where $\vec{\nu}(x)$ denotes the outward normal at x.

Then it is required in [AGH] that

$$\inf_{x\in\partial\Omega_\perp} \left|\det\, D^2 V_\partial(x)\right| > 0\,,$$

where V_∂ denotes the restriction of V to $\partial\Omega$, and $D^2 V_\partial$ denotes its Hessian matrix.

Note that the techniques employed in [AGH] are not applicable for potentials satisfying (3). The reason is that the ensuing approximate operators near the boundaries are not separable.

Main results

We seek an approximation for $\inf \mathrm{Re}\,\sigma(\mathcal{A}_h)$ in the limit $h \to 0$. Let

$$J_m = \min_{x \in \partial\Omega_D} |\nabla V(x)|\,. \tag{5}$$

Denote by $\mathfrak{S}$ the set

$$\mathfrak{S} = \{x \in \partial\Omega_D \,:\, |\nabla V(x)| = J_m\}\,. \tag{6}$$

Type V1 potentials

In this case, any $x \in \mathfrak{S}$ is a minimum point of $|\partial V/\partial\nu|$ on $\partial\Omega_D$. Thus,

$$\partial_{\|}\partial_{\nu} V(x) = 0\,, \quad \forall x \in \mathfrak{S}\,, \tag{7}$$

where $\partial_{\|}$ represents the derivative with respect to the arclength along the boundary in the positive trigonometric direction.
We next introduce

$$\alpha(x) = \partial_{\|}^2\partial_{\nu} V(x)\,, \quad \forall x \in \mathfrak{S}\,. \tag{8}$$

and

$$\alpha_m = \min_{x \in \mathfrak{S}} |\alpha(x)|\,. \tag{9}$$

We then introduce

$$\mathfrak{S}^m = \{x \in \mathfrak{S}\,|\,|\alpha(x)| = \alpha_m\}\,, \tag{10}$$

and assume

$$\alpha_m > 0\,. \tag{11}$$

Consequently any $x \in \mathfrak{S}$ is a non-degenerate minimum point of $|\partial V/\partial\nu|$.

Our main result in the case $V1$ is:

Main theorem Case V1

Let $\partial\Omega$ satisfies (R1) and (R2), and let V, the solution of (3), be of type V1 and satisfy (4). Suppose further that (11) is satisfied. Then

$$\lim_{h\to 0} \frac{1}{h^{2/3}} \inf\{\mathrm{Re}\ \sigma(\mathcal{A}_h)\} = J_m^{2/3} \frac{|\nu_1|}{2}\,, \tag{12}$$

where $\nu_1 < 0$ is the rightmost zero of Airy's function.

Type V2 potentials

In this case we similarly define

$$\hat{\alpha}(x) = \partial_{\|}\partial_\nu V(x)\,, \tag{13}$$

where (we are at the corners)

$$\hat{\alpha}_m = \min_{x \in \mathfrak{S}} |\hat{\alpha}(x)|\,. \tag{14}$$

We then define in $\mathfrak{S}$ a new subset

$$\hat{\mathfrak{S}}^m = \{x \in \mathfrak{S} \,|\, |\hat{\alpha}(x)| = \hat{\alpha}_m\}\,. \tag{15}$$

Main theorem Case V2

Let $\partial\Omega$ satisfies (R1) and (R2), let V, the solution of (3), be of type V2 and satisfy (4). Suppose further that $\hat{\alpha}_m > 0$. Then

$$\lim_{h\to 0} \frac{1}{h^{2/3}} \inf\{\mathrm{Re}\, \sigma(\mathcal{A}_h)\} = J_m^{2/3} \frac{|\nu_1|}{2}\,. \tag{16}$$

The proof goes through the following steps:

- Obtain the leading order asymptotic behaviour of a lower bound of $\inf \mathrm{Re}\,\sigma(\mathcal{A}_h)$ in the limit $h \to 0$.
- Construction of a a quasimode state for $\mathcal{A}_h$ for potentials of type $V1$ and $V2$.
- Obtain some auxiliary resolvent estimates for models in one dimension: complex Airy operators, complex harmonic oscillators,..
- Obtain resolvent estimates for (2D)-approximating operators.
- Complete the proof of the main theorems by proving the existence of an eigenvalue close to the quasimode.

WKB constructions-Type V1

We want to construct a two terms expansion of a quasimode.
Had $\mathcal{A}_h$ been self-adjoint, we could have used from here the spectral theorem to obtain the existence of an eigenvalue.

Alternatively, we can use in the self-adjoint case the Min-max Theorem to obtain an upper bound for the left margin of the spectrum.

This is, of course, not possible in the non selfadjoint case which is considered in this work.

Local coordinates and approximate operator

Let $x_0 \in \mathfrak{S}^m \subset \partial\Omega_D$. In the curvilinear coordinate system (s, ρ) centered at x_0 we have

$$\Delta = \Big(\frac{1}{g}\frac{\partial}{\partial s}\Big)^2 + \frac{1}{g}\frac{\partial}{\partial \rho}\Big(g\frac{\partial}{\partial \rho}\Big) = \frac{1}{\tilde{g}^2}\frac{\partial^2}{\partial s^2} + \frac{\partial^2}{\partial \rho^2} - \frac{\rho\kappa'(s)}{\tilde{g}^3}\frac{\partial}{\partial s} - \frac{\kappa(s)}{\tilde{g}}\frac{\partial}{\partial \rho}, \tag{17}$$

where

$$g(x) := \widetilde{g}(s, \rho) = 1 - \rho\,\kappa(s)\,, \tag{18}$$

and $\kappa(s)$ is the curvature at s on $\partial\Omega$.

We next expand V in the curvilinear coordinates (s,ρ),

$$V(x)-V(x_0)=\widetilde{V}(s,\rho)-V(x_0)=c\,\rho+\frac{1}{2}\hat{\beta}\rho^2+\frac{1}{2}\alpha s^2\rho+\delta\widetilde{V}(s,\rho)\,, \quad (19)$$

where

$$c=\widetilde{V}_\rho(0)\quad,\quad \alpha=\widetilde{V}_{ss\rho}(0)\quad,\quad \hat{\beta}=\widetilde{V}_{\rho\rho}(0)\,, \quad (20)$$

and (here is involved the regularity of the boundary)

$$|\delta\widetilde{V}(s,\rho)|\leq C\,(|s|\rho^2+|\rho|^3)\,,\ \text{for } (s,\rho)\in(-s_0,s_0)\times[0,\rho_0)\,. \quad (21)$$

($c\neq 0$, we assume in the next slides that $c>0$).

Blowup

Applying the transformation

$$\tau = \left(\frac{J_m}{h^2}\right)^{1/3} \rho\,,\ \sigma = \left(\frac{\alpha_m^3}{8J_m h^4}\right)^{1/12} s\,, \qquad (22)$$

to (17) with

$$u(x) = \widetilde{u}(s,\rho) = \check{u}(\sigma,\tau)\,,$$

yields the identity

$$h^2\Delta u = (hJ_m)^{2/3}\left(\check{u}_{\tau\tau} + \varepsilon(h)\,\check{u}_{\sigma\sigma} - \varepsilon(h)\check{\kappa}(\sigma)[2J_m/\alpha_m]^{1/2}\,\check{u}_\tau + \delta\,u\right)\,, \quad (23)$$

where

$$\varepsilon(h) = \frac{\alpha_m^{1/2} h^{2/3}}{2^{\frac{1}{2}} J_m^{5/6}}\,. \qquad (24)$$

Here $\check{\kappa}(\sigma) = \kappa(s(\sigma))$ and

$$\delta u = \varepsilon(h)\Big(\frac{1}{\tilde{g}^2} - 1\Big)\,\check{u}_{\sigma\sigma} + \varepsilon(h)^{5/2}\frac{2J_m}{\alpha_m}\frac{\tau\kappa'(s(\sigma))}{\tilde{g}^3}\,\check{u}_\sigma - \varepsilon(h)\check{\kappa}(\sigma)[2J_m/\alpha_m]^{1/2}\Big(\frac{1}{\tilde{g}} - 1\Big)\,\check{u}_\tau\,. \quad (25)$$

Note that we have

$$\|\delta u\|_2 \leq C\varepsilon(h)^2\|\check{u}\|_{B^3(\mathbb{R}^2_+)}\,, \quad (26)$$

where for $\ell \in \mathbb{N}$,

$$B^\ell(\mathbb{R}^2_+) = \{\check{u} \in L^2\,,\ \sigma^p\tau^q\partial_\sigma^m\partial_\tau^n\check{u} \in L^2\,,\ \forall\, p,q,m,n \geq 0 \text{ s.t. } p+q+m+n \leq \ell\}\,.$$

Converting (19) to the coordinates (σ, τ) via (22) yields

$$\check{V}(\sigma,\tau) - V(x_0) = (hJ_m)^{2/3}\left(\tau + \varepsilon(h)\left[\sigma^2\tau + \frac{\hat{\beta}}{2^{1/2}[\alpha_m J_m]^{1/2}}\tau^2\right] + \delta\check{V}\right).$$

where

$$\|\delta V u\| \leq C\,\varepsilon(h)^{\frac{3}{2}}\,\|\check{u}\|_{B^3(\mathbb{R}^2_+)}\,. \tag{27}$$

We thus obtain the approximate problem

$$\begin{cases} -\check{u}_{\tau\tau} + i\tau\check{u} + \varepsilon(-\check{u}_{\sigma\sigma} + i\sigma^2\tau\check{u} + i\beta\tau^2\check{u} + 2\omega\check{u}_\tau) + \mathcal{O}(\varepsilon^{3/2}) = \lambda\check{u} & \text{in } \mathbb{R}^2_+ \\ \check{u}(0,\sigma) = 0 \text{ for } \sigma \in \mathbb{R} & \end{cases}, \tag{28}$$

where

$$\omega = \kappa(0)\left[\frac{J_m}{2\alpha_m}\right]^{1/2} \quad ; \quad \beta = \frac{\hat{\beta}}{[2\alpha_m J_m]^{1/2}}\,. \tag{29}$$

The formal construction

We look, in the (σ,τ) variables, for an approximate spectral pair in the form (modulo a multiplication by a cut-off function)

$$u = u_0 + \varepsilon u_1 \,,\; \lambda = \lambda_0 + \varepsilon \lambda_1 \,,$$

with u_0, u_1 in $\mathcal{S}(\overline{\mathbb{R}^2_+})$.
The leading order balance reads

$$(\mathcal{L}^+ - \lambda_0)u_0 = 0 \,, \tag{30}$$

where

$$\mathcal{L}^+ = -\frac{\partial^2}{\partial \tau^2} + i\tau \,, \tag{31a}$$

with domain

$$D(\mathcal{L}^+) = \{u \in H^2(\mathbb{R}_+) \cap H^1_0(\mathbb{R}_+) \,|\, \tau u \in L^2(\mathbb{R}_+)\} \,, \tag{31b}$$

For u_0 in the form

$$u_0(\sigma,\tau) = v(\tau)\, w_0(\sigma)\,, \tag{32}$$

(30) leads to

$$(\mathcal{L}^+ - \lambda_0)\, v = 0 \tag{33}$$

in $L^2(\mathbb{R}_+)$ and hence

$$v(\tau) = v_0(\tau), \quad \lambda_0 = -e^{-i2\pi/3}\nu_1\,.$$

Here ν_k denotes, for $k \geq 1$ the kth zero of Airy's function, and

$$v_0(\tau) = C_0\, A_i(e^{i\pi/6}\tau + \nu_1)\,, \tag{34}$$

where C_0 is determined by the condition

$$\langle \bar{v}_0, v_0 \rangle = 1\,.$$

We thus conclude that u_0 has the form

$$u_0(\sigma,\tau) = v_0(\tau)\, w_0(\sigma)\,, \tag{35}$$

where $w_0 \in \mathcal{S}(\mathbb{R})$ will be determined w_0 from the next order balance.

It has the form

$$(\mathcal{L}^+ - \lambda_0) u_1 = -\Big(-\frac{\partial^2}{\partial \sigma^2} + 2\omega \frac{\partial}{\partial \tau} + i(\sigma^2 \tau + \beta \tau^2) - \lambda_1 \Big) u_0 \quad ; \quad u_1(0,\sigma) = 0 \,. \tag{36}$$

Taking the inner product of (36) with $\bar{v}_0$ in $L^2(\mathbb{R}_+, \mathbb{C})$ we obtain that the pair (λ_1, w_0) should satisfy

$$(\mathcal{P} - \lambda_1) w_0 = 0 \,,$$

where

$$D(\mathcal{P}) = \{ u \in H^2(\mathbb{R}) \,|\, \sigma^2 u \in L^2(\mathbb{R}) \}$$

and

$$\mathcal{P} := -\frac{\partial^2}{\partial \sigma^2} + e^{i\pi/6} \tau_m \sigma^2 + \beta \tau_{m,2} \,, \tag{37}$$

with

$$\tau_m = e^{i\pi/3} \langle \bar{v}_0, \tau v_0 \rangle \,, \quad \tau_{m,2} = i \langle \bar{v}_0, \tau^2 v_0 \rangle \,. \tag{38}$$

where

$$\tau_m = \int_{\mathbb{R}_+} \tau \mathrm{Ai}^2(\tau + \nu_1) \, d\tau > 0 \text{ and } \tau_{m,2} = \int_{\mathbb{R}_+} \tau^2 \mathrm{Ai}^2(\tau + \nu_1) \, d\tau > 0 \,. \tag{39}$$

We choose λ_1 as the eigenvalue with smallest real part of the complex harmonic operator $\mathcal{P}$ and w_0 as the corresponding eigenfunction

$$w_0(\sigma) = \hat{C}_0 \exp\left\{ - \left[\frac{\tau_m}{2}\right]^{1/2} e^{i\frac{\pi}{12}} \sigma^2 \right\} \quad , \quad \lambda_1 = \sqrt{2\tau_m} e^{i\tau_m \frac{\pi}{12}} + \beta \tau_{m,2} \,, \tag{40}$$

where $\hat{C}_0$ is chosen so that

$$\int_{\mathbb{R}} w_0(\sigma)^2 \, d\sigma = 1 \,.$$

Note that this is NOT $\int_{\mathbb{R}} |w_0(\sigma)|^2 \, d\sigma$.

With this choice of λ_1, the function $u_1 \in \mathcal{S}(\overline{\mathbb{R}^2_+})$ must satisfy

$$(\mathcal{L}^+ - \lambda_0)u_1 = -i[\sigma^2(\tau - e^{-i\pi/3}\tau_m) + \beta(\tau^2 - i\tau_{m,2}) + 2i\omega\partial_\tau]u_0 \quad ; \quad u_1(0,\sigma) = 0\,. \tag{41}$$

Let Π_0 denote the spectral projection of $L^2(\mathbb{R}_+,\mathbb{C})$ on $\mathrm{span}\, v_0$, defined by:

$$\Pi_0 u = \langle u, \bar{v}_0\rangle_\tau\, v_0\,, \tag{42}$$

where $\langle\cdot,\cdot\rangle_\tau$ denotes the inner product in $L^2(\mathbb{R}_+,\mathbb{C})$ with respect to the τ variable.

Consequently we may write

$$u_1(\sigma,\tau) = w_1(\sigma)v_0(\tau) + \hat{u}_1(\sigma,\tau)\,,$$

where $\hat{u}_1 \in (I - \Pi_0)L^2(\mathbb{R}^2_+)$ and we set $w_1 = 0$ in the sequel, as a two-term expansion satisfies our needs.

With Fredholm alternative in mind, we look for $u_1(\sigma,\tau)$ in the form

$$u_1(\sigma,\tau) = -i\sigma^2 w_0(\sigma)u_{11}(\tau) + w_0(\sigma)\big(\beta u_{12}(\tau) + \omega u_{13}(\tau)\big)\,,$$

where $u_{11}(\tau)$ is the unique solution in $\mathrm{Im}\,(I-\Pi_0)$ of

$$(\mathcal{L}^+ - \lambda_0)u_{11}(\tau) = (\tau - e^{-i\pi/3}\tau_m)v_0(\tau)\,,\ u_{11}(0) = 0\,,$$

$u_{12}(\tau)$ is the unique solution in $\mathrm{Im}\,(I-\Pi_0)$ of

$$(\mathcal{L}^+ - \lambda_0)u_{12}(\tau) = (\tau^2 - i\tau_{m,2})v_0(\tau)\,,\ u_{12}(0) = 0\,.$$

and $u_{13}(\tau)$ is the unique solution in $\mathrm{Im}\,(I-\Pi_0)$ of

$$(\mathcal{L}^+ - \lambda_0)u_{13}(\tau) = v_0'(\tau)\,,\ u_{13}(0) = 0\,.$$

The above equations are uniquely solvable, since their right hand sides are all orthogonal to $\bar{v}_0$, and since $(\mathcal{L}^+ - \lambda_0)_{/\mathrm{Im}\,(I-\Pi_0)}$ is invertible.

We have thus determined λ_1 and $u_1 \in \mathcal{S}(\overline{\mathbb{R}^2_+})$, providing sufficient accuracy for the derivation of the upper bound.

We can actually continue the construction to any order but this is not needed for the proof of the main theorems.

From a quasimode construction to the existence of an eigenvalue

A very technical analysis (using localized Resolvent estimates) (30 pages!) leads to the following semi-classical result:

Proposition–Case V1

For any $\hat{q} < \frac{1}{9}$, there exist positive constants C and h_0 such that, for all $h \in (0, h_0]$,

$$\sup_{\lambda \in \partial B(\hat{\Lambda}(h), h^{\frac{2}{3}+\hat{q}})} \|(\mathcal{A}_h - \lambda)^{-1}\| \leq C\, h^{-(\frac{2}{3}+\hat{q})}, \tag{43}$$

where $\hat{\Lambda}(h)$ is the above constructed approximate eigenvalue.

We can now prove the upper bound for the spectrum.

Proposition (Case V1)

There exist $h_0 > 0$ and, for $h \in (0, h_0]$ an eigenvalue $\lambda \in \sigma(\mathcal{A}_h)$ satisfying

$$\lambda - \hat{\Lambda}(h) = o(h^{\frac{2}{3}}) \quad \text{as } h \to 0 . \tag{44}$$

Proof

Let U be the quasimode state. Clearly,

$$(\mathcal{A}_h - \lambda)U = f + (\hat{\Lambda}(h) - \lambda)U\,,$$

with $||f|| = \mathcal{O}(h)||U||$.

Hence, for $\lambda \in \partial B(\hat{\Lambda}(h), h^{\frac{2}{3}+\hat{q}})$, we can write

$$\langle U, (\mathcal{A}_h - \lambda)^{-1} U\rangle = -\frac{1}{\lambda - \hat{\Lambda}(h)}[\langle U, U\rangle - \langle U, (\mathcal{A}_h - \lambda)^{-1} f\rangle]\,.$$

We then obtain

$$\|(\mathcal{A}_h-\lambda)^{-1}f\|_2 \leq \frac{C}{h^{\frac{2}{3}+\hat{q}}}\,\|f\|_2 \leq C\,h^{\frac{1}{3}-\hat{q}}\|U\|\,.$$

Consequently,

$$\Big|\frac{1}{2\pi i}\oint_{\partial B(\Lambda(h),h^{\frac{2}{3}+\hat{q}})}\langle U,(\mathcal{A}_h-\lambda)^{-1}f\rangle\,d\lambda+\|U\|^2\Big| \leq C\,h^{\frac{1}{3}-\hat{q}}\,\|U\|^2\,.$$

Hence there exists $h_0>0$ such that, for $h\in(0,h_0]$, $(\mathcal{A}_h-\lambda)^{-1}$ is not holomorphic in $B(\Lambda(h),h^{\frac{2}{3}+\hat{q}})$ and the proposition follows. □

The existence of this eigenvalue provides an effective upper bound for $\inf \mathrm{Re}\,\sigma(\mathcal{A}_h)$.

The most difficult is to have resolvent estimates for approximating models (after blowing up) in $\tau > 0$

$$u \mapsto -u_{\tau\tau} + i\tau u + \varepsilon(-u_{\sigma\sigma} + i\sigma^2\tau u + i\beta\tau^2 u + 2\omega u_\tau)\,,$$

where $\varepsilon > 0$ is small enough.
We just mention, various difficulties:

- The operators are "strongly" non self-adjoint.
- The term $\sigma^2\tau$ is mixing the variables.
- We have no control of the sign of β. Hence we should introduce a cut-off.

On the contrary, we know a lot for the resolvent estimates dot the complex Airy operator $D_t^2 + it$ (on the line or the half-line with Dirichlet condition) or the complex harmonic oscillator $D_s^2 + is^2$.

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