Integrability of the Brouwer degree and chain rules for distributional Jacobians

Heiner Olbermann

26 April 2018

Workshop “Transitions de phase et équations non-locales”,
Institut de l’Académie Roumaine

UNIVERSITÄT LEIPZIG

Overview

The Brouwer degree

- paracompact oriented manifold M of dimension n
- $U \subset \mathbb{R}^n$ bounded
- $u \in C^\infty(\overline{U}; M)$
- $z \in M \setminus u(\partial U)$
- smooth n-form μ on M with support in the same connected component of $M \setminus u(\partial U)$ as z with $\int_M \mu = 1$

Then

$$\deg(u, U, z) = \int_U u^*(\mu)\,,$$

For regular $z \in M \setminus u(\partial U)$,

$$\deg(u, U, z) = \sum_{x \in u^{-1}(\{z\})} \operatorname{sgn} \det Du(x)\,.$$

Statement of main result

Let $U \subset \mathbb{R}^n$ be open and bounded with $\mathrm{dim}_{\mathrm{box}} \partial U = d \in [n-1, n]$, $\alpha \in (d/n, 1]$ and $u \in C^{0,\alpha}(U; \mathbb{R}^n)$.

Theorem (O. '15)

Then

$$\|\deg(u, U, \cdot)\|_{L^p} \leq C(n, U, \alpha, d, p) \|u\|_{C^{0,\alpha}(U;\mathbb{R}^n)}^{n/p}$$

for any $1 < p < \frac{n\alpha}{d}$.

Motivation: The $C^{1,\alpha}$ isometric embedding problem

Why are the integrability properties of $\deg$ interesting?

Theorem (Borisov 1960, Conti, De Lellis, Székelyhidi 2012)

Let $\alpha > \frac{2}{3}$, let (S^2, g) be a Riemannian manifold with positive Gauss curvature, and let $y \in C^{1,\alpha}(S^2; \mathbb{R}^3)$ be an isometric immersion. Then y is rigid, i.e. the unique isometric immersion up to Euclidean motions.

Crucial part in the proof of this Theorem: Show that the normal ν_y has *bounded extrinsic curvature*, i.e.,

$$\sup\left\{\sum_{i=1}^{N}\mathcal{H}^2(\nu_y(E_i)) :\, E_i \subset M \text{ closed disjoint for } i = 1,\dots,N\right\} < \infty$$

ν_y=unit normal to the the immersed manifold

"Phase transition" h-principle/rigidity

- Nash, Kuiper 1950's: For every short immersion $y : \mathbb{R}^2 \supset U \to \mathbb{R}^3$, there exists a C^1-isometric immersion arbitrarily close in C^0
- De Lellis, Inauen, Székelyhidi '15: Also valid for $C^{1,\alpha}$ isometric immersions with $\alpha < \frac{1}{5}$

Isometric embedding of the flat torus in $\mathbb{R}^3$

(Borrelli, Jabrane, Lazarus, Thibert, PNAS '12)

Question (e.g. Yau, "Open problems in geometry")

Does there exist a critical exponent $\bar{\alpha}$ such that the $C^{1,\alpha}$-Weyl problem is rigid for $\alpha > \bar{\alpha}$ and there is an *h*-principle for $\alpha < \bar{\alpha}$?

$C^{1,\alpha}$ rigidity: Main step in the proof

Look at $E_i \subset U$ closed disjoint for $i = 1, \dots, N$
Want to estimate

$$\sum_{i=1}^{N} \mathcal{H}^2(\nu_y(E_i)) = \sum_{i=1}^{N} \int_{S^2} \chi_{\nu_y(E_i)}(x) \mathrm{d}\mathcal{H}^2(x)$$

Proposition

Under the assumption $\nu_y \in C^{0,\alpha}$, we have for all $\psi \in C^\infty(S^2)$,

$$\int_{S^2} \psi(z) \deg(\nu, E_i, z) \mathrm{d}\mathcal{H}^2(z) = \int_{E_i} \psi(\nu_y(x)) \kappa_y(x) \sqrt{\det g_y(x)} \mathrm{d}x \,,$$

and for $\kappa_y > 0$, we have additionally $\chi_{\nu_y(E_i)} \leq \deg(\nu, E_i, \cdot)$.

κ_y=Gauss curvature of the immersed manifold
Euclidean version: $\int_{\mathbb{R}^n} \psi(z) \deg(u, U, z) \mathrm{d}z = \int_U \psi(u(x)) \det Du(x) \mathrm{d}x$

Plan of proof

- For smooth u, we have

$$\int_{\mathbb{R}^n} \varphi(z)\deg(u, U, z)\mathrm{d}z = \int_U \varphi(u(x)) \det Du(x)\mathrm{d}x \,,$$

  and this allows for an estimate of

$$\|\deg(u, U, \cdot)\|_{L^p} = \sup_{\|\varphi\|_{L^{p'}} \leq 1} \langle \deg(u, U, \cdot), \varphi \rangle \ .$$

- Multiply $\deg(u, U, \cdot) \in L^p$ with a test function $\varphi \in L^{p'}$, and use the trick

$$\varphi(u(x)) \det Du(x) = \sum_{i=1}^{n} \det Dv^i(x) \,,$$

  where $v^i := (u_1, \dots, u_{i-1}, \psi_i \circ u, u_{i+1}, \dots, u_n)$, and ψ is a solution of $\operatorname{div} \psi = \varphi$.

Plan of the proof continued

- For $u \in C^{0,\alpha}$, define the Jacobian determinant $\det Du$ by multi-linear interpolation, and choose α high enough, so that "$\det Du = \operatorname{div} f$ with $f \in C^{0,\beta}$"
- In this way, make rigorous sense of the change of variables formula

$$\langle [Ju], \chi_U \rangle = \int_U \det Du \mathrm{d}x = \int_{\mathbb{R}^n} \deg(u, U, y) \mathrm{d}y$$

- For Hölder functions $f \in C^{0,\beta}$ and bounded sets $U \subset \mathbb{R}^n$ with $\dim_{\mathrm{box}} \partial U = d$, we may define the integral

$$\int_U \operatorname{div} f \mathrm{d}x$$

with the help of the Gauss-Green theorem if $\beta > d - (n-1)$ (see `Harrison, Norton, 1991`)

Definition of the weak Jacobian

- Define

$$\begin{aligned}\mathbf{j} : C^\infty(U;\mathbb{R}^n) &\to C^\infty(U;\mathbb{R}^n)\\ u &\mapsto \frac{1}{n} u \operatorname{cof} Du\end{aligned}$$

- $\mathbf{j}$ is chosen such that $\operatorname{div}\mathbf{j}u = \det Du$.
- Define $[Ju] \in (C^1_c(U))^*$ by $\langle [Ju], \varphi\rangle := \int_U \mathbf{j}u \cdot D\varphi \, \mathrm{d}x$

Theorem (Brezis, Nguyen '14)

For $u, v \in C^\infty(U;\mathbb{R}^n)$ and $\varphi \in C^1_c(U)$, we have

$$\langle [Ju]-[Jv], \varphi\rangle \lesssim |u-v|_{W^{(n-1)/n,n}} \left(|u|^{n-1}_{W^{(n-1)/n,n}} + |v|^{n-1}_{W^{(n-1)/n,n}}\right) \|D\varphi\|_{L^\infty}\,.$$

Distributional Jacobian as a trace

- Interpolation spaces as trace spaces:

$$u \in W^{(n-1)/n,n}(U) \quad \Leftrightarrow \begin{cases} \varepsilon \mapsto \frac{\mathrm{d}}{\mathrm{d}\varepsilon} u_\varepsilon \in L^n(\mathbb{R}^+, L^n(U)) \\ \varepsilon \mapsto u_\varepsilon \in L^n(\mathbb{R}^+, W^{1,n}(U)) \end{cases}$$

- Estimates are computed via the equation $\langle [Ju], \varphi \rangle = \int_0^\infty \frac{\mathrm{d}}{\mathrm{d}\varepsilon} \langle \mathbf{j} u_\varepsilon, D\varphi \rangle \,\mathrm{d}\varepsilon$
- In this representation, the Null Lagrangian property of the determinant can be exploited to shift the derivatives

Definition of the weak Jacobian, Hölder setting

Set $X_{00} = \{\omega \in C^\infty(U;\mathbb{R}^n) : \operatorname{div}\omega = 0\}$ and define two norms on the quotient space $C^\infty(U;\mathbb{R}^n)/X_{00}$:

$$\|\omega\|_{X_0} := \inf\{\|\omega+\alpha\|_{C^0} : \alpha \in X_{00}\}$$
$$\|\omega\|_{X_1} := \|\operatorname{div}\omega\|_{C^0}$$

Lemma

Let $U \subset \mathbb{R}^n$ be bounded and open, let $u_1,\dots,u_n \in C^\infty(U)$, and for $i = 1,\dots,n$, let $\alpha_i \in (0,1)$ such that $\theta := \left(\sum_{i=1}^n \alpha_i\right) - (n-1) > 0$. Then

$$\|\mathbf{j}u\|_{(X_0,X_1)_{\theta,\infty}} \leq C(n,\alpha_1,\dots,\alpha_n)\prod_{i=1}^n \|u_i\|_{C^{0,\alpha_i}(U)},$$

and hence $\mathbf{j}$ extends to a multi-linear operator $C^{0,\tilde\alpha_1}(U)\times\cdots\times C^{0,\tilde\alpha_n}(U) \to (X_0,X_1)_{\theta,\infty}$ for $\tilde\alpha_i > \alpha_i$, $i = 1,\dots,n$.

Whitney decomposition

Lemma (Whitney∼1930s)

There exists a countable collection $W = \{Q_i : i \in \mathbb{N}\}$ of cubes Q_i with the following properties:

- *For every $Q \in W$, there exist $m \in \mathbb{Z}^n, k \in \mathbb{Z}$ such that $Q = 2^{-k}(m + (0,1)^n)$. For fixed k, the union of cubes for which this holds (for some m) is denoted by W_k.*
- $U \subset \cup_{Q \in W} \overline{Q}$
- *The cubes in W are mutually disjoint*
- $\operatorname{dist}(Q, \partial U) \leq \operatorname{diam} Q \leq 4 \operatorname{dist}(Q, \partial U)$ *for all* $Q \in W$

With $u \equiv u(t)$, $t > 0$, a representative of $u \in C^{0,\alpha} = (C^0, C^1)_{\alpha,\infty}$, we are going to estimate

$$\left| \int_U \operatorname{div} \mathbf{j} u dx \right| \leq \sum_{i \in \mathbb{N}} \left| \int_{Q_i} \operatorname{div} \mathbf{j} u dx \right| .$$

Estimate on a cube

- Consider Whitney decomposition of U into cubes
- Let Q be such a cube. We have

$$
\begin{aligned}
\mathbf{j}u(0) =& \mathbf{j}u(0) - \mathbf{j}u(t) + \mathbf{j}u(t) \\
=& - \int_0^t (\mathbf{j}u)'(s)\mathrm{d}s + \mathbf{j}u(t)
\end{aligned}
$$

$$
\begin{aligned}
\Rightarrow \left| \int_Q \operatorname{div} \mathbf{j}u(0)\mathrm{d}x \right| &\le \int_0^t \mathrm{d}s \left| \int_{\partial Q} (\mathbf{j}u)'(s)\mathrm{d}\sigma \right| + \left| \int \operatorname{div} \mathbf{j}u(t)\mathrm{d}x \right| \\
&\le \int_0^t \mathrm{d}s\, s^{-\theta} \mathcal{H}^{n-1}(\partial Q) \|\mathbf{j}u\|_{(X_0,X_1)_{\theta,\infty}} \\
&\quad + \mathcal{L}^n(Q) t^{-\theta} \|\mathbf{j}u\|_{(X_0,X_1)_{\theta,\infty}} \\
&\overset{t:=\mathcal{L}^n(Q)/\mathcal{H}^{n-1}(\partial Q)}{\le} C\mathcal{H}^{n-1}(\partial Q)^{1-\theta} \mathcal{L}^n(Q)^{\theta} \|\mathbf{j}u\|_{(X_0,X_1)_{\theta,\infty}} .
\end{aligned}
$$

How many cubes of given size are there?

$N_r(\partial U) :=$ number of n-dimensional cubes of side length r that is required to cover ∂U

Definition

$$\dim_{\text{box}}(\partial U) = \lim_{r \to 0} \frac{\log N_r(\partial U)}{-\log r},$$

Theorem (Martio, Vuorinen 1987)

$$\lim_{k \to \infty} \frac{\log_2 \# W_k}{k} = \dim_{\text{box}}(\partial U).$$

Summing up Whitney cubes

From the estimate for a single cube and the estimate on $\#W_k$, we get (with $d = \mathrm{dim}_{\mathrm{box}}\partial U$)

$$\sum_{i\in\mathbb{N}} \left| \int_{Q_i} \mathrm{div}\, \mathbf{j}u \mathrm{d}x \right| \leq C \sum_{k\geq k_0} 2^{kd}(2^{-k(n-1)})^{1-\theta}(2^{-kn})^{\theta} \|\mathbf{j}u\|_{(X_0,X_1)_{\theta,\infty}}$$
$$\leq \sum 2^{k(d-(n-1)-\theta)} \|\mathbf{j}u\|_{(X_0,X_1)_{\theta,\infty}} .$$

Now we use the trick

$$\mathrm{div}\,(\psi(u)\mathrm{cof}\, Du) = \mathrm{Tr}(D\psi(u)\mathrm{Id}_{n\times n} \det Du) = (\mathrm{div}\,\psi)(u) \det Du$$
$$= \sum_{i=1}^{n} \det Dv^i$$

with $v^i(x) = (u_1, \dots, u_{i-1}, \psi_i \circ u, u_{i+1}, \dots, u_n)$ and $\mathrm{div}\,\psi = \varphi \in L^{p'}$.

- By standard L^p theory, $\|\psi\|_{W^{1,p'}} \lesssim \|\varphi\|_{L^{p'}}$
- By the Sobolev embedding for $p' > n$, $\|\psi\|_{C^{0,1-n/p'}} \lesssim \|\psi\|_{W^{1,p'}}$
- With $\tilde{\alpha} := (1 - n/p')\alpha$, we have $\|\psi_i \circ u\|_{C^{0,\tilde{\alpha}}} \lesssim \|\psi\|_{C^{0,1-n/p'}} \|u\|_{C^{0,\alpha}}^{1-n/p'}$.

Hence, with $\theta = n(1 - 1/p')\alpha - (n-1) = n\alpha/p - (n-1)$,

$$\|\mathbf{j}v^i\|_{(X_0,X_1)_{\theta,\infty}} \leq C\|\varphi\|_{L^{p'}}\|u\|_{C^{0,\alpha}}^{n/p}.$$

Thus we get for $\|\varphi\|_{L^{p'}} \leq 1$,

$$\begin{aligned}\int_{\mathbb{R}^n} \varphi(z)\deg(u, U, z)\mathrm{d}z &\overset{\text{morally}}{=} \int_U \varphi(u(x)) \det Du(x)\mathrm{d}x \\ &= \sum_i \int_U \operatorname{div} \mathbf{j}v^i \mathrm{d}x \\ &\leq \sum_i \sum_k 2^{k(d-(n-1)-\theta)} \|\mathbf{j}v^i\|_{(X_0,X_1)_{\theta,\infty}} \\ &\leq C\|u\|_{C^{0,\alpha}}^{n/p}. \quad \square\end{aligned}$$

A slight improvement

Let $U \subset \mathbb{R}^n$ be open and bounded with $\dim_{\text{box}} \partial U = d \in [n-1, n]$, $\alpha \in (d/n, 1]$ and $u \in C^{0,\alpha}(U; \mathbb{R}^n)$.

Theorem (O. '15)

Then

$$\|\deg(u, U, \cdot)\|_{L^p} \leq C(n, U, \alpha, d, p)\|u\|^{n/p}_{C^{0,\alpha}(U;\mathbb{R}^n)}$$

for any $1 < p < \frac{n\alpha}{d}$.

Theorem (De Lellis, Inauen '17)

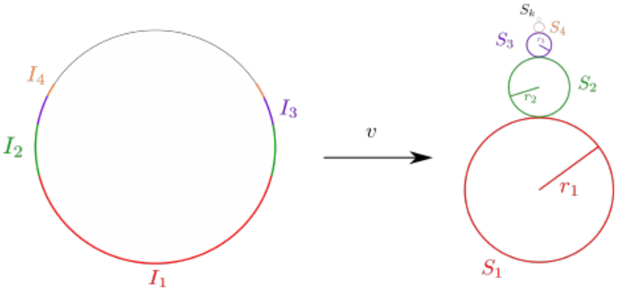
Then

$$|\deg(u, U, \cdot)|_{W^{\beta,p}} \leq C(U, n, \alpha, \beta, p)|u|^{n/p-\beta}_{C^{0,\alpha}}$$

for any pair (β, p) *with* $p \geq 1$ *and* $0 \leq \beta < \frac{n}{p} - \frac{d}{\alpha}$.

Counterexample for $d = n - 1 = 1$, $n\alpha < pd$

- De Lellis, Inauen '17: Consider $U = B(0,1) \subset \mathbb{R}^2$, and define $u \in C^{0,\alpha}(\partial B_1; \mathbb{R}^2)$ as in the sketch:



- $u(I_k)$ needs to cover S_k, k times. Then $u \in C^{0,\alpha}$, $\deg(u, U, \cdot) \notin L^1(\mathbb{R}^n)$.

Extrinsic curvature in higher dimension

With similar methods, one can show:

Theorem (Behr, O. '16)

Let (M,g) be a $2m$-dimensional Riemannian manifold with positive Pfaffian form, $\alpha > 2m/(2m+1)$, and let $y \in C^{1,\alpha}(M;\mathbb{R}^{2m+1})$ be an isometric immersion. Then the surface $y(M)$ has bounded extrinsic curvature.

- Pfaffian form:

$$\mathrm{Pf}(\Omega) = \frac{1}{2^m m!} \sum_{\sigma \in \mathrm{Sym}(2m)} \Omega_{\sigma(1)}^{\sigma(2)} \wedge \cdots \wedge \Omega_{\sigma(2m-1)}^{\sigma(2m)},$$

  where Ω_i^j are the curvature forms on (M,g).
- Crucial additional ingredient: Gauss-Bonnet-Chern.

Chain rules for distributional Jacobians

- With $C^0 \ni \varphi = \operatorname{div} \psi$ as before, define $\Psi_i(x) = (x_1, \dots, x_{i-1}, \psi_i(x), x_{i+1}, \dots, x_n)$. We have shown

$$\sum_i \int_U \operatorname{div} \mathbf{j}(\Psi_i \circ u) \mathrm{d}x = \int_{\mathbb{R}^n} \varphi(z) \deg(u, U, z) \mathrm{d}z\,.$$

  This is also true for $n = 2$, $u \in C^{0,\alpha}$ with $\alpha > \frac{1}{2}$.

- If we have that $[Ju]$ is a Radon measure, and validity of the chain rule

$$\langle [J(\Psi_i \circ u)], \varphi \rangle_{(C_c^1)^*, C_c^1} = \langle [Ju], \varphi \det D\Psi_i(u) \rangle_{\mathcal{M}, C^0}$$

  for all test function φ, then we can use the same arguments as Conti, De Lellis, Székelyhidi to prove rigidity in the $C^{1,\alpha}$ Weyl problem with $\alpha > \frac{1}{2}$. This is the so-called strong chain rule.

Strong chain rule

- For $a \in \mathbb{R}^n$, let $u^a = \frac{u-a}{|u-a|}$.
- De Lellis, '03: Let $u \in W^{1,p}(U;\mathbb{R}^n)$ be continuous with $p > n-1$. If
$$\int_{\mathbb{R}^n} |[Ju^a]|_{\mathcal{M}} \,\mathrm{d}a < \infty\,,$$
then the strong chain rule holds. In particular, this is true for $u \in W^{1,n} \cap C^0(U;\mathbb{R}^n)$

Theorem (Strong chain rule; Gladbach, O. '18)

Let $u \in W^{n/(n+1),n+1}(U;\mathbb{R}^n)$ such that $[Ju]$ defines a Radon measure, and $F \in C^2_c(\mathbb{R}^n;\mathbb{R}^n)$. Then

$$\langle [J(F\circ u)], \varphi\rangle_{(C^1_c)^*, C^1_c} = \langle [Ju], \varphi \det DF(u)\rangle_{\mathcal{M}, C^0}$$

for all $\varphi \in C^1_c(U)$.

Weak coarea formula

For smooth $u, \partial U$ and $a \notin u(\partial U)$,

$$\deg(u, U, a) = \frac{1}{\mathcal{L}^n(B(0,1))} \int_{\partial U} \mathbf{j} u^a \mathrm{d}s \quad ,$$

where $u^a(x) = \frac{u(x)-a}{|u(x)-a|}$.

Theorem (Jerrard, Soner, '02)

Let $u \in W^{1,n-1} \cap L^\infty(U; \mathbb{R}^n)$. Then $u^a \in W^{1,n-1} \cap L^\infty(U; \mathbb{R}^n)$ for a.e. $a \in \mathbb{R}^n$ and the weak coarea formula holds:

$$\langle [Ju], \varphi \rangle = \frac{1}{\mathcal{L}^n(B(0,1))} \int_{\mathbb{R}^n} \langle [Ju^a], \varphi \rangle \, \mathrm{d}a \, .$$

Weak chain rule

- Jerrard, Soner '02: Let $u \in W^{1,n-1} \cap L^\infty(U;\mathbb{R}^n)$. Then $u^a \in W^{1,n-1} \cap L^\infty(U;\mathbb{R}^n)$ for a.e. $a \in \mathbb{R}^n$ and the *weak* chain rule holds:

$$\langle [J(F\circ u)], \varphi\rangle = \frac{1}{\mathcal{L}^n(B(0,1))}\int_{\mathbb{R}^n} \det DF(a)\,\langle [Ju^a],\varphi\rangle\,\mathrm{d}a\,.$$

Theorem (Weak coarea formula + chain rule; Gladbach, O. '18)

Let $u \in W^{s,n} \cap L^\infty(U;\mathbb{R}^n)$ with $s > (n-1)/n$, and $F \in C^1(\mathbb{R}^n;\mathbb{R}^n)$. Then we have that for all $\varphi \in C^1_c(\mathbb{R}^n)$,

$$\langle [Ju], \varphi\rangle = \frac{1}{\mathcal{L}^n(B(0,1))}\int_{\mathbb{R}^n} \langle [Ju^a],\varphi\rangle\,\mathrm{d}a$$

$$\langle [J(F\circ u)], \varphi\rangle = \frac{1}{\mathcal{L}^n(B(0,1))}\int_{\mathbb{R}^n} \det DF(a)\,\langle [Ju^a],\varphi\rangle\,\mathrm{d}a\,.$$

Literature

- H. O., "Integrability of the Brouwer degree for irregular arguments", *Ann. Inst. H. Poincaré Analyse Non-Linéaire* 34-4, 933-959, 2017.
- S. Behr, H. O., "Extrinsic curvature of codimension one isometric immersions with Hölder continuous derivatives" *arXiv preprint* 1601.05959, 2016.
- C. De Lellis, D. Inauen, "Fractional Sobolev regularity for the Brouwer degree", *Communications in Partial Differential Equations* 42-10, 1510-1523, 2017.