# Asymptotic behaviour for fractional diffusion-convection equations

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Joint work with Diana Stan (BCAM-Spain)

# Fractional Diffusion Convection

We study the following nonlocal model:

$$u_t(t,x) + (-\Delta)^{\alpha/2} u(t,x) + (f(u))_x = 0 \quad \text{(CD)}$$

for $t > 0$ and $x \in \mathbb{R}$,

where

- $u : (0,\infty) \times \mathbb{R} \to \mathbb{R}$
- $(-\Delta)^{\alpha/2}$ is the Fractional Laplacian operator of order $\alpha \in (0,2)$

$$(-\Delta)^{\alpha/2} u(x) = C_{n,\alpha} \int_{\mathbb{R}^d} \frac{u(x) - u(y)}{|x-y|^{n+\alpha}} dy$$

- $f(\cdot)$ is a locally Lipschitz function whose prototype is

$$f(s) = |s|^{q-1} s/q$$

  with $q > 1$.

# Few words about local diffusion problems

$$\begin{cases} u_t - \Delta u = 0 \quad \text{in } \mathbb{R}^d \times (0,\infty), \\ u(0) = u_0. \end{cases}$$

For any $u_0 \in L^1(\mathbb{R})$ the solution $u \in C([0,\infty), L^1(\mathbb{R}^d))$ is given by:

$$u(t,x) = (G(t,\cdot) * u_0)(x)$$

where

$$G(t,x) = (4\pi t)^{-d/2} \exp(-\frac{|x|^2}{4t})$$

Smoothing effect

$$u \in C^\infty((0,\infty), \mathbb{R}^d)$$

Decay of solutions, $1 \leq p \leq q \leq \infty$:

$$\|u(t)\|_{L^q(\mathbb{R}^d)} \lesssim t^{-\frac{d}{2}(\frac{1}{p}-\frac{1}{q})} \|u_0\|_{L^p(\mathbb{R}^d)}$$

# Asymptotics

**Theorem**

*For any* $u_0 \in L^1(\mathbb{R}^d)$ *and* $p \geq 1$ *we have*

$$t^{\frac{d}{2}(1-\frac{1}{p})}\|u(t) - MG_t\|_{L^p} \to 0,$$

*where* $M = \int u_0$.

Proof:

$$(G_t * u_0)(x) - MG_t(x) = \frac{1}{(4\pi t)^{d/2}} \int_{\mathbb{R}^d} (\exp(-\frac{|x-y|^2}{4t}) - \exp(-\frac{|x|^2}{4t}))u_0(y)dy$$

$+$ Taylor expansion with integral reminder, etc...

# A linear nonlocal problem

E. Chasseigne, M. Chaves and J. D. Rossi, *Asymptotic behavior for nonlocal diffusion equations*, J. Math. Pures Appl., 86, 271–291, (2006).

$$\begin{cases} u_t(x,t) & = J * u - u(x,t) = \int_{\mathbb{R}^d} J(x-y)u(y,t)\,dy - u(x,t), \\ & = \int_{\mathbb{R}^d} J(x-y)(u(y,t)-u(x,t))dy \\ u(x,0) & = u_0(x), \end{cases}$$

where $J : \mathbb{R}^N \to \mathbb{R}$ be a nonnegative, radial function with $\int_{\mathbb{R}^N} J(r)dr = 1$

# Nonlocal models

There are two different models

Case 1: $s \in (0,1)$,

$$\frac{c_1}{|y-x|^{d+2s}} \leq J(x,y) \leq \frac{c_2}{|y-x|^{d+2s}}$$

Case 2: essentially $J$ is a nice function, $(1+|x|^2)J(x) \in L^1(\mathbb{R})$,
$J = \frac{1}{1+x^2}, J = e^{-|x|}$

- L.I, J.D. Rossi, JFA2007, JEE2008, JMPA2009,
- L.I, T. Ignat, D. Stancu, SIAM 2015
- L.I., C. Cazacu, A. Pazoto, Nonlinearity 2017

# Local Case

$$u_t - \Delta u + \bar{b} \cdot \nabla(|u|^{q-1}u) = 0$$

- EZ for the supercritical case $q > 1 + 1/N$ and critical case $q = 1 + 1/N$ in $\mathbb{R}^N$.
- EVZ for the subcritical case $1 < q < 2$ in dimension $N = 1$.
- Subcritical case $q < 1 + 1/N$ in any dimension $N \geq 1$: EVZ for nonnegative solutions and Carpio for changing sign solutions.


M. Escobedo and E. Zuazua, "Large time behavior for convection-diffusion equations in $\mathbf{R}^N$," *J. Funct. Anal.*, vol. 100, no. 1, pp. 119–161, 1991.

M. Escobedo, J. L. Vázquez, and E. Zuazua, "Asymptotic behaviour and source-type solutions for a diffusion-convection equation," *Arch. Rational Mech. Anal.*, vol. 124, no. 1, pp. 43–65, 1993.

M. Escobedo, J. L. Vázquez, and E. Zuazua, "A diffusion-convection equation in several space dimensions," *Indiana Univ. Math. J.*, vol. 42, no. 4, pp. 1413–1440, 1993.

A. Carpio, "Large time behaviour in convection-diffusion equations," *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, vol. 23, no. 3, pp. 551–574, 1996.

# Large time asymptotic expansion (1D), $u_0 \in L^1(\mathbb{R})$

$$\begin{cases} u_t(t,x) - \Delta u(t,x) + (|u|^{q-1}u)_x = 0 & \text{for } t>0 \text{ and } x \in \mathbb{R}, \\ u(0,x) = u_0(x) & \text{for } x \in \mathbb{R}. \end{cases}$$

Then

$$t^{\frac{1}{\alpha(q)}(1-\frac{1}{p})} \|u(t,\cdot) - U_M(t,\cdot)\|_{L^p(\mathbb{R})} \to 0, \quad \text{as } t \to \infty$$

where

- (EZ) If $q > 2$ then $\alpha(q) = 2$, $U_M$ is the fundamental solution of the Heat Equation:

$$\begin{cases} U_t(t,x) = \Delta U(t,x) & \text{for } t>0 \text{ and } x \in \mathbb{R}, \\ U(0,x) = M\delta(x) & \text{for } x \in \mathbb{R}. \end{cases}$$

- (EVZ) If $1 < q < 2$ then $\alpha(q) = q$, $U_M$ is the unique entropy solution of the Conservation law

$$\begin{cases} U_t(t,x) + (f(U))_x = 0 & \text{for } t>0 \text{ and } x \in \mathbb{R}, \\ U(0,x) = M\delta(x) & \text{for } x \in \mathbb{R}. \end{cases}$$

- (EZ) If $q = 2$ then $U_M$ is a self-similar solution of viscous Burger's eq:

$$U(x,t;M) = t^{-1/2}F(xt^{1/2};M) \quad \text{with } F(\eta,M) = \frac{e^{-\eta^2/4}}{K + \frac{1}{2}\int_0^{\eta} e^{-\xi^2/4}} d\xi.$$

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# A nonlinear model: convection-diffusion

For $q \geq 1$

$$\begin{cases} u_t - \Delta u + (|u|^{q-1}u)_x = 0 \text{ in } (0,\infty) \times \mathbb{R} \\ u(0) = u_0 \end{cases}$$

- Decay of the solutions by using

$$\frac{d}{dt}\int_{\mathbb{R}} |u|^p dx = -\frac{4(p-1)}{p}\int_{\mathbb{R}} |\nabla(|u|^{p/2})|^2 dx.$$


M. Schonbek, *Uniform decay rates for parabolic conservation laws*, Nonlinear Anal., 10(9), 943–956, (1986).

M. Escobedo and E. Zuazua, *Large time behavior for convection-diffusion equations in* $\mathbb{R}^N$, J. Funct. Anal., 100(1), 119–161, (1991).

# Some ideas of the proof

• For $q > 2$

$$u(t) = S(t)u_0 + \int_0^t S(t-s)(u^q)_x(s)ds$$

and use that the nonlinear part decays faster than the linear one

• $q = 2$ scaling: introduce $u_\lambda(x,t) = \lambda u(\lambda x, \lambda^2 t)$, write the equation for $u_\lambda$ and observe that the estimates for $u$ are equivalent to the fact that

$$u_\lambda(x,1) \to f_M(x) \text{ in } L^1(\mathbb{R})$$

Proof: the so-called "four step method" :

- scaling - write the equation for $u_\lambda$
- estimates and compactness of $\{u_\lambda\}$
- passage to the limit
- identification of the limit

• $1 < q < 2$, read EVZ's paper, entropy solutions, Main ideea: Oleinik estimate

$$(u^{q-1})_x \leq \frac{1}{t}.$$

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# Nonlocal general model

The general model is

$$\begin{cases} u_t(t,x) + \mathcal{L}[u](t,x) + \overline{b}\cdot\nabla(f(u)) = 0 & \text{for } t>0 \text{ and } x\in\mathbb{R}^N, \\ u(0,x) = u_0(x) & \text{for } x\in\mathbb{R}^N, \end{cases} \tag{1}$$

where $\mathcal{L}$ is a Lévy type operator, $\widehat{\mathcal{L}v}(\xi) = a(\xi)\hat{v}(\xi)$, whose symbol $a$ is written in the form

$$a(\xi) = ik\xi + \mu(\xi) + \int_{\mathbb{R}^N} \big(1 - e^{-i\eta\xi} - i\eta\xi\mathbf{1}_{|\eta|<1}\big)\Pi(d\eta).$$

Usually $k\in\mathbb{R}^N$, $\mu$ is a positive semi-definite quadratic form on $\mathbb{R}^N$ and $\Pi$ is a positive Radon measure satisfying

$$\int_{\mathbb{R}^N} \min\{|z|^2,1\}\Pi(dz) < \infty.$$

Two particular cases are the Laplacian, $\mathcal{L} = -\Delta$ and $\mathcal{L} = (-\Delta)^{\alpha/2}$ corresponding to $k=0$, $\mu(\xi)=|\xi|^2$, $\Pi=0$ and $k=0$, $\mu(\xi)=0$, $\Pi(dz) = |z|^{-N-\alpha}dz$ respectively.

# Existence of solutions for the GM

For all ranges or parameters $\alpha \in (0,2)$, $q > 1$, the model admits a unique entropy solution.

• Droniou, Gallouet, Vovelle 2003: existence and uniqueness of entropy solutions for $\alpha \in (1,2)$ and $f$ locally Lipshitz. Alibaud 2007 : $\alpha \in (0,2)$.

• Droniou, Gallouet, Vovelle: If $f \in C^\infty$, $\alpha \in (1,2)$ and $q > 1 \Longrightarrow$ there exists a unique mild solution with good regularity properties.

• When the diffusion is smaller, $\alpha \in (0,1]$, there is non-uniqueness of weak solutions, as proved by Alibaud and Andreianov.

# Asymptotic expansion for the Nonlocal Model

$$\begin{cases} u_t(t,x) + (-\Delta)^{\alpha/2} u(t,x) + a \cdot \nabla(|u|^{q-1}u) = 0 & \text{for} t > 0 \text{ and } x \in \mathbb{R}^N, \\ u(0,x) = u_0(x) & \text{for } x \in \mathbb{R}^N. \end{cases}$$

Then

$$t^{a(p,q,\alpha)} \|u(t,\cdot) - U_M(t,\cdot)\|_{L^p(\mathbb{R})} \to 0, \quad \text{as } t \to \infty$$

where

- Critical case: $q = 1 + \frac{\alpha-1}{N} \implies U_M$ is the unique self-similar solution $U(t,x) = t^{-N/\alpha} U(1, xt^{-1/\alpha})$ with data $U(0,x) = M\delta(x)$. Biler, Karch and Woyczyński 2001.
- Supercritical case $q > 1 + \frac{\alpha-1}{N}$, $\alpha \in (1,2)$ : $U_M$ is the fundamental solution of the Fractional Heat Equation:

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  1D case: Biler, Funaki and Woyczyński 1998
  multi-D case: . Biler, Karch and Woyczyński 2001.
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# The case $\alpha \in (0,1)$

**Theorem**

*For any $\alpha \in (0,1)$, $q > 1$, $f(u) = |u|^{q-1}u/q$ and $u_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ there exists a unique entropy solution $u$ of (CD). Moreover, for any $1 \le p < \infty$, the solution $u$ satisfies*

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*where $U$ is the unique weak solution of the equation*

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Proof. It follows as in Alibaud, Imbert, Karch, SIAM 2010, $q = 2$, by using the technique of approximation with a vanishing viscosity term:

$$(u_\epsilon)_t + (-\Delta)^{\alpha/2} u_\epsilon + (f(u_\epsilon))_x = \epsilon \Delta u_\epsilon.$$

Then, the asymptotic behavior is proved for this approximating problem.
We could also work directly with entropy solutions + scaling arguments in this present work.

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# Asymptotic behavior in the subcritical case

**Theorem**

*For any* $1 < q < \alpha < 2$, $f(u) = |u|^{q-1}u/q$ *and* $u_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ *nonnegative there exists a unique mild solution* $u \in C([0,\infty), L^1(\mathbb{R})) \cap C_b((0,\infty), L^\infty(\mathbb{R}))$ *of system (CD). Moreover, for any* $1 \leq p < \infty$, *solution* $u$ *satisfies*

$$\lim_{t\to\infty} t^{\frac{1}{q}(1-\frac{1}{p})} \|u(t) - U_M(t)\|_{L^p(\mathbb{R})} = 0,$$

*where* $M$ *is the mass of the initial data and* $U_M$ *is the unique entropy solution of the equation*

$$\begin{cases} u_t + (f(u))_x = 0 \quad \text{for } t > 0 \text{ and } x \in \mathbb{R}, \\ u(0) = M\delta_0. \end{cases} \qquad (C)$$

L. Ignat and D. Stan. *Asymptotic behaviour for fractional diffusion-convection equations.* JLMS 2018.

# Entropy solutions of the limit problem

$$w \in L^\infty((0,\infty), L^1(\mathbb{R})) \cap L^\infty((\tau,\infty)\times\mathbb{R}),\ \forall \tau \in (0,\infty)$$

such that:

C1) For every constant $k \in \mathbb{R}$ and $\varphi \in C_c^\infty((0,\infty)\times\mathbb{R})$, $\varphi \geq 0$, the following inequality holds

$$\int_0^\infty \int_{\mathbb{R}} \left( |w-k| \frac{\partial \varphi}{\partial t} + \operatorname{sgn}(w-k)(f(w)-f(k))\frac{\partial \varphi}{\partial x}\right) dxdt \geq 0.$$

C2) For any bounded continuous function $\psi$

$$\lim_{t\downarrow 0} \operatorname{ess} \int_{\mathbb{R}} w(t,x)\psi(x)dx = M\psi(0).$$

The existence of a unique entropy solution of system (C): Liu and Pierre.
System (B) has an unique entropy solution $U_M$, given by the $N$-wave profile

$$U_M(t,x) = \begin{cases} (x/t)^{\frac{1}{q-1}}, & 0 < x < r(t), \\ 0, & otherwise, \end{cases}$$

with $r(t) = (\frac{q}{q-1})^{\frac{q-1}{q}} M^{(q-1)/q} t^{1/q}$.

T.-P. Liu and M. Pierre, "Source-solutions and asymptotic behavior in conservation laws," *J. Differential Equations*, vol. 51, no. 3, pp. 419–441, 1984.

# Mild Solution of (CD), $\alpha \in (1,2)$

We say that $u(t,x) : (0,\infty) \times \mathbb{R} \to \mathbb{R}$ is a *mild solution* of Problem (CD) if

$$u(t,x) = (K_t^\alpha \star u_0)(x) + \int_0^t (K_{t-\sigma}^\alpha)_x \star f(u)(\sigma,x) d\sigma,$$

for all $x \in \mathbb{R}$, $t > 0$.

Here $K_t^\alpha$ is the Fractional Heat Kernel:

$$\frac{d}{dt} K_t^\alpha + (-\Delta)^{\alpha/2} K_t^\alpha = 0, \quad K_t^\alpha(0,x) = \delta(x).$$

# Decay of the Fractional Heat Kernel

- $\widehat{K}_t^\alpha(\xi) = e^{-|\xi|^\alpha t}$.
- $K_t^\alpha$ has the self-similar form

$$K_t^\alpha(x) = t^{-1/\alpha} F_\alpha(|x| t^{-1/\alpha}),$$

  for some profile function, $F_\alpha(r)$.
- For any $\alpha \in (0,2)$ the profile $F_\alpha$ is $C^\infty(\mathbb{R})$, positive and decreasing on $(0,\infty)$, and behaves at infinity like $F_\alpha(r) \sim r^{-(1+\alpha)}$.

**Lemma**

*For any $\alpha \in (0,2)$, $s \geq 0$ and $1 \leq p \leq \infty$ the kernel $K_t^\alpha$ satisfies the following estimates for any positive $t$:*

$$\|K_t^\alpha\|_{L^p(\mathbb{R})} \simeq \mathcal{K} t^{-\frac{1}{\alpha}(1-\frac{1}{p})}, \tag{2}$$

$$\||D|^s K_t^\alpha\|_{L^p(\mathbb{R})} \lesssim t^{-\frac{1}{\alpha}(1-\frac{1}{p})-\frac{s}{\alpha}}, \tag{3}$$

$$\||D|^s \partial_x K_t^\alpha\|_{L^p(\mathbb{R})} \lesssim t^{-\frac{1}{\alpha}(1-\frac{1}{p})-\frac{s+1}{\alpha}}. \tag{4}$$

We used the notation $|D|^s := (-\Delta)^{s/2}$.

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## Mild solutions

For any $u_0 \in L^\infty(\mathbb{R})$ there exists a unique global mild solution $u$ of Problem (CD). Moreover $u$ satisfies:

1. $\inf u_0 \le u(t,x) \le \sup u_0$.
2. If $u_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ then $u(t) \in C([0,+\infty) : L^1(\mathbb{R})) \cap C_b((0,\infty), L^\infty(\mathbb{R}))$ and $\|u(t)\|_{L^1(\mathbb{R})} \le \|u_0\|_{L^1(\mathbb{R})}$.
3. For any $s < \alpha + \min\{\alpha, q\} - 1$ and $1 < p < \infty$ solution $u$ satisfies $u_t \in C((0,\infty), L^p(\mathbb{R}))$ and $u \in C((0,\infty), H^{s,p}(\mathbb{R}))$.

Remark. Since $\alpha + \min\{\alpha, q\} - 1 > 1$ we have for any $t > 0$ that $u_x(t) \in L^p(\mathbb{R})$ for any $1 < p < \infty$. Moreover for any $t > 0$, the map $x \mapsto u(t,x)$ is continuous. The last property also guarantees that various integrations by parts used in the paper are allowed.

(1) and (2) are proved by DGV.

(3) we prove it by using fractional chain rule + technical tricks

J. Droniou, T. Gallouet, and J. Vovelle, "Global solution and smoothing effect for a non-local regularization of a hyperbolic equation," *J. Evol. Equ.*, vol. 3, no. 3, pp. 499–521, 2003.

# Approximated problem

$$\begin{cases} (u_\epsilon)_t(t,x) + (-\Delta)^{\alpha/2} u_\epsilon(t,x) + |u_\epsilon|^{q-1}(u_\epsilon)_x = 0 & \text{for } t>0 \text{ and } x\in\mathbb{R}, \\ u_\epsilon(0,x) = u_{0,\epsilon}(x) & \text{for } x\in\mathbb{R}, \end{cases} \qquad (P_\epsilon)$$

where $u_{0,\epsilon} > \epsilon$ is an approximation of $u_0$.

## Oleinik type Estimate

Let $1 < q, \alpha \leq 2$. For any $\epsilon > 0$ solution $u_\epsilon$ of Problem $P_\epsilon$ satisfies the *Oleinik type estimate*:

$$(u_\epsilon^{q-1})_x(t,x) \leq \frac{1}{t}, \quad \forall t>0,\ x\in\mathbb{R}.$$

# Main Ideea

Denoting $z = u^{q-1}$ we have

$$z_t + (q-1)z^{1-\frac{1}{q-1}}(-\Delta)^{\alpha/2}[z^{\frac{1}{q-1}}] + zz_x = 0.$$

Moreover $w = z_x$ satisfies

$$w_t + w^2 + zw_x + z^{-\beta-1}A(w,z) = 0$$

where

$$A(w,z) = -(2-q)w(-\Delta)^{\alpha/2}[z^{\beta+1}] + z(-\Delta)^{\alpha/2}[z^\beta w]$$

Question: $w \leq \frac{1}{t}$?

# Main trick

Cordoba & Corboda PNAS 2003

$$(-\Delta)^{\alpha/2}(u^2) - 2u(-\Delta)^{\alpha/2}(u) \leq 0$$

**key estimate**

Let us assume that

$$Lu = \int_{\mathbb{R}} K(x-y)(u(x)-u(y))dy.$$

For any $\beta \geq 0$ and $z \geq 0$ there exists $A_z : \mathbb{R} \to \mathbb{R}$, $A_z \leq 0$ such that

$$\Big(\frac{\beta}{\beta+1} wL(z^{\beta+1}) - zL(z^{\beta}w)\Big)(x_0) \leq A_z(x_0)w(x_0) \tag{5}$$

at the point $x_0 \in \mathbb{R}$ where $w$ attains its maximum.

Obs: $w \equiv 1$, $\beta = 1$ we can take $A_z \equiv 0$.
Obs: $L = -u_{xx}$, then $A_z = -\beta z^{\beta-1} z_x^2$

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# "Proof"

$$
\begin{aligned}
&\left(\frac{\beta}{\beta+1} wL(z^{\beta+1}) - zL(z^{\beta}w)\right)(x_0) \\
&= \frac{\beta}{\beta+1} w(x_0) \int_{\mathbb{R}} K(x_0 - y)(z^{\beta+1}(x_0) - z^{\beta+1}(y))dy \\
&\qquad - z(x_0) \int_{\mathbb{R}} K(x_0 - y)(z^{\beta} w(x_0) - z^{\beta} w(y))dy \\
&\leq \frac{\beta}{\beta+1} w(x_0) \int_{\mathbb{R}} K(x_0 - y)(z^{\beta+1}(x_0) - z^{\beta+1}(y))dy \\
&\qquad - z(x_0) w(x_0) \int_{\mathbb{R}} K(x_0 - y)(z^{\beta}(x_0) - z^{\beta}(y))dy \\
&= -w(x_0) \int_{\mathbb{R}} K(x_0 - y)\Big(\frac{z^{\beta+1}(x_0)}{\beta+1} + \frac{\beta z^{\beta+1}(y)}{\beta+1} - z(x_0) z^{\beta}(y)\Big)dy
\end{aligned}
$$

Let $u$ be the solution of (CD) with data $u_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$, $u_0 \geq 0$.

1. Mass conservation: $\int_{\mathbb{R}} u(t,x)dx = M, \quad \forall t \geq 0.$
2. Hyperbolic estimate: $(u^{q-1})_x(t,x) \leq \dfrac{1}{t}$ for all $t > 0$ in $\mathcal{D}'(\mathbb{R})$.
3. Upper bound: $0 \leq u(t,x) \leq \left(\dfrac{q}{q-1}M\right)^{1/q} t^{-1/q}$
4. Decay of the spatial derivative: $u_x(t,x) \leq C(q)M^{\frac{2-q}{q}}t^{-\frac{2}{q}}$
5. $W^{1,1}_{\text{loc}}(\mathbb{R})$ estimate:

$$\int_{|x|\leq R} |u_x(t,x)|dx \leq 2R\,C(q)M^{\frac{2-q}{q}}t^{-\frac{2}{q}} + 2\left(\frac{q}{q-1}M\right)^{1/q} t^{-1/q} \quad \forall t > 0.$$

6. Energy estimate: for every $0 < \tau < T$,

$$\int_\tau^T \int_{\mathbb{R}} |(-\Delta)^{\alpha/4}u(t,x)|^2 dxdt \leq \frac{1}{2}\int_{\mathbb{R}} u^2(\tau,x)dx \leq \frac{1}{2}\left(\frac{q}{q-1}\right)^{1/q} \tau^{-1/q}M$$

# Asymptotic Behavior. Proof

4-steps method (Kamin and Vázquez)

**Step I.** Rescale: $u_\lambda(t,x) := \lambda u(\lambda^q t, \lambda x) \Longrightarrow$

It follows that $u_\lambda$ is a solution of the problem

$$\begin{cases} (u_\lambda)_t + \lambda^{q-\alpha}(-\Delta)^{\alpha/2}[u_\lambda] + (u_\lambda)^{q-1}(u_\lambda)_x = 0, & x \in \mathbb{R},\ t > 0, \\ u_\lambda(0,x) = \lambda u_0(\lambda x), & x \in \mathbb{R}. \end{cases} \tag{$P_\lambda$}$$

Compactness of family $(u_\lambda)_{\lambda>0}$ in $C([t_1,t_2], L^2_{loc}(\mathbb{R}))$ and apply the Aubin-Lions-Simon compactness argument to get

$$u_\lambda \to U \quad \text{in} \quad C([t_1,t_2] : L^2_{loc}(\mathbb{R})) \quad \text{as} \quad \lambda \to \infty.$$

S. Kamin and J. L. Vázquez, "Fundamental solutions and asymptotic behaviour for the $p$-Laplacian equation," *Rev. Mat. Iberoamericana*, vol. 4, no. 2, pp. 339–354, 1988.

Step II. Tail control and convergence in $C([t_1, t_2], L^1(\mathbb{R}))$.

Step III. Identifying the limit: $U \in C_{\mathrm{loc}}((0, \infty), L^1(\mathbb{R}))$ obtained above is an entropy solution of system (C).
We know there exists a unique entropy solution of (C).

Step IV. Prove that $U(t) \in L^p(\mathbb{R})$ and $\|u_\lambda(t) - U_M(t)\|_{L^p(\mathbb{R})} \to 0$ as $\lambda \to \infty$.
Then take $t = 1$ and obtain the result.

# To be continued...

1. Multidimensional case

$$u_t + (-\Delta)^{\alpha/2} u + \partial_y(|u|^{q-1}u) = 0$$

Here the profile may be related with the solutions of

$$u_t + (-\Delta_x)^{\alpha/2} u + \partial_y(|u|^{q-1}u) = 0, u_0 = M\delta_0$$

Warning $(-\Delta)^{\alpha/2}(u(\lambda x, \lambda^2 y)) =???$

2. Nonlinear Fractional Diffusion + Nonlinear convection ?

$$u_t + (-\Delta)^{\alpha/2} u^m + u^{q-1} u_x = 0.$$

3. Even nonlinearities

$$u_t + (-\Delta)^{\alpha/2} u + (|u|^q)_x = 0.$$

1. Nonlocal convection

$$u_t = J * u - u + G * u^q - u^q, 1 < q < 2$$

2. Nonlocal Oleinik's estimates: fake models with J. Rossi
3. Step like initial data + rarefaction waves

$$\varphi = \begin{cases} \varphi_- + L^1((-\infty, 0)), \\ \varphi_+ + L^1((0, \infty)). \end{cases}$$

4. CFL conditions for global solutions, $|G| \leq C|K|$ and small initial data depending on $C$

$$\begin{aligned} u_t(t,x) &= \int_{\mathbb{R}} K(x-y)(u(t,y) - u(t,x))dy \\ &\quad + \int_{\mathbb{R}} G(x-y) f\Big(\frac{u(t,y)+u(t,x)}{2}\Big) dy, t > 0, x \in \mathbb{R}, \end{aligned}$$

5. Understand the competition between diffusion and the nonlocal convection

# To be continued... NUMERICS

Many ideas from the nonlocal world have been used in the numerical context


- L.I., A. Pozo, *A splitting method for the augmented Burgers equation.* BIT Numerical Mathematics (2018)
- L.I., A. Pozo, *A semi-discrete large-time behavior preserving scheme for the augmented Burgers equation.* ESAIM: M2AN (2017)
- L.I., A. Pozo, E. Zuazua *Large-time asymptotics, vanishing viscosity and numerics for 1-D scalar conservation laws.* Math. Comp. (2015)

# Asymptotic behavior and numerical simulations

L.I. & A. Pozo & E. Zuazua, Math of Comp., 2015

$$u_t + \left(\frac{u^2}{2}\right)_x = 0, \qquad x \in \mathbb{R}, t > 0.$$

For large time the solution behavios as a N-wave

$$w_{p,q}(x,t) = \begin{cases} \frac{x}{t}, & -\sqrt{2pt} < x < \sqrt{2qt}, \\ 0, & \text{elsewhere.} \end{cases} \tag{6}$$

Lor the Lax-Friedrichs scheme, $w = w_{M_\Delta}$ is the unique solution of the continuous viscous Burgers equation

$$\begin{cases} w_t + \left(\frac{w^2}{2}\right)_x = \frac{(\Delta x)^2}{2} w_{xx}, \quad x \in \mathbb{R}, t > 0, \\ w(0) = M_\Delta \delta_0, \end{cases} \tag{7}$$

with $M_\Delta = \int_{\mathbb{R}} u_\Delta^0$.

$w$ - parabolic profile

For Engquist-Osher and Godunov schemes, $w = w_{p_\Delta, q_\Delta}$ is the unique solution of the hyperbolic Burgers equation

$$\begin{cases} w_t + \left(\frac{w^2}{2}\right)_x = 0, \quad x \in \mathbb{R}, t > 0, \\ w(0) = M_\Delta \delta_0, \quad \lim_{t \to 0} \int_{-\infty}^{x} w(t,z)dz = \begin{cases} 0, & x < 0, \\ -p_\Delta, & x = 0, \\ q_\Delta - p_\Delta, & x > 0, \end{cases} \end{cases} \tag{8}$$

with $M_\Delta = \int_{\mathbb{R}} u_\Delta^0$ and

$$p_\Delta = -\min_{x \in \mathbb{R}} \int_{-\infty}^{x} u_\Delta^0(z)dz \quad \text{and} \quad q_\Delta = \max_{x \in \mathbb{R}} \int_{x}^{\infty} u_\Delta^0(z)dz.$$

$w$ - hyperbolic profile

THANKS for your attention !!!