Recent approaches for the study of the Navier-Stokes equations with discontinuous density

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May 21, 2018

The classical incompressible Navier-Stokes equations:

$$(NS): \begin{cases} u_t + \operatorname{div}(u \otimes u) - \mu\Delta u + \nabla P = 0 & \text{in } \mathbb{R}_+ \times \Omega \\ \operatorname{div} u = 0 & \text{in } \mathbb{R}_+ \times \Omega \\ u = 0 & \text{on } \mathbb{R}_+ \times \partial\Omega \\ u|_{t=0} = u_0 & \text{in } \Omega. \end{cases}$$

Here $u = u(t,x) \in \mathbb{R}^d$ and $P = P(t,x) \in \mathbb{R}$ with $t \geq 0$ and $x \in \Omega \subset \mathbb{R}^d$, $d \geq 2$.

- Energy balance: $\frac{1}{2}\|u(t)\|_{L^2}^2 + \mu \int_0^t \|\nabla u\|_{L^2}^2 \, d\tau = \frac{1}{2}\|u_0\|_{L^2}^2.$
- Scaling invariance: If $\Omega = \mathbb{R}^d$ then the System (NS) is invariant (up to a change of P and u_0) by the family of dilations:

$$T_\lambda u(t,x) := \lambda u(\lambda^2 t, \lambda x).$$

Global weak solutions

The classical incompressible Navier-Stokes equations:

$$(NS):\begin{cases} u_t+\operatorname{div}(u\otimes u)-\mu\Delta u+\nabla P=0 & \text{in } \mathbb{R}_+\times\Omega\\ \operatorname{div} u=0 & \text{in } \mathbb{R}_+\times\Omega\\ u=0 & \text{on } \mathbb{R}_+\times\partial\Omega\\ u|_{t=0}=u_0 & \text{in } \Omega.\end{cases}$$

Theorem (J. Leray, 1934)

Any divergence free $u_0\in L^2(\Omega)$ generates at least one global weak solution of (NS) satisfying the energy inequality:

$$\frac12\|u(t)\|_{L^2}^2+\mu\int_0^t\|\nabla u\|_{L^2}^2\,d\tau\leq\frac12\|u_0\|_{L^2}^2.$$

- The proof relies essentially on the energy balance and on compactness arguments (or, equivalently, Schauder-Tikhonov theorem).
- Unless $d=2$, uniqueness of Leray's solutions is (still) an open question.

'Mild solutions' of NS equations

Let $A = -\mu\Delta u + \nabla P$ be the Stokes operator. Then, formally,

$$u(t) = \underbrace{e^{-tA}u_0}_{u_L} - \underbrace{\int_0^t e^{-(t-\tau)A}(\mathrm{div}\,(u\otimes u)(\tau))\,d\tau}_{\mathcal{B}(u,u)}.$$

Lemma (based on the fixed point theorem in a Banach spaces)

Let X be a Banach space and $\mathcal{B} : X\times X \to X$, a continuous bilinear map with norm M. Then equation $\boxed{u = u_L - \mathcal{B}(u,u)}$ *has a unique solution in the closed ball $\overline{B}(0, 2\|u_L\|_X)$ whenever*

$$4M\|u_L\|_X < 1.$$

- The largest spaces in which one may expect $\mathcal{B}$ to be continuous are *scaling invariant* by the family of dilations $(T_\lambda)_{\lambda>0}$.

- Examples : small initial data in Sobolev spaces $\dot H^{\frac d2-1}(\mathbb{R}^d)$ (Fujita-Kato), Lebesgue space $L^d(\mathbb{R}^d)$ (Giga- Kato), Besov spaces $\dot B^{\frac dp-1}_{p,r}(\mathbb{R}^d)$, etc.

The **inhomogeneous incompressible Navier-Stokes equations** read:

$$(INS): \begin{cases} (\rho u)_t + \mathrm{div}\,(\rho u \otimes u) - \mu\Delta u + \nabla P = 0 & \text{in } \mathbb{R}_+ \times \Omega \\ \mathrm{div}\, u = 0 & \text{in } \mathbb{R}_+ \times \Omega \\ \rho_t + \mathrm{div}\,(\rho u) = 0 & \text{in } \mathbb{R}_+ \times \Omega. \end{cases}$$

- Energy balance : $\frac{1}{2}\|\sqrt{\rho(t)}\, u(t)\|_{L^2}^2 + \mu \int_0^t \|\nabla u\|_{L^2}^2\, d\tau = \frac{1}{2}\|\sqrt{\rho_0}\, u_0\|_{L^2}^2.$
- Conservation of L^p norms of functions of the density.
- Scaling invariance if $\Omega = \mathbb{R}^d$:

$$\rho(t,x) \to \rho(\lambda^2 t, \lambda x), \qquad u(t,x) \to \lambda u(\lambda^2 t, \lambda x), \qquad P(t,x) \to \lambda^2 P(\lambda^2 t, \lambda x).$$

- **Global weak solutions with finite energy** for any pair (ρ_0, u_0) such that $\rho_0 \in L^\infty(\Omega)$ with $\rho_0 \geq 0$, and $\sqrt{\rho_0} u_0 \in L^2(\Omega)$ with $\mathrm{div}\, u_0 = 0$ (Kazhikhov, 1974, J. Simon 1990, P.-L. Lions 1996: 'renormalized solutions').
- Even if $d = 2$, uniqueness in the class of finite energy solutions is a widely open question.
- **Strong solutions for smooth data with no vacuum**: global if $d = 2$ or $d = 3$ and u_0 small (Ladyzhenskaya and Solonnikov ,1978).

Can (INS) be a model for mixture of nonreacting incompressible fluids ?

Initial data: u_0 sufficiently smooth and ρ_0 discontinuous along some interface:

$$\rho_0 = \rho_1 1_{D_0} + \rho_2 1_{^cD_0} \quad \text{with} \quad \rho_1, \rho_2 \geq 0 \quad \text{and} \quad D_0 \subset \Omega.$$

According to Lions' result on weak solutions, the velocity has a generalized flow X, and

$$\rho(t) = \rho_1 1_{D_t} + \rho_2 1_{^cD_t} \quad \text{with} \quad D_t := X(t, D_0).$$

Lions' question: is the regularity of D_0 preserved by the time evolution for any $\rho_1 \geq 0$ and $\rho_2 \geq 0$?

According to Cauchy-Lipschitz theorem, the minimal requirement is

$$\nabla u \in L^1(0, T; L^\infty(\Omega)).$$

As (INS) has a hyperbolic part, **it is also needed for uniqueness.**

Aim of the talk

Presenting three different approaches that are based respectively on :

1. Critical functional framework and endpoint maximal regularity;
2. Classical maximal regularity;
3. Energy approach.

I. An approach based on the endpoint maximal regularity

Assume that $\Omega = \mathbb{R}^d$ $(d \geq 2)$ and that $\rho \to 1$ at ∞, and set $a := \rho - 1$. System for (a, u, P) reads:

$$(\widetilde{INS}) : \begin{cases} u_t - \mu\Delta u + \nabla P = -au_t - (1+a)\mathrm{div}\,(u \otimes u) & \text{in } \mathbb{R}_+ \times \mathbb{R}^d \\ \mathrm{div}\, u = 0 & \text{in } \mathbb{R}_+ \times \mathbb{R}^d \\ a_t + u \cdot \nabla a = 0 & \text{in } \mathbb{R}_+ \times \mathbb{R}^d. \end{cases}$$

Scaling invariance is the same as for (INS):

$$a(t,x) \to a(\lambda^2 t, \lambda x), \qquad u(t,x) \to \lambda u(\lambda^2 t, \lambda x), \qquad P(t,x) \to \lambda^2 P(\lambda^2 t, \lambda x).$$

Abstract Maximal Regularity

Let Y and Z be two Banach spaces. Consider the evolution equation

$$u_t + Au = f \in Z, \qquad u(0) = 0$$

where A is an unbounded operator with domain $D(A) \subset Y$.

Maximal regularity means that *both* u_t *and* Au *are in* Z and

$$(MR) \qquad \|u_t, Au\|_Z \leq C\|f\|_Z.$$

In our case, A is the stokes operator, that is

$$\begin{cases} u_t - \mu\Delta u + \nabla P = f & \text{in } \mathbb{R}_+ \times \mathbb{R}^d \\ \operatorname{div} u = 0 & \text{in } \mathbb{R}_+ \times \mathbb{R}^d. \end{cases}$$

- (MR) is true if $Z = L^q(\mathbb{R}_+; L^r(\Omega))$ with $1 < p, r < \infty$ and Ω is the whole space, half-space, smooth bounded or exterior domain,...
- **Endpoint maximal regularity**: We have for any $s \in \mathbb{R}$ and $p \in [1, \infty]$:

$$\|u\|_{L^\infty(\mathbb{R}_+;\dot{B}^s_{p,1})} + \|u_t, \mu\nabla_x^2 u, \nabla_x P\|_{L^1(\mathbb{R}_+;\dot{B}^s_{p,1})} \lesssim \|u_0\|_{\dot{B}^s_{p,1}} + \|f\|_{L^1(\mathbb{R}_+;\dot{B}^s_{p,1})}.$$

Scaling invariance pushes us to take $s=\frac{d}{p}-1$, and thus $(u,\nabla P)\in E_p$ with

$$E_p=\Bigl\{(u,\nabla P)\in\mathcal{C}_b(\mathbb{R}_+;\dot B^{\frac{d}{p}-1}_{p,1})\times L^1(\mathbb{R}_+;\dot B^{\frac{d}{p}-1}_{p,1})\ \text{ with }\ u_t,\nabla^2u\in L^1(\mathbb{R}_+;\dot B^{\frac{d}{p}-1}_{p,1})\Bigr\}.$$

- *Stability of the Besov space* $\dot B^{\frac{d}{p}}_{p,1}$ by product if $p<\infty$:

$$\|\mathrm{div}\,(u\otimes u)\|_{\dot B^{\frac{d}{p}-1}_{p,1}}\lesssim\|u\otimes u\|_{\dot B^{\frac{d}{p}}_{p,1}}\lesssim\|u\|^2_{\dot B^{\frac{d}{p}}_{p,1}}.$$

- *Multiplier spaces:* $\|a\|_{\mathcal{M}(\dot B^{\frac{d}{p}-1}_{p,1})}:=\sup_{\|z\|_{\dot B^{\frac{d}{p}-1}_{p,1}}=1}\|az\|_{\dot B^{\frac{d}{p}-1}_{p,1}}<\infty.$

- *Estimates for the transport equation* (deduced from the ones in Besov spaces):

$$\|a(t)\|_{\mathcal{M}(\dot B^{\frac{d}{p}-1}_{p,1})}\leq\|a_0\|_{\mathcal{M}(\dot B^{\frac{d}{p}-1}_{p,1})}\exp\Bigl\{C\int_0^t\|\nabla u\|_{\dot B^{\frac{d}{p}}_{p,1}}\,d\tau\Bigr\}.$$

Taking $f=-au_t-(1+a)\mathrm{div}\,(u\otimes u)$ in (S), we deduce that

$$\|(u,\nabla P)\|_{E_p}\lesssim\|u_0\|_{\dot B^{\frac dp-1}_{p,1}}+\|a\|_{L^\infty(\mathbb{R}_+;\mathcal{M}(\dot B^{\frac dp-1}_{p,1}))}\|u_t\|_{L^1(\mathbb{R}_+;\dot B^{\frac dp-1}_{p,1})}+\Big(1+\|a\|_{L^\infty(\mathbb{R}_+;\mathcal{M}(\dot B^{\frac dp-1}_{p,1}))}\Big)\|u\|^2_{E_p}.$$

Combining with

$$\|a\|_{L^\infty(\mathbb{R}_+;\mathcal{M}(\dot B^{\frac dp-1}_{p,1}))}\leq\|a_0\|_{\mathcal{M}(\dot B^{\frac dp-1}_{p,1})}\exp\Big\{C\|(u,\nabla P)\|_{E_p}\Big\},$$

one may close the estimates *if both* $\|a_0\|_{\mathcal{M}(\dot B^{\frac dp-1}_{p,1})}$ *and* $\|u_0\|_{\dot B^{\frac dp-1}_{p,1}}$ *are small.*

Theorem (D & P.B. Mucha, 2012)

Assume that $1 \leq p < 2d$. *There exists a constant* $c > 0$ *such that if*

$$\mu\|a_0\|_{\mathcal{M}(\dot{B}^{\frac{d}{p}-1}_{p,1})\cap L^\infty} + \|u_0\|_{\dot{B}^{\frac{d}{p}-1}_{p,1}} \leq c\mu \qquad (1)$$

then (INS) *has a unique solution with* $(u, \nabla P) \in E_p$ *and* $a \in \mathcal{C}(\mathbb{R}_+; \mathcal{M}(\dot{B}^{\frac{d}{p}-1}_{p,1}))$.

Example: 1_D is in $\mathcal{M}(\dot{B}^{\frac{d}{p}-1}_{p,1})$ if $d/p - 1 < 1/p$.

Corollary (The density patch problem)

Let D *be a* C^1 *bounded domain. If* u_0 *fulfills* (1) *with* $d-1 < p < 2d$ *and* $\rho_0 = c_1 1_D + c_2 1_{^cD}$ *with* $|c_1 - c_2| \ll 1$ *then* (INS) *has a unique global solution as above, and* $\rho(t) = c_1 1_{D_t} + c_2 1_{^cD_t}$. *Furthermore* D_t **remains** C^1.

About the uniqueness issue

Let $(\rho^1, u^1, \nabla P^1)$ and $(\rho^2, u^2, \nabla P^2)$ be two solutions of (INS).

Then $(\delta\rho, \delta u, \delta P) := (\rho^2 - \rho^1, u^2 - u^1, P^2 - P^1)$ fulfills

$$\begin{cases} \delta\rho_t + u^1 \cdot \nabla \delta\rho = -\delta u \cdot \nabla \rho^2 \longleftarrow \text{Loss of one derivative here} \\ \delta u_t - \mu \Delta \delta u + \nabla \delta P = (1-\rho^1)\delta u_t + \delta\rho\,(u_t^2 + u^2 \cdot \nabla u^2) + \rho^1 (u^1 \cdot \nabla \delta u + \delta u \cdot \nabla u^2). \end{cases}$$

Lagrangian coordinates: Assume $\nabla u \in L^1_{loc}(\mathbb{R}_+; L^\infty)$ and set

$$\bar\rho(t,y) := \rho(t,x), \quad \bar u(t,y) := u(t,x) \ \text{ and } \ \bar P(t,y) := P(t,x) \ \text{ with } \ \boxed{x := X(t,y)}$$

where X **is the flow of** u defined by

$$\boxed{X(t,y) = y + \int_0^t u(\tau, X(\tau,y))\,d\tau.}$$

(INS) in Lagrangian coordinates:

- $\bar\rho$ is time independent.
- $(\bar u, \bar P)$ satisfies

$$(\widetilde{INS}) : \begin{cases} \rho_0 \bar u_t - \mathrm{div}\,(A^T A \nabla \bar u) + {}^T\!A \cdot \nabla \bar P = 0, \\ \mathrm{div}\,(A \bar u) = {}^T\!A : \nabla \bar u = 0, \end{cases}$$

with $A = (D_y X)^{-1} = \displaystyle\sum_{k=0}^{+\infty} (-1)^k \left(\int_0^t D\bar u(\tau,\cdot)\,d\tau \right)^k.$

- $(\widetilde{INS})$ may be solved *by means of the fixed point theorem.*
- Uniqueness may be proved at the level of Lagrangian coordinates.

II. An approach based on the classical maximal regularity

Consider a solution $(u, \nabla P)$ to

$$(S) : \begin{cases} u_t - \mu\Delta u + \nabla P = f & \text{in } \mathbb{R}_+ \times \mathbb{R}^d \\ \operatorname{div} u = 0 & \text{in } \mathbb{R}_+ \times \mathbb{R}^d. \end{cases}$$

Then, for all $1 < p, r < \infty$,

$$|(u, \nabla P)\|_{E_p^r} := \|(u_t, \mu\nabla^2 u, \nabla P)\|_{L^r(\mathbb{R}_+;L^p)} + \|u\|_{L^\infty(\mathbb{R}_+;\dot{B}^{2-\frac{2}{r}}_{p,r})} \lesssim \|u_0\|_{\dot{B}^{2-\frac{2}{r}}_{p,r}} + \|f\|_{L^r(\mathbb{R}_+;L^p)}.$$

- *Critical regularity* for (INS) corresponds to

$$2 - \frac{2}{r} = \frac{d}{p} - 1.$$

which gives us the constraint $\frac{d}{3} < p < d.$

- We want to apply this to $f = -au_t - (1+a)u \cdot \nabla u.$

So we have

$$\|(u,\nabla P)\|_{E_p^r} := \|(u_t,\mu\nabla^2 u,\nabla P)\|_{L^r(\mathbb{R}_+;L^p)} + \|u\|_{L^\infty(\mathbb{R}_+;\dot B^{2-\frac{2}{r}}_{p,r})} \lesssim \|u_0\|_{\dot B^{2-\frac{2}{r}}_{p,r}}$$
$$+\|a\|_{L^\infty(\mathbb{R}_+\times\mathbb{R}^d)}\|u_t\|_{L^r(\mathbb{R}_+;L^p)} + (1+\|a\|_{L^\infty(\mathbb{R}_+\times\mathbb{R}^d)})\|u\cdot\nabla u\|_{L^r(\mathbb{R}_+;L^p)}.$$

Note that $\|a\|_{L^\infty(\mathbb{R}_+\times\mathbb{R}^d)} = \|a_0\|_{L^\infty}$. Hence, if $\|a_0\|_{L^\infty}$ is small, then we get

$$\|(u,\nabla P)\|_{E_p^r} \lesssim \|u_0\|_{\dot B^{2-\frac{2}{r}}_{p,r}} + \|u\cdot\nabla u\|_{L^r(\mathbb{R}_+;L^p)}.$$

If *critical regularity*: $2-\frac{2}{r}=\frac{d}{p}-1$ then we have

$$\|u\cdot\nabla u\|_{L^r(\mathbb{R}_+;L^p)} \le \|u\|_{L^{2r}(\mathbb{R}_+;L^{\frac{dr}{r-1}})}\|\nabla u\|_{L^{2r}(\mathbb{R}_+;L^{\frac{dr}{2r-1}})}$$

and

$$\|u\|_{L^{\frac{dr}{r-1}}} \lesssim \|\nabla u\|_{L^{\frac{dr}{2r-1}}} \qquad \text{(Sobolev embedding)}$$
$$\|\nabla u\|_{L^{\frac{dr}{2r-1}}} \lesssim \|\nabla^2 u\|_{L^p}^{\frac12}\|u\|^{\frac12}_{\dot B^{2-\frac{2}{r}}_{p,r}} \qquad \text{(Interpolation).}$$

Hence

$$\|(u,\nabla P)\|_{E_p^r} \lesssim \|u_0\|_{\dot B^{2-\frac{2}{r}}_{p,r}} + \|(u,\nabla P)\|^2_{E_p^r}.$$

Results

Theorem (Huang, Paicu & Zhang, 2013)

Let $a_0 \in L^\infty(\mathbb{R}^d)$ and $u_0 \in \dot{B}^{-1+\frac{d}{p}}_{p,r}(\mathbb{R}^d)$ with $d \geq 2$, $p := \frac{dr}{3r-2}$ and $r \in (1,\infty)$. There exists a positive constant $c_0 = c_0(r,d)$ so that if

$$\mu\|a_0\|_{L^\infty} + \|u_0\|_{\dot{B}^{-1+\frac{d}{p}}_{p,r}} \leq c_0\mu \tag{2}$$

then (NSI) has a global solution $(a,u,\nabla P)$ satisfying $\|a(t)\|_{L^\infty} = \|a_0\|_{L^\infty}$ for all $t \geq 0$, and $(u,\nabla P) \in E^r_p$.

Since $r > 1$ and $p < d$, we do not have $\nabla u \in L^1_{loc}(\mathbb{R}_+; L^\infty)$ which precludes our using Lagrangian coordinates for proving uniqueness.

Theorem (Huang, Paicu & Zhang, 2013)

If, in addition, $u_0 \in \dot{B}^{-1+\frac{d}{p}}_{\widetilde{p},r}$ for some $d < \widetilde{p} \leq \frac{dr}{r-1}$, then $(u,\nabla P)$ also belongs to $E^r_{\widetilde{p}}$, and the solution $(a,u,\nabla P)$ is unique in $L^\infty(\mathbb{R}_+ \times \mathbb{R}^d) \times \big(E^r_p \cap E^r_{\widetilde{p}}\big)$. Besides, the $C^{1,\alpha}$ (with $\alpha = 1 - d/\widetilde{p}$) regularity of interfaces is preserved.

An approach based on energy estimates

Our assumptions:

- $\Omega = \mathbb{T}^2$;
- $0 \leq \rho_0 \leq \rho^*$;
- $u_0 \in H^1$ and $\operatorname{div} u_0 = 0$;
- and (with no loss of generality),

$$\int_{\mathbb{T}^2} \rho_0 \, dx = \mu = 1 \quad \text{and} \quad \int_{\mathbb{T}^2} \rho_0 u_0 \, dx = 0.$$

Remember:

- Energy balance : $\frac{1}{2}\|\sqrt{\rho(t)}\, u(t)\|_{L^2}^2 + \mu \int_0^t \|\nabla u\|_{L^2}^2 \, d\tau = \frac{1}{2}\|\sqrt{\rho_0}\, u_0\|_{L^2}^2.$
- Conservation of L^p norms of functions of the density.

H^1 estimates for the velocity

- Take the L^2 scalar product of $\rho(u_t+u\cdot\nabla u)-\Delta u+\nabla P=0$ with u_t:

$$\frac12\frac{d}{dt}\int_{\mathbb{T}^2}|\nabla u|^2dx+\int_{\mathbb{T}^2}\rho|u_t|^2\,dx\leq\frac12\int_{\mathbb{T}^2}\rho|u_t|^2\,dx+\frac12\int_{\mathbb{T}^2}\rho|u\cdot\nabla u|^2\,dx.$$

From $-\Delta u+\nabla P=-(\rho u_t+\rho u\cdot\nabla u)$ and $\operatorname{div}\Delta u=0$, we have

$$\|\nabla^2u\|_{L^2}^2+\|\nabla P\|_{L^2}^2=\|\rho(\partial_tu+u\cdot\nabla u)\|_{L^2}^2\leq2\rho^*\biggl(\int_{\mathbb{T}^2}\rho|u_t|^2\,dx+\int_{\mathbb{T}^2}\rho|u\cdot\nabla u|^2\,dx\biggr)\cdot$$

Hence

$$\frac{d}{dt}\|\nabla u\|_{L^2}^2+\frac12\|\sqrt{\rho}\,u_t\|_{L^2}^2+\frac{1}{4\rho^*}\bigl(\|\nabla^2u\|_{L^2}^2+\|\nabla P\|_{L^2}^2\bigr)\leq\frac32\int_{\mathbb{T}^2}\rho|u\cdot\nabla u|^2\,dx.$$

- Apply Hölder and Gagliardo-Nirenberg inequality:

$$\begin{aligned}\int_{\mathbb{T}^2}\rho|u\cdot\nabla u|^2\,dx\leq\rho^*\|u\|_{L^4}^2\|\nabla u\|_{L^4}^2\;&\leq C\rho^*\|u\|_{L^2}\|\nabla u\|_{L^2}^2\|\nabla^2u\|_{L^2}\\&\leq\frac{1}{12\rho^*}\|\nabla^2u\|_{L^2}^2+C(\rho^*)^3\|u\|_{L^2}^2\|\nabla u\|_{L^2}^2\|\nabla u\|_{L^2}^2.\end{aligned}$$

- If $\rho\geq\rho_*>0$, then $\|u\|_{L^2}^2\leq\rho_*^{-1}\|\sqrt{\rho}u\|_{L^2}^2\leq\rho_*^{-1}\|\sqrt{\rho_0}u_0\|_{L^2}^2.$

H^1 estimates (continued)

Lemma (B. Desjardins, 1997)

If $\int_{\mathbb{T}^2}\rho\,dx=1$ *and* $\int_{\mathbb{T}^2}\rho z\,dx=0$ *then*

$$\biggl(\int_{\mathbb{T}^2}\rho z^4\,dx\biggr)^{\frac12}\leq C\|\sqrt{\rho}z\|_{L^2}\|\nabla z\|_{L^2}\log^{\frac12}\biggl(e+\|\rho-1\|_{L^2}^2+\frac{\rho^*\|\nabla z\|_{L^2}^2}{\|\sqrt{\rho}z\|_{L^2}^2}\biggr)\cdot \qquad (3)$$

- Write $\displaystyle\int_{\mathbb{T}^2}\rho|u\cdot\nabla u|^2dx\leq\sqrt{\rho^*}\biggl(\int_{\mathbb{T}^2}\rho|u|^4\,dx\biggr)^{\frac12}\|\nabla u\|_{L^4}^2$

and use (3) with $z=u,$ energy balance and $ab\leq a^2/2+b^2/2$:

$$\int_{\mathbb{T}^2}\rho|u\cdot\nabla u|^2dx\leq\frac{1}{12\rho^*}\|\nabla^2u\|_{L^2}^2$$
$$+C(\rho^*)^2\|\sqrt{\rho_0}\,u_0\|_{L^2}^2\|\nabla u\|_{L^2}^2\|\nabla u\|_{L^2}^2\log\biggl(e+\|\rho_0-1\|_{L^2}^2+\rho^*\frac{\|\nabla u\|_{L^2}^2}{\|\sqrt{\rho_0}\,u_0\|_{L^2}^2}\biggr)\cdot$$

H^1 estimates (end)

We eventually get

$$\frac{d}{dt}X \leq f X \log(e+X),$$

with $f(t) := C_0\|\nabla u(t)\|_{L^2}^2$ for some suitable $C_0 = C(\rho_0, u_0)$ and

$$X(t) := \int_{\mathbb{T}^2} |\nabla u(t)|^2\,dx + \frac{1}{2}\int_{\mathbb{T}^2}\Big(\rho|u_t|^2 + \frac{1}{4\rho^*}\big(|\nabla^2 u|^2 + |\nabla P|^2\big)\Big)dx.$$

Hence

$$(e+X(t)) \leq (e+X(0))^{\exp(\int_0^t f(\tau)\,d\tau)} \leq (e+X(0))^{\exp\,(C_0\|\sqrt{\rho_0}u_0\|_{L^2}^2)}.$$

• So far, we only proved $\nabla u \in L^1_{loc}(\mathbb{R}_+; H^1(\mathbb{T}^2))$, hence we do not know if $\nabla u \in L^1_{loc}(\mathbb{R}_+; L^\infty(\mathbb{T}^2))$.

BUT $u_0 \in H^1(\mathbb{T}^2)$ implies almost $t \mapsto \nabla e^{t\Delta}u_0$ in $L^1_{loc}(\mathbb{R}_+; H^2(\mathbb{T}^2))$.

Regularity of the first time derivative of u

Case of the heat equation: we know that $u_0 \in \dot{H}^1$ implies that $v := e^{t\Delta} u_0$ satisfies $\sqrt{t} v_t \in L^\infty(\mathbb{R}_+; L^2) \cap L^2(\mathbb{R}_+; \dot{H}^1)$ since

$$\|u_0\|_{\dot{H}^1} \approx \Big\| \sqrt{t} \|\Delta e^{t\Delta} u_0\|_{L^2} \Big\|_{L^2(\frac{dt}{t})} = \Big\| \|\sqrt{t} v_t\|_{L^2} \Big\|_{L^2(\frac{dt}{t})}.$$

Hint: Estimating $\sqrt{t} u_t$ in $L^\infty(\mathbb{R}_+; L^2) \cap L^2(\mathbb{R}_+; \dot{H}^1)$.

- Take the L^2 scalar product of $\rho(u_t + u \cdot \nabla u) - \Delta u + \nabla P = 0$ with $t u_{tt}$:

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}^2} \rho t |u_t|^2\,dx + \int_{\mathbb{T}^2} t|\nabla u_t|^2\,dx = \frac{1}{2}\int_{\mathbb{T}^2} \rho |u_t|^2\,dx + \int_{\mathbb{T}^2} (\rho_t u_t - \rho_t u \cdot \nabla u - \rho u_t \cdot \nabla u) \cdot (t u_t)\,dx.$$

- Using the previous estimates and the energy balance, we get

$$\|\sqrt{\rho t}\, u_t\|_{L^2} + \int_0^t \|\nabla \sqrt{\tau}\, u_t\|_{L^2}^2\,d\tau \le h(t),$$

where h is a nondecreasing nonnegative function with $h(0) = 0$.

Shift of regularity from time to space variable

- Step 1 gives $\nabla u \in L^\infty(\mathbb{R}_+; L^2)$, $\nabla u \in L^2(\mathbb{R}_+; H^1)$, $\nabla P, \sqrt{\rho} u_t \in L^2(\mathbb{R}_+ \times \mathbb{T}^2)$.
- Step 2 gives $\sqrt{\rho t}\, u_t \in L^\infty_{loc}(\mathbb{R}_+; L^2)$ and $\nabla \sqrt{t}\, u_t \in L^2_{loc}(\mathbb{R}_+; L^2)$.
- Use Stokes equation:

$$\begin{cases} -\Delta \sqrt{t}\, u + \nabla \sqrt{t}\, P = -\sqrt{t}\, \rho u_t - \sqrt{t}\, \rho u \cdot \nabla u, \\ \operatorname{div} \sqrt{t}\, u = 0. \end{cases}$$

Steps 1,2 + embedding imply that the r.h.s. is almost $L^2_{loc}(\mathbb{R}_+; L^\infty)$.
Hence $\nabla^2 \sqrt{t}\, u$ and $\nabla \sqrt{t}\, P$ are almost in $L^2(0,T; L^\infty)$.

- Use embedding and Hölder inequality to conclude that $\nabla u \in L^1_{loc}(0,T; L^\infty)$ (and in fact much better).

The main statement

Theorem (Global existence and uniqueness in $\mathbb{T}^2$, R.D & P.B. Mucha, 2017)

Consider any data (ρ_0, u_0) *in* $L^\infty(\mathbb{T}^2) \times H^1(\mathbb{T}^2)$ *with* $\rho_0 \geq 0$ *and* $\operatorname{div} u_0 = 0$. *Then System* (INS) *supplemented with data* (ρ_0, u_0) *admits a* **unique global solution** $(\rho, u, \nabla P)$ *that satisfies the energy equality, the conservation of total mass and momentum,*

$$\rho \in L^\infty(\mathbb{R}_+; L^\infty), \quad u \in L^\infty(\mathbb{R}_+; H^1), \quad \sqrt{\rho} u_t, \nabla^2 u, \nabla P \in L^2(\mathbb{R}_+; L^2)$$

and also, for all $1 \leq r < 2$, $1 \leq m < \infty$ *and* $T \geq 0$,

$$\nabla(\sqrt{t}P),\ \nabla^2(\sqrt{t}u) \in L^\infty(0,T;L^r) \cap L^2(0,T;L^m).$$

Furthermore, we have $\sqrt{\rho}u \in \mathcal{C}(\mathbb{R}_+; L^2)$ *and* $\rho \in \mathcal{C}(\mathbb{R}_+; L^p)$ *for all* $p < \infty$.

Corollary (Answer to Lions' question)

Take $\rho_0 = \rho_1 1_{D_0} + \rho_2 1_{^cD_0}$ **with** $\rho_1, \rho_2 \geq 0$ **arbitrary**, *and* $u_0 \in H^1(\mathbb{T}^2)$. *Then the regularity* $C^{1,\alpha}$ *of* D_0 *(with* $0 < \alpha < 1$*) is* **preserved for all time.**