Anneaux tourbillonnaires visqueux

Th. Gallay (Institut Fourier, Université Grenoble Alpes)

Analyse, analyse numérique et contrôle des milieux continus

Faculté de Mathématique et Informatique de l'Université de Bucarest

Bucarest, 22 mai 2018

Cet exposé repose sur des travaux en collaboration

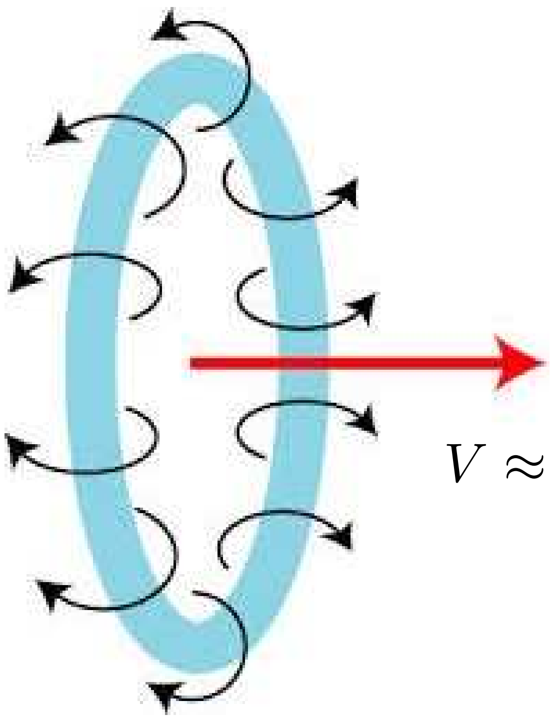
avec Vladimir Sverak (Minneapolis)

What is a vortex ring ?

‘‘Smoke’’ ring over Mount Etna in November 2013
photographed by volcanologist Tom Pfeiffer

Introduction : vortex rings and filaments

A vortex ring is a three-dimensional flow in which the vorticity is essentially concentrated in a solid torus, so that the fluid particles spin around an imaginary line that forms a closed loop.



$$V \approx \frac{\Gamma}{4\pi R} \log \frac{R}{r}$$

Axisymmetric flows without swirl

We use cylindrical coordinates (r, θ, z) in $\mathbb{R}^3$.

- Unit vectors :

$$e_r = \begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \end{pmatrix}, \qquad e_\theta = \begin{pmatrix} -\sin\theta \\ \cos\theta \\ 0 \end{pmatrix}, \qquad e_z = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

"radial" "toroidal" "vertical"

- Velocity field : $u = u_r(r,z,t)e_r + u_z(r,z,t)e_z$.

- Vorticity distribution : $\omega = \omega_\theta(r,z,t)e_\theta$, $\omega_\theta = \partial_z u_r - \partial_r u_z$.

- Incompressibility condition : $\mathrm{div}\, u = \partial_r u_r + \frac{1}{r} u_r + \partial_z u_z = 0$.

Please note : we always assume that the "swirl" $u \cdot e_\theta$ vanishes identically.

Vortex rings in ideal fluids

A) Historical example : Hill's spherical vortex (Hill, 1894)

Vortices bifurcating from that particular solution were studied by Norbury (1974), Amick & Fraenkel (1986, 1988), Amick & Turner (1988).

B) Existence of stationary solutions by variational or fixed point methods :

- Fraenkel (1970, 1972), Fraenkel & Berger (1974)
- Benjamin (1976)
- Ni (1980)
- Friedman & Turkington (1981)
- Ambrosetti & Mancini (1981), Ambrosetti & Struwe (1989)

C) General solutions with concentrated vorticity :

- Benedetto, Caglioti & Marchioro (2000)
- Slightly viscous case : Marchioro (2007), Brunelli & Marchioro (2011)

Overview

We consider the axisymmetric Navier-Stokes equations without swirl, assuming that the initial vorticity is either an integrable function or a finite measure. In the latter case, we concentrate on circular vortex filaments.

Part I
- The axisymmetric (viscous) vorticity equation
- Global well-posedness for integrable data
- Comparison with previous results
- A priori estimates

Part II
- Vorticities represented by finite measures
- Global well-posedness for small data
- Existence of solutions originating from large vortex filaments
- Uniqueness of arbitrarily large viscous vortex rings

Part I : The axisymmetric vorticity equation

The axisymmetric vorticity $\omega_\theta(r,z,t)$ satisfies the evolution equation :

$$\partial_t \omega_\theta + u\cdot\nabla\omega_\theta - \frac{u_r}{r}\omega_\theta \;=\; \nu\Big(\partial_r^2 + \partial_z^2 + \frac{1}{r}\partial_r - \frac{1}{r^2}\Big)\omega_\theta\;, \qquad (1)$$

where $\nu > 0$ is the kinematic viscosity.

The velocity field $u = (u_r, u_z)$ is determined by solving the elliptic system

$$\partial_r u_r + \tfrac{1}{r}u_r + \partial_z u_z \;=\; 0\;, \qquad \partial_z u_r - \partial_r u_z \;=\; \omega_\theta\;.$$

The boundary conditions on the symmetry axis $r = 0$ are

$$\omega_\theta(0,z) \;=\; u_r(0,z) \;=\; \partial_r u_z(0,z) \;=\; 0\;, \qquad z \in \mathbb{R}\;.$$

Important remark : the related quantity $\eta(r,z,t) \;=\; \dfrac{1}{r}\,\omega_\theta(r,z,t)$ satisfies

$$\partial_t \eta + u\cdot\nabla\eta \;=\; \nu\Big(\Delta\eta + \frac{2}{r}\partial_r\eta\Big)\;, \qquad \Delta \;=\; \partial_r^2 + \partial_z^2 + \frac{1}{r}\partial_r\;. \qquad (2)$$

Scale invariant function spaces

Equations (1) and (2) are invariant under the rescaling

$$
\begin{aligned}
u(r,z,t) &\mapsto \lambda\, u(\lambda r, \lambda z, \lambda^2 t)\ , \\
\omega_\theta(r,z,t) &\mapsto \lambda^2\, \omega_\theta(\lambda r, \lambda z, \lambda^2 t)\ , \\
\eta(r,z,t) &\mapsto \lambda^3\, \eta(\lambda r, \lambda z, \lambda^2 t)\ .
\end{aligned}
$$

Natural scale invariant function spaces :

- $\eta \in L^1(\mathbb{R}^3)$, $\quad \|\eta\|_{L^1(\mathbb{R}^3)} = \int_\Omega |\eta(r,z)|\, r\, \mathrm{d}r\, \mathrm{d}z$ (3D measure)
- $\omega_\theta \in L^1(\Omega)$, $\quad \|\omega_\theta\|_{L^1(\Omega)} = \int_\Omega |\omega_\theta(r,z)|\, \mathrm{d}r\, \mathrm{d}z$ (2D measure)

Here Ω denotes the half-space $\Omega = \{(r,z)\,|\, r > 0\,,\ z \in \mathbb{R}\} \subset \mathbb{R}^2$.

Global well-posedness for integrable data

Our first result (ThG & V. Sverak, Confluentes Mathematici, 2015) shows that the axisymmetric vorticity equation (1) is globally well-posed in $L^1(\Omega)$.

Theorem 1 *For any initial data* $\omega_0 \in L^1(\Omega)$, *the vorticity equation*

$$\partial_t \omega_\theta + \partial_r(u_r \omega_\theta) + \partial_z(u_z \omega_\theta) \;=\; \nu\Big(\Delta - \frac{1}{r^2}\Big)\omega_\theta \qquad (1)$$

has a unique global solution $\omega_\theta \in C^0([0,\infty), L^1(\Omega)) \cap C^0((0,\infty), L^\infty(\Omega))$.

Moreover $\|\omega_\theta(t)\|_{L^1(\Omega)} \le \|\omega_0\|_{L^1(\Omega)}$ *for all* $t > 0$, *and*

- $\lim_{t\to 0} t^{1-1/p}\,\|\omega_\theta(t)\|_{L^p(\Omega)} = 0$, $\qquad 1 < p \le \infty$,
- $\lim_{t\to \infty} t^{1-1/p}\,\|\omega_\theta(t)\|_{L^p(\Omega)} = 0$, $\qquad 1 \le p \le \infty$.

Comparison with (some) previous results I

A) Local well-posedness results for general initial data :

- If $\omega_\theta \in L^1(\Omega)$, then $\omega = \omega_\theta\, e_\theta$ belongs to the Morrey space $M^{3/2}(\mathbb{R}^3)$:

$$\sup_{x\in\mathbb{R}^3}\ \sup_{R>0}\ \frac{1}{R}\int_{B(x,R)} |\omega(x)|\,\mathrm{d}x \;<\; \infty\,.$$

$\Rightarrow$ Local well-posedness was established by Giga & Miyakawa (1989).

- If $\omega_\theta \in L^1(\Omega)$, the velocity field u belongs to $\mathrm{BMO}^{-1}(\mathbb{R}^3)$:

$$\sup_{x\in\mathbb{R}^3}\ \sup_{R>0}\ \frac{1}{R^3}\int_{B(x,R)}\int_0^{R^2} |e^{t\Delta}u|^2\,\mathrm{d}t\,\mathrm{d}x \;<\; \infty\,.$$

$\Rightarrow$ Local well-posedness was established by Koch & Tataru (2001).

- If $\omega_\theta \in L^1(\Omega)$, the velocity field u belongs to $\dot{B}^{-1+3/p}_{p,q}(\mathbb{R}^3)$ iff $p = q = \infty$.

Comparison with (some) previous results II

B) Global well-posedness results for axisymmetric initial data :

- Ladyzhenskaya (1968), Ukhovskii & Yudovich (1968) :

$$u \in H^2(\mathbb{R}^3)\,, \quad \omega_\theta \in L^\infty(\mathbb{R}^3)\,, \quad \eta \in L^1(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)\,.$$

- Leonardi, Malek, Necas, & Pokorny (1999) : $u \in H^2(\mathbb{R}^3)$.

- Abidi (2008), Abidi, Hmidi, & Keraani (2010) : $u \in H^{1/2}(\mathbb{R}^3)$.

All these global well-posedness results consider finite energy solutions.

C) Local well-posedness for axisymmetric data with swirl satisfying

$$\int_{\mathbb{R}^3} \frac{|u(x)|^2}{r}\,\mathrm{d}x \,=\, 2\pi \int_\Omega |u(r,z)|^2\,\mathrm{d}r\,\mathrm{d}z \,<\, \infty\,.$$

Gallagher, Ibrahim, & Majdoub (2001, 2002).

A priori estimates I

A) Estimates for the auxiliary quantity $\eta = \omega_\theta / r$:

Applying Nash's method to the advection-diffusion equation

$$\partial_t \eta + u \cdot \nabla \eta \;=\; \Delta \eta + \frac{2}{r} \partial_r \eta \;, \tag{2}$$

with initial data $\eta_0 = \omega_0 / r$, we obtain for $t > 0$:

$$\|\eta(t)\|_{L^p(\mathbb{R}^3)} \;\le\; \frac{C}{t^{\frac{3}{2}(1-\frac{1}{p})}} \|\eta_0\|_{L^1(\mathbb{R}^3)} \;, \qquad 1 \le p \le \infty \;.$$

Moreover $t \mapsto \|\eta(t)\|_{L^p(\mathbb{R}^3)}$ is non-increasing for $1 \le p \le \infty$.

This provides estimates in weighted norms for the axisymmetric vorticity :

$$\|r^{\frac{1}{p}-1} \omega_\theta(t)\|_{L^p(\Omega)} \;\le\; \frac{C}{t^{\frac{3}{2}(1-\frac{1}{p})}} \|\omega_0\|_{L^1(\Omega)} \;, \qquad t > 0 \;.$$

The particular case $p = \infty$ is especially useful.

A priori estimates II

B) Estimates for the axisymmetric vorticity ω_θ :

Proposition 1 *Any solution of the axisymmetric vorticity equation (1) with initial data* $\omega_0 \in L^1(\Omega)$ *satisfies, for* $1 \le p \le \infty$ *:*

$$\|\omega_\theta(t)\|_{L^p(\Omega)} \le \frac{C(\|\omega_0\|_{L^1(\Omega)})}{t^{1-\frac{1}{p}}}, \qquad t > 0,$$

where $C(s) = \mathcal{O}(s)$ *as* $s \to 0$*. Moreover the map* $t \mapsto \|\omega_\theta(t)\|_{L^1(\Omega)}$ *is strictly decreasing if* $\omega_\theta \not\equiv 0$*.*

Proof: We know that $t \mapsto \|\omega_\theta(t)\|_{L^1(\Omega)} = \|\eta(t)\|_{L^1(\mathbb{R}^3)}$ is non-increasing.

For nontrivial positive solutions, we compute

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \omega_\theta(r,z,t)\,\mathrm{d}r\,\mathrm{d}z = -2\int_{\mathbb{R}} \partial_r\omega_\theta(0,z,t)\,\mathrm{d}z < 0,$$

hence $t \mapsto \|\omega_\theta(t)\|_{L^1(\Omega)}$ is strictly decreasing.

A priori estimates III

When $p = 2$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \omega_\theta^2 \,\mathrm{d}r\,\mathrm{d}z \;=\; -2\int_\Omega |\nabla\omega_\theta|^2 \,\mathrm{d}r\,\mathrm{d}z + \int_\Omega \Big(\frac{u_r}{r} - \frac{1}{r^2}\Big)\omega_\theta^2 \,\mathrm{d}r\,\mathrm{d}z \;.$$

Denoting $M = \|\omega_0\|_{L^1(\Omega)}$, we use Nash's inequality :

$$\|\omega_\theta(t)\|_{L^2(\Omega)}^2 \;\le\; C\|\omega_\theta(t)\|_{L^1(\Omega)}\|\nabla\omega_\theta(t)\|_{L^2(\Omega)} \;\le\; CM\|\nabla\omega_\theta(t)\|_{L^2(\Omega)} \;,$$

and the following estimate on the velocity field

$$\|u_r(t)/r\|_{L^\infty(\Omega)} \;\le\; C\,\|\omega_\theta(t)\|_{L^1(\Omega)}^{1/3}\,\|\omega_\theta(t)/r\|_{L^\infty(\Omega)}^{2/3} \;\le\; CM/t \;.$$

If $f(t) = \|\omega_\theta(t)\|_{L^2(\Omega)}^2$, we thus obtain the differential inequality

$$f'(t) \;\le\; -\frac{K_1}{M^2}\,f(t)^2 + \frac{K_2 M}{t}\,f(t) \;, \qquad K_1, K_2 > 0 \;,$$

which gives the bound $f(t) = \|\omega_\theta(t)\|_{L^2(\Omega)}^2 \le K_1^{-1}(1 + K_2 M)M^2/t$ for $t > 0$.

A priori estimates IV

Since, for $p > 1$, an upper bound on $\|\omega_0\|_{L^p(\Omega)}$ gives a lower bound on the local existence time T, we deduce :

Corollary *All solutions of the vorticity equation (1) in* $L^1(\Omega)$ *are global for positive times.*

C) Estimate for the velocity field u :

$$\|u(t)\|_{L^\infty(\Omega)} \le C\|\omega_\theta(t)\|_{L^1(\Omega)}^{1/2} \|\omega_\theta(t)\|_{L^\infty(\Omega)}^{1/2} \le \frac{C(\|\omega_0\|_{L^1(\Omega)})}{\sqrt{t}} . \qquad (3)$$

D) Estimates for the vorticity gradient $\nabla\omega_\theta$:

$$\|\nabla\omega_\theta(t)\|_{L^p(\Omega)} \le \frac{C_p(\|\omega_0\|_{L^1(\Omega)})}{t^{\frac{3}{2}-\frac{1}{p}}} , \qquad 1 \le p \le \infty . \qquad (4)$$

This follows from Proposition 1, estimate (3), and standard smoothing properties of the Navier-Stokes equations.

Part II : The space of finite measures

As in the 2D case, we can take the initial vorticity in the space $\mathcal{M}(\Omega)$ of all finite, real-valued measures on Ω. Given $\mu = \omega_0 \in \mathcal{M}(\Omega)$, we decompose

$$\mu \;=\; \mu_{ac} + \mu_{sc} + \mu_{pp}\;, \qquad \text{where}$$

- μ_{ac} is absolutely continuous with respect to Lebesgue's measure;
- μ_{pp} is a countable collection of point masses;
- μ_{sc} has no atoms, yet is supported on a set of zero Lebesgue measure.

The Banach space $\mathcal{M}(\Omega)$ is equipped with the total variation norm :

$$\|\mu\|_{\mathrm{tv}} \;=\; \sup\left\{\int_\Omega \varphi\,\mathrm{d}\mu \;\middle|\; \varphi \in C_0(\Omega)\,,\; \|\varphi\|_{L^\infty(\Omega)} \le 1\right\}.$$

For any $\mu \in \mathcal{M}(\Omega)$ one has $\mu_{ac} \perp \mu_{sc} \perp \mu_{pp}$, hence

$$\|\mu\|_{\mathrm{tv}} \;=\; \|\mu_{ac}\|_{\mathrm{tv}} + \|\mu_{sc}\|_{\mathrm{tv}} + \|\mu_{pp}\|_{\mathrm{tv}}\;.$$

Well-posedness for measure-valued initial data

Theorem 2 *There exists positive constants* ε *and* C *such that, for any initial data* $\mu \in \mathcal{M}(\Omega)$ *satisfying* $\|\mu_{pp}\|_{\mathrm{tv}} \le \varepsilon\nu$*, the axisymmetric vorticity equation*

$$\partial_t \omega_\theta + \partial_r(u_r \omega_\theta) + \partial_z(u_z \omega_\theta) = \nu\Big(\Delta - \frac{1}{r^2}\Big)\omega_\theta \qquad (1)$$

has a unique global (mild) solution

$$\omega_\theta \in C^0((0,\infty), L^1(\Omega) \cap L^\infty(\Omega))$$

such that

$$\lim_{t\to 0} \|\omega_\theta(t)\|_{L^1(\Omega)} < \infty \,, \qquad \limsup_{t\to 0} \,(\nu t)^{1/4} \|\omega_\theta(t)\|_{L^{4/3}(\Omega)} \le C\varepsilon\nu \,,$$

and $\omega_\theta(t)\,\mathrm{d}r\,\mathrm{d}z \rightharpoonup \mu$ *as* $t \to 0$*. Moreover,*

$$\lim_{t\to\infty} t^{1-1/p} \|\omega_\theta(t)\|_{L^p(\Omega)} = 0 \,, \qquad 1 \le p \le \infty \,.$$

Large vortex rings I : existence

The following existence result was obtained by Feng & Sverak (ARMA, 2015) :

Theorem 3 *Fix* $\Gamma > 0$, $\bar{r} > 0$, $\bar{z} \in \mathbb{R}$, *and* $\nu > 0$. *Then the axisymmetric vorticity equation*

$$\partial_t \omega_\theta + \partial_r(u_r \omega_\theta) + \partial_z(u_z \omega_\theta) = \nu\Big(\Delta - \frac{1}{r^2}\Big)\omega_\theta \,, \qquad (1)$$

has a non-negative global solution such that $\omega_\theta(t)\,\mathrm{d}r\,\mathrm{d}z \rightharpoonup \Gamma\,\delta_{(\bar{r},\bar{z})}$ *as* $t \to 0$. *Moreover, this solution satisfies, for all* $t > 0$,

$$\int_\Omega \omega_\theta(r,z,t)\,\mathrm{d}r\,\mathrm{d}z \le \Gamma \,, \qquad \int_\Omega r^2 \omega_\theta(r,z,t)\,\mathrm{d}r\,\mathrm{d}z = \Gamma\,\bar{r}^2 \,.$$

The proof is based on an approximation procedure, which is reminiscent of the works of Cottet (1986) and Giga, Miyakawa, & Osada (1988) in the two-dimensional case.

Large vortex rings II : uniqueness

Our final result is (ThG & V. Sverak, to appear in Annales de l'ENS) :

Theorem 4 *Fix* $\Gamma > 0$, $\bar{r} > 0$, $\bar{z} \in \mathbb{R}$, $\nu > 0$. *Then the axisymmetric vorticity eq.*

$$\partial_t \omega_\theta + \partial_r(u_r \omega_\theta) + \partial_z(u_z \omega_\theta) \;=\; \nu\Big(\Delta - \frac{1}{r^2}\Big)\omega_\theta \qquad (1)$$

has a unique global solution ω_θ *such that :*

i) $\sup_{t>0} \|\omega_\theta(t)\|_{L^1(\Omega)} < \infty$, *and*

ii) $\omega_\theta(t)\,\mathrm{d}r\,\mathrm{d}z \rightharpoonup \Gamma\,\delta_{(\bar{r},\bar{z})}$ *as* $t \to 0+$.

Moreover the solution ω_θ *is non-negative and satisfies*

$$\int_\Omega \Big|\omega_\theta(r,z,t) - \frac{\Gamma}{4\pi\nu t}\,e^{-\frac{(r-\bar{r})^2+(z-\bar{z})^2}{4\nu t}}\Big|\,\mathrm{d}r\,\mathrm{d}z \;\le\; C\,\Gamma\,\frac{\sqrt{\nu t}}{\bar{r}}\log\frac{\bar{r}}{\sqrt{\nu t}}\;, \qquad (5)$$

as long as $\sqrt{\nu t} \le \bar{r}/2$, *where* $C > 0$ *depends only on* Γ/ν.

Comments on the uniqueness result

- Assumptions i), ii) on the axisymmetric vorticity equation are arguably the weakest ones under which estimate (5) is expected to hold.
- Existence of a viscous vortex ring with initial data $\Gamma\,\delta_{(\bar{r},\bar{z})}$ is already established in Theorem 3. Uniqueness is the main new assertion in Theorem 4, together with the short time asymptotic expansion (5).
- The short time estimate (5) is sharp in the sense that the logarithmic correction in the right-hand side cannot be dispensed with, except if the position of the viscous vortex evolves in time according to

$$\bar{z}(t) \;=\; \bar{z} + \frac{\Gamma t}{4\pi\bar{r}} \log \frac{\bar{r}}{\sqrt{\nu t}}\,.$$

- An important open problem is to control the viscous vortex ring over a finite time interval $t \in [0,T]$ in the vanishing viscosity limit $\nu \to 0$.
- Uniqueness is only asserted within the class of axisymmetric solutions without swirl !

Sketch of the uniqueness proof

Assume that $\omega_\theta \in C^0((0,\infty), L^1(\Omega) \cap L^\infty(\Omega))$ is a mild solution of the axisymmetric vorticity equation (1) satisfying

i) $\sup_{t>0} \|\omega_\theta(t)\|_{L^1(\Omega)} < \infty$, and

ii) $\omega_\theta(t)\,\mathrm{d}r\,\mathrm{d}z \rightharpoonup \Gamma\,\delta_{(\bar r,\bar z)}$ as $t \to 0+$.

Step 1 : Localization. For any $\eta > 0$ there exists $C > 0$ such that

$$|\omega_\theta(r,z,t)| \le \frac{C\Gamma}{\nu t} \exp\Big(-\frac{(r-\bar r)^2 + (z-\bar z)^2}{(4+\eta)\nu t}\Big)\,, \qquad (r,z) \in \Omega\,, \quad t > 0\,. \quad (6)$$

Moreover $\int_\Omega \omega_\theta(r,z,t)\,\mathrm{d}r\,\mathrm{d}z \longrightarrow \Gamma$ as $t \to 0$.

This is proved using a Gaussian upper bound on the fundamental solution of the vorticity equation (1), where the velocity field u is considered as given.

The proof of the Gaussian bound (6) relies on the study of the adjoint equation

$$\partial_t \varphi + u \cdot \nabla \varphi + \nu \Big(\Delta \varphi - \frac{2}{r} \varphi \Big) \;=\; 0\,, \tag{7}$$

which defined so that

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \varphi(r,z,t)\, \omega_\theta(r,z,t)\, \mathrm{d}r\, \mathrm{d}z \;=\; 0\,,$$

whenever ω_θ solves (1). Eq. (7) can be solved backwards in time with "terminal condition" at time $T > 0$. Boundary conditions are $\varphi = \partial_r \varphi = 0$ on $\partial\Omega$.

Proposition *Assume that u is the velocity field associated with a mild solution ω_θ of (1) satisfying i), ii). Given $T > 0$ and $\varphi_1 \in C_0(\Omega)$, the unique solution φ of the adjoint equation (7) with terminal condition $\varphi(\cdot,\cdot,T) = \varphi_1$ can be extended to a continuous function on $\bar{\Omega} \times [0,T]$ satisfying $\varphi(0,z,0) = 0$ for all $z \in \mathbb{R}$. Moreover one has $\varphi(\cdot,\cdot,t) \in C_0(\Omega)$ for all $t \in [0,T]$, and*

$$\sup_{(r,z)\in\Omega} |\varphi(r,z,t) - \varphi(r,z,0)| \;\longrightarrow\; 0\,, \qquad \textit{as} \quad t \to 0\,.$$

The proposition itself relies on the regularity theory for drift-diffusion equations of the form

$$\partial_t h + b(x,t)\cdot\nabla h \,=\, \nu\Delta h\,, \qquad x\in\mathbb{R}^n\,, \quad t>0\,,$$

where

- $b \in L^\infty_t(L^\infty)^{-1}_x$ (Osada, 1987), or
- $b \in L^\infty_t(\mathrm{BMO})^{-1}_x$ (Koch, Nadirashvili, Seregin, Sverak, 2009).

In the present case, we have the estimate

$$\|u\|_{(L^\infty)^{-1}(\mathbb{R}^3)} \,\le\, C\|\omega_\theta\|_{L^1(\Omega)}\,,$$

which can be checked directly using the axisymmetric Biot-Savart law.

Consequences : Under the assumptions of Theorem 4,

- $\omega_\theta(r,z,t) > 0$ for all $t>0$;
- $\|\omega_\theta(t)\|_{L^1(\Omega)} \to \Gamma$ as $t\to 0$;
- the sequence $(\omega_\theta(t)\,\mathrm{d}r\,\mathrm{d}z)_{t\in(0,T)}$ is tight.

Evolution equation for the vector-valued vorticity $\omega(x,t) = \omega_\theta(r,z,t)e_\theta$:

$$\partial_t\omega + (U\cdot\nabla)\omega - V\omega \,=\, \nu\Delta\omega\,, \qquad x\in\mathbb{R}^3\,, \qquad t>0\,, \qquad (*)$$

where

- $U = u_r e_r + u_z e_z$ satisfies $K_1 \,:=\, \sup_{t>0}\Bigl(\frac{t}{\nu}\Bigr)^{1/2}\|U(\cdot,t)\|_{L^\infty(\mathbb{R}^3)} \,<\, \infty\,,$
- $V = u_r/r$ satisfies $K_2 \,:=\, \int_0^\infty \|V(\cdot,t)\|_{L^\infty(\mathbb{R}^3)}\,\mathrm{d}t \,<\, \infty$ (not obvious !)

Proposition (Aronson's estimate)
The fundamental solution of $(*)$ *satisfies*

$$0 \,<\, \Phi(x,t;y,s) \,\le\, \frac{C}{(\nu(t-s))^{3/2}}\,\exp\Bigl(-\frac{|x-y|^2}{4\nu(t-s)} + K_1\frac{|x-y|}{\sqrt{\nu(t-s)}} + K_2\Bigr)\,,$$

for $x,y\in\mathbb{R}^3$ *and* $t>s>0$*, where* $C>0$ *is a universal constant.*

Conclusion of step 1 : "integrating over θ" yields the Gaussian upper bound.

Step 2 : Self-similar variables. We make the change of variables

$$\begin{cases} \omega_\theta(r,z,t) \;=\; \dfrac{\Gamma}{\nu t}\, f\Big(\dfrac{r-\bar{r}}{\sqrt{\nu t}}\,,\,\dfrac{z-\bar{r}}{\sqrt{\nu t}}\,,\,t\Big)\,, \\[2ex] u_\theta(r,z,t) \;=\; \dfrac{\Gamma}{\sqrt{\nu t}}\, U^\varepsilon\Big(\dfrac{r-\bar{r}}{\sqrt{\nu t}}\,,\,\dfrac{z-\bar{z}}{\sqrt{\nu t}}\,,\,t\Big)\,. \end{cases}$$

We also introduce the dimensionless quantities

$$R \;=\; \frac{r-\bar{r}}{\sqrt{\nu t}}\,, \quad Z \;=\; \frac{z-\bar{z}}{\sqrt{\nu t}}\,, \quad \varepsilon \;=\; \frac{\sqrt{\nu t}}{\bar{r}}\,, \qquad \gamma \;=\; \frac{\Gamma}{\nu}\,.$$

The evolution equation for the new function $f(R,Z,t)$ reads

$$t\partial_t f + \gamma\Big(\partial_R(U_R^\varepsilon f) + \partial_Z(U_Z^\varepsilon f)\Big) \;=\; \mathcal{L}f + \varepsilon\partial_R\Big(\frac{f}{1+\varepsilon R}\Big)\,, \tag{8}$$

where

$$\mathcal{L} \;=\; \partial_R^2 + \partial_R^2 + \frac{R}{2}\,\partial_R + \frac{Z}{2}\,\partial_Z + 1\,.$$

Remarks :

- Eq. (8) is now defined in the time-dependent domain

$$\Omega_\varepsilon \;=\; \Bigl\{(R,Z) \in \mathbb{R}^2 \,|\, 1+\varepsilon R > 0\Bigr\}\,.$$

  Note that $\Omega_\varepsilon \to \mathbb{R}^2$ as $\varepsilon \to 0$ and $\Omega_\varepsilon \to \Omega$ as $\varepsilon \to \infty$.

- The velocity U^ε is reconstructed from the vorticity f by solving the linear elliptic system

$$\partial_Z U_r^\varepsilon - \partial_R U_z^\varepsilon \;=\; f\,, \qquad \partial_R U_r^\varepsilon + \frac{\varepsilon U_r^\varepsilon}{1+\varepsilon R} + \partial_Z U_z^\varepsilon \;=\; 0\,,$$

  which interpolates between the Biot-Savart law in $\mathbb{R}^2$ and in Ω.

- As $t \to 0$, i.e. $\varepsilon \to 0$, equation (8) reduces to the two-dimensional vorticity equation in $\mathbb{R}^2$, expressed in self-similar variables.

- The Gaussian bound in step 1 implies, for any $\eta > 0$, the a priori estimate

$$0 \;<\; f(R,Z,t) \;\le\; C_\eta \exp\Bigl(-\frac{R^2+Z^2}{(4+\eta)}\Bigr)\,, \qquad (R,Z) \in \Omega_\varepsilon\,, \quad t > 0\,.$$

Step 3: Compactness. The solution $f(t)$ of (8) is uniformly bounded for $t \in (0,1]$ and relatively compact in the space X_t defined by the norm

$$\|f(t)\|_{X_t}^2 = \int_{\Omega_\varepsilon} f(R,Z,t)^2 \, e^{(R^2+Z^2)/4} \, \mathrm{d}R \, \mathrm{d}Z \, .$$

This follows from the Gaussian bound above, thanks to parabolic regularity.

Step 4: Alpha-limit set. As $t \to 0$ we have

$$\lim_{t\to 0} \|f(t) - G\|_{X_t} = 0 \,, \qquad \text{where} \quad G(R,Z) = \frac{1}{4\pi} \, e^{-\frac{1}{4}(R^2+Z^2)} \, .$$

Intuitively, any f_0 in the α-limit set of the trajectory $(f(t))_{t\in(0,1]}$ in X_t is the value at $\tau = 0$ of an ancient solution to the rescaled vorticity equation in $\mathbb{R}^2$:

$$\partial_\tau f + \gamma U \cdot \nabla f = \mathcal{L} f \,, \qquad U = K_{BS} * f \, . \tag{9}$$

Moreover $|f_0(R,Z)| \le C \, e^{-(R^2+Z^2)/5}$ and $\int_{\mathbb{R}^2} f_0(R,Z) \, \mathrm{d}R \, \mathrm{d}Z = 1$.

Liouville theorem (ThG & C.E. Wayne, 2005): $f_0 = G$.

Step 5 : Proof of estimate (5). We decompose

$$
\begin{aligned}
f(R,Z,t) &= G(R,Z) + \tilde{f}(R,Z,t)\,, \\
U^\varepsilon(R,Z,t) &= U_G^\varepsilon(R,Z,t) + \tilde{U}^\varepsilon(R,Z,t)\,,
\end{aligned}
$$

and we define

$$
\begin{aligned}
E(t) &= \frac{1}{2}\int_{\Omega_\varepsilon} \tilde{f}(R,Z,t)^2 G^{-1}(R,Z)\,\mathrm{d}R\,\mathrm{d}Z\,, \\
\mathcal{E}(t) &= \frac{1}{2}\int_{\Omega_\varepsilon} \Bigl(|\nabla\tilde{f}|^2 + (1+R^2+Z^2)\tilde{f}^2\Bigr) G^{-1}\,\mathrm{d}R\,\mathrm{d}Z \;\ge\; E(t)\,.
\end{aligned}
$$

Proposition *There exist* $\delta > 0$ *and* $\kappa > 0$ *such that, if* $\varepsilon > 0$ *is small enough,*

$$
tE'(t) \;\le\; -2\delta\mathcal{E}(t) + \kappa\varepsilon|\log\varepsilon|\,E(t)^{1/2} + \kappa E(t)^{1/2}\mathcal{E}(t) + \mathcal{O}(e^{-1/(36\varepsilon^2)})\,.
$$

The proof relies on the stability of the Oseen vortex γG as an equilibrium of the rescaled vorticity equation (9), for arbitrary values of the circulation γ.

Step 6 : Uniqueness. If $f_1(t), f_2(t)$ are two solutions of (8) which converge to G as $t \to 0$, we define $\tilde{f} = f_1 - f_2$ and denote as above

$$E(t) \;=\; \frac{1}{2}\int_{\Omega_\varepsilon} \tilde{f}(R,Z,t)^2 G^{-1}(R,Z)\,\mathrm{d}R\,\mathrm{d}Z\,,$$

$$\mathcal{E}(t) \;=\; \frac{1}{2}\int_{\Omega_\varepsilon} \Big(|\nabla\tilde{f}|^2 + (1+R^2+Z^2)\tilde{f}^2\Big)G^{-1}\,\mathrm{d}R\,\mathrm{d}Z \;\geq\; E(t)\,.$$

Proposition *There exist* $\delta, \kappa, K > 0$ *such that, if* $\varepsilon > 0$ *is small enough,*

$$tE'(t) \;\leq\; -2\delta\mathcal{E}(t) + \kappa(E_1(t)^{1/2} + E_2(t)^{1/2})\,\mathcal{E}(t) + \mathcal{O}(e^{-1/(36\varepsilon^2)})\,,$$

$$tE'(t) \;\leq\; -\delta\mathcal{E}(t) + KE(t) + \kappa(E_1(t)^{1/2} + E_2(t)^{1/2})\,\mathcal{E}(t)\,.$$

The first inequality shows that $E(t) = \mathcal{O}(e^{-1/(36\varepsilon^2)})$ as $t \to 0$.

The second inequality implies $E(t) \leq (t/t_0)^K E(t_0)$ for $0 < t_0 < t$, hence

$$E(t) \;\equiv\; 0 \quad \text{for sufficiently small} \quad t > 0\,.$$

Merci de votre attention !