Stabilisation feedback d'un système de transition de phase avec effets de viscosité

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Atelier "Analyse, analyse numérique et contrôle des milieux continus"

Introduction

- Let us consider the problem

$$\begin{aligned} y'(t) + \mathcal{A}y(t) &= 0, \text{ a.e. } t > 0, \\ y(0) &= y_0 \end{aligned}$$

$$\mathcal{A} : \mathcal{H} \to \mathcal{H}$$

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- Let y_∞ be a stationary solution

$$\mathcal{A}y_\infty = 0$$

Introduction

A stationary solution y_∞ is asymptotically stable if

$$\textit{for any } y_0 \in \mathcal{H}, \ \ \|y_0 - y_\infty\|_{\mathcal{H}} \leq \varepsilon$$

$$\lim_{t\to\infty} y(t) = y_\infty \text{ in } \mathcal{H}.$$

- If y_∞ is not asymptotically stable, one can attempt to stabilize it by a feedback controller $U(t) = \mathcal{F}(y(t))$

$$\begin{aligned} y'(t) + \mathcal{A}y(t) &= BU(t), \text{ a.e. } t > 0, \\ y(0) &= y_0 \end{aligned}$$

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- U can act on a part of the boundary; boundary control

Problem presentation

The transformation

$$y \longrightarrow y - y_\infty$$

implies to stabilize the stationary solution $y_\infty = 0$ for y_0 in a neighborhood of 0.

Problem presentation

The Cahn-Hilliard system (CH) in the Caginalp approach

φ =*order parameter*, μ =*chemical potential*, θ =*temperature*

$$\begin{aligned} \varphi_t - \Delta\mu &= 0, \text{ in } (0,\infty)\times\Omega, \\ \mu &= \tau\varphi_t - \nu\Delta\varphi + F'(\varphi) - \gamma\theta, \text{ in } (0,\infty)\times\Omega, \end{aligned}$$

$$(\theta + l\varphi)_t - \Delta\varphi = 0, \text{ in } (0,\infty)\times\Omega,$$

$$\varphi(0) = \varphi_0, \ \theta(0) = \theta_0, \text{ in } \Omega,$$

$$\frac{\partial\varphi}{\partial\nu} = \frac{\partial\mu}{\partial\nu} = \frac{\partial\theta}{\partial\nu} = 0, \text{ on } (0,\infty)\times\partial\Omega,$$

$$\nu, \ l, \ \gamma > 0, \ \tau = \textit{viscosity} > 0$$

The Cahn-Hilliard system (CH)

$$\varphi = \textit{order parameter}, \ \ \theta = \textit{temperature}, \ \ \sigma = \theta + l\varphi$$

$$(1 - \tau\Delta)\varphi_t + \nu\Delta^2\varphi - \Delta F'(\varphi) - \gamma l\Delta\varphi + \gamma\Delta\sigma = 0, \ \text{in } (0, \infty) \times \Omega,$$

$$\sigma_t - \Delta\sigma + l\Delta\varphi = 0, \ \text{in } (0, \infty) \times \Omega,$$

$$\varphi(0) = \varphi_0, \ \sigma(0) = \sigma_0, \ \text{in } \Omega,$$

$$\frac{\partial\varphi}{\partial\nu} = \frac{\partial\Delta\varphi}{\partial\nu} = \frac{\partial\sigma}{\partial\nu} = 0, \ \text{on } (0, \infty) \times \partial\Omega,$$

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Problem presentation

Double well potential

$$F(r) = \frac{(r^2 - 1)^2}{4}$$

Logarithmic potential

$$F(r) = (1 + r)\ln(1 + r) + (1 - r)\ln(1 - r) - ar^2, \ r \in (-1, 1), \ a > 0.$$

Problem presentation

Problem

The aim is to stabilize exponentially a stationary solution $(\varphi_\infty, \sigma_\infty)$ *by means of an internal feedback control*

$$(\mathrm{v}, \mathbf{u}) = \mathcal{F}(\varphi, \sigma),$$

namely

$$\lim_{t\to\infty} (\varphi(t), \sigma(t)) = (\varphi_\infty, \sigma_\infty),$$

with exponential decay, as the initial datum (φ_0, σ_0) *is in a neighborhood of* $(\varphi_\infty, \sigma_\infty)$.

[1]

[1] G. M., Pure Appl. Funct. Analysis 1 (2018)

Problem presentation

Lemma

The stationary system

$$\begin{aligned}
\nu\Delta^2\varphi_\infty - \Delta F'(\varphi_\infty) - \gamma l\Delta\varphi_\infty + \gamma\Delta\sigma_\infty &= 0, \text{ in } \Omega, \\
-\Delta\sigma_\infty + l\Delta\varphi_\infty &= 0, \text{ in } \Omega, \\
\frac{\partial\varphi_\infty}{\partial\nu} &= \frac{\partial\Delta\varphi_\infty}{\partial\nu} = \frac{\partial\sigma_\infty}{\partial\nu} = 0, \text{ on } \partial\Omega
\end{aligned}$$

has at least a solution

$$\varphi_\infty \in H^4(\Omega),\ \theta_\infty = \text{constant, for the regular potential}$$

$$\varphi_\infty = \text{constant},\ \sigma_\infty = \text{constant, for the singular potential}$$

Problem presentation

The controlled Cahn-Hilliard system

- $$(1-\tau\Delta)\varphi_t + \nu\Delta^2\varphi - \Delta F'(\varphi) - \gamma l\Delta\varphi + \gamma\Delta\sigma = (1-\tau\Delta)(f_\omega v)$$

$$\sigma_t - \Delta\sigma + l\Delta\varphi = f_\omega u$$

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- $$f_\omega \in C_0^\infty(\Omega),\quad \text{supp } f_\omega \subset \omega,\quad f_\omega > 0 \text{ on } \omega_0 \subset \omega$$

  $$\omega \text{ open bounded subset of } \Omega \subset \mathbb{R}^d,\ d = 1, 2, 3$$

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6. Limit case $\tau = 0$ for the regular potential

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- The stabilization technique already used for Navier-Stokes equations and nonlinear parabolic systems is based on the design of the feedback controller as linear combination of the unstable modes of the corresponding linearized system.

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- Introduce

$$A = I - \tau\Delta,\ \ A : D(A) \subset H \times H \to H \times H,$$

$$D(A) = \left\{ w \in H^2(\Omega);\ \frac{\partial w}{\partial \nu} = 0 \text{ on } \partial\Omega \right\}$$

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- A is linear continuous, self-adjoint and m-accretive on H.
- Define $A^\alpha : D(A^\alpha) \subset H \to H,\ \ \alpha \geq 0$

$$D(A^\alpha) = \{w \in H; \|A^\alpha w\|_H < \infty\}, \quad \|w\|_{D(A^\alpha)} = \|A^\alpha w\|_H$$

$$D(A^\alpha) \subset H^{2\alpha}(\Omega) \text{ with equality if } 2\alpha < 3/2,\ \ \alpha \neq 1/4.$$

Preliminaries: the transformed nonlinear system (NS)

- $$\begin{aligned} (1-\tau\Delta)\varphi_t + \nu\Delta^2\varphi - \Delta F'(\varphi) - \gamma l\Delta\varphi + \gamma\Delta\sigma &= (1-\tau\Delta)(f_\omega v) \\ \sigma_t - \Delta\sigma + l\Delta\varphi &= f_\omega u \end{aligned}$$

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- develop $F'(y+\varphi_\infty)$ in Taylor series

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- develop $F'(y + \varphi_\infty)$ in Taylor series
- apply A^{-1} to the first equation

Preliminaries: the transformed nonlinear system (NS)

$$
\begin{aligned}
&y_t + \frac{\nu}{\tau^2}(A + A^{-1} - 2)y - \frac{1}{\tau}(A^{-1} - I)(F''(\varphi_\infty)y) \\
&+\frac{\gamma}{\tau}(A^{-1} - I)z - \frac{\gamma I}{\tau}(A^{-1} - I)y = f_\omega v + \frac{1}{\tau}(A^{-1} - I)F_r(y),
\end{aligned}
$$

$$
z_t + \frac{1}{\tau}(A - I)z + \frac{I}{\tau}(I - A)y = f_\omega u,
$$

$$
y(0) = y_0,\ z(0) = z_0.
$$

$F_r(y)$ is the rest of the Taylor series

Preliminaries: the transformed nonlinear system (NS)

- Denote $U(t) = (v(t), u(t))$

$$\frac{d}{dt}(y(t), z(t)) + \mathcal{A}(y(t), z(t)) = f_\omega U(t) + \mathcal{G}(y(t)), \text{ a.e. } t \in (0, \infty), \quad (NS)$$

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- $\mathcal{A} : D(\mathcal{A}) \subset H \times H \to H \times H$
- $$D(\mathcal{A}) = \left\{ w = (y, z) \in L^2(\Omega) \times L^2(\Omega);\ \mathcal{A}w \in H \times H, \frac{\partial y}{\partial \nu} = \frac{\partial z}{\partial \nu} = 0 \text{ on } \partial\Omega \right\}.$$

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- Set

$$\mathcal{H} = H \times H,\ \mathcal{V} = D(A^{1/2}) \times D(A^{1/2}),\ \mathcal{V}' = (D(A^{1/2}) \times D(A^{1/2}))'$$

Preliminaries: the linear system (LS)

$$\frac{d}{dt}(y(t), z(t)) + \mathcal{A}(y(t), z(t)) = f_\omega U(t), \text{ a.e. } t \in (0, \infty) \qquad (NS)$$

Preliminaries: The linear system

Proposition

The operator $\mathcal{A}$ is quasi m-accretive on $H \times H$ and its resolvent is compact. Moreover, $-\mathcal{A}$ generates a C_0-analytic semigroup.
Let $(y_0, z_0) \in H \times H$ and $U = (v, u) \in L^2(0, T; H \times H)$.
Then, the linear Cauchy problem (LS) has, for all $T > 0$, a unique solution

$$(y, z) \in C([0, T]; H \times H) \cap L^2(0, T; H^2(\Omega) \times H^1(\Omega)) \cap C((0, T]; H^2(\Omega) \times H^1(\Omega)).$$

Preliminaries: The linear system

Fact

The resolvent of $\mathcal{A}$ is compact $\Longrightarrow$ there exists a finite number of eigenvalues with nonpositive real parts $\mathrm{Re}\,\lambda_i < 0$, each having the order of multiplicity l_i, $i = 1, ..., p$.

$$\mathrm{Re}\,\lambda_1 \leq \mathrm{Re}\,\lambda_2 \leq ... \leq \mathrm{Re}\,\lambda_N \leq 0$$

$$N = l_1 + l_2 + ... + l_p$$

Denote $(\varphi_i, \psi_i)\}_{i\geq 1}$ *the complex eigenfunctions of* $\mathcal{A}$
Denote $(\varphi_i^*, \psi_i^*)\}_{i\geq 1}$ *the complex eigenfunctions of* $\mathcal{A}^*$.

2. Stabilization of the linear system by a finite dimensional controller

- $$f_\omega U(t,x) = \sum_{j=1}^{N} f_\omega \operatorname{Re}(\widetilde{w}_j(t)(\varphi_j^*(x), \psi_j^*(x))), \ t \geq 0, \ x \in \Omega, \quad (C)$$

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- The open loop linear system (LS)

$$\begin{aligned} \frac{d}{dt}(y(t), z(t)) + \mathcal{A}(y(t), z(t)) &= \sum_{j=1}^{N} f_\omega \operatorname{Re}(\widetilde{w}_j(t)(\varphi_j^*(x), \psi_j^*(x))), \\ (y(0), z(0)) &= (y^0, z^0). \end{aligned}$$

2. Stabilization of the linear system by a finite dimensional controller

Proposition

Let λ_i be semi-simple and φ_∞ be an analytic function in Ω.
Then, there exist $w_j \in L^2(\mathbb{R}^+)$, $j = 1, ..., 2N$, such that the controller (C) stabilizes exponentially system (LS), that is,

$$\|y(t)\|_H + \|z(t)\|_H \leq C_\infty e^{-k_\infty t} \left(\|y^0\|_H + \|z^0\|_H \right), \text{ for all } t \geq 0.$$

Moreover, we have

$$\left(\sum_{j=1}^{2N} \int_0^\infty |w_j(t)|^2 \, dt \right)^{1/2} \leq C \left(\|y^0\|_H + \|z^0\|_H \right),$$

where C_∞ and k_∞ depend on the problem parameters ν, γ, l and Ω and $\|F''(\varphi_\infty)\|_\infty$.

2. Stabilization of the linear system by a finite dimensional controller

Proof idea

- Work in the complexified space $\widetilde{\mathcal{H}} = \mathcal{H} + i\mathcal{H}$, $i = \sqrt{-1}$

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Proof idea

- Work in the complexified space $\widetilde{\mathcal{H}} = \mathcal{H} + i\mathcal{H}$, $i = \sqrt{-1}$
- Set $(\widetilde{y}, \widetilde{z}) = (y, z) + i(Y, Z)$
- Introduce the system

$$\begin{aligned} \frac{d}{dt}(\widetilde{y}(t), \widetilde{z}(t)) + \mathcal{A}(\widetilde{y}(t), \widetilde{z}(t)) &= \sum_{j=1}^{N} f_{\omega}(\widetilde{w}_j(t)(\varphi_j^*(x), \psi_j^*(x)), \\ (\widetilde{y}(0), \widetilde{z}(0)) &= (y^0, z^0). \end{aligned}$$

2. Stabilization of the linear system by a finite dimensional controller

- Represent the solution $(\xi_j \in C([0,\infty);\mathbb{C}))$

$$(\widetilde{y}(t,x), \widetilde{z}(t,x)) = \sum_{j=1}^{\infty} \xi_j(t)(\varphi_j(x), \psi_j(x)), \ (t,x) \in (0,\infty) \times \Omega,$$

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-

$$\begin{aligned}
\xi_i' + \lambda_i \xi_i &= \sum_{j=1}^{N} \widetilde{w}_j d_{ij}, \quad i \geq 1, \\
\xi_i(0) &= \xi_{i0} = \int_{\Omega} (y^0 \overline{\varphi_j^*}(x) + z^0 \overline{\psi_j^*}(x))dx,\ i \geq 1, \\
d_{ij} &= \int_{\Omega} f_\omega(\varphi_i^* \overline{\varphi_j^*} + \psi_i^* \overline{\psi_j^*})dx, \quad j = 1, ..., N, \quad i \geq 1
\end{aligned}$$

2. Stabilization of the linear system by a finite dimensional controller

- (i) System from $i = 1, ..., N$ is null controllable in $T_0 > 0$

$$\xi_i(T_0) = 0 \text{ and } \xi_i(t) = 0 \text{ for } t > T_0, \; i = 1, ..., N.$$

Ingredients: Kalman Lemma, system $\{\sqrt{f_\omega}\varphi_j, \sqrt{f_\omega}\psi_j\}_{j=1}^N$ is linearly independent on ω,

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Ingredients: Kalman Lemma, system $\{\sqrt{f_\omega}\varphi_j, \sqrt{f_\omega}\psi_j\}_{j=1}^{N}$ is linearly independent on ω,

- (ii) System from $i = N + 1, ...$is stabilized exponentially in origin.

2. Stabilization of the linear system by a finite dimensional controller

$$w_j := \mathrm{Re}\, \widetilde{w}_j, \text{ for } j = 1, ..., N, \quad w_j := \mathrm{Im}\, \widetilde{w}_j, \text{ for } j = N+1, ..., 2N.$$

$$v(t,x) = \sum_{j=1}^{N} w_j(t)\, \mathrm{Re}\, \varphi_j^*(x) - \sum_{j=N+1}^{2N} w_j(t)\, \mathrm{Im}\, \varphi_j^*(x),$$

$$u(t,x) = \sum_{j=1}^{N} w_j(t)\, \mathrm{Re}\, \psi_j^*(x) - \sum_{j=N+1}^{2N} w_j(t)\, \mathrm{Im}\, \psi_j^*(x)),$$

$$\begin{aligned}
&\Phi(y^0, z^0) && (P)\\
= \quad & \mathrm{Min}\left\{\frac{1}{2}\int_0^\infty \left(\|Ay(t)\|_H^2 + \|Az(t)\|_H^2 + \|W(t)\|_{\mathbb{R}^{2N}}^2\right) dt\right\}
\end{aligned}$$

subject to (OLS),
for all $W = (w_1, ..., w_N, w_{N+1}, ..., w_{2N}) \in L^2(0, \infty; \mathbb{R}^{2N})$

3. Construction of the feedback control and properties

Proposition

For each pair $(y^0, z^0) \in D(A^{1/2}) \times D(A^{1/2})$, *problem* (P) *has a unique optimal solution* $(\{w_j^*\}_{j=1}^{2N}, y^*, z^*)$.

$$\begin{aligned} c_1\left(\left\|A^{1/2}y^0\right\|_H^2 + \left\|A^{1/2}z^0\right\|_H^2\right) &\leq \Phi(y^0, z^0) \\ &\leq c_2\left(\left\|A^{1/2}y^0\right\|_H^2 + \left\|A^{1/2}z^0\right\|_H^2\right) \\ \forall(y^0, z^0) &\in D(A^{1/2}) \times D(A^{1/2}) \end{aligned}$$

Corollary

There exists a linear positive operator

$$R \in \mathcal{L}(D(A^{1/2}) \times D(A^{1/2}); (D(A^{1/2}) \times (D(A^{1/2}))')$$

such that

$$\Phi(y^0, z^0) = \frac{1}{2}\langle R(y^0, z^0), (y^0, z^0)\rangle_{\mathcal{V}',\mathcal{V}} \text{ for } (y^0, z^0) \in D(A^{1/2}) \times D(A^{1/2}).$$

Moreover, $R(y^0, z^0)$ *is the Gâteaux derivative of the function* Φ *at* (y^0, z^0)

$$\Phi'(y^0, z^0) = R(y^0, z^0), \text{ for all } (y^0, z^0) \in D(A^{1/2}) \times D(A^{1/2})$$

and R *restricted to* $H \times H$ *is self-adjoint.*

3. Construction of the feedback control and properties

- The open loop system

$$\begin{aligned} \frac{d}{dt}(y(t),z(t)) + \mathcal{A}(y(t),z(t)) &= BW(t), \text{ a.e. } t>0, \\ (y(0),z(0)) &= (y^0,z^0). \end{aligned}$$

3. Construction of the feedback control and properties

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- The open loop system

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- $$B : \mathbb{R}^{2N} \to H \times H,$$

- $$BW = \begin{bmatrix} f_\omega \left(\sum_{j=1}^{N} (w_j \operatorname{Re} \varphi_j^* - \sum_{j=N+1}^{2N} w_j \operatorname{Im} \varphi_j^* \right) \\ f_\omega \left(\sum_{j=1}^{N} (w_j \operatorname{Re} \psi_j^* - \sum_{j=N+1}^{2N} w_j \operatorname{Im} \psi_j^* \right) \end{bmatrix}, \ W = \begin{bmatrix} w_1 \\ \dots \\ w_{2N} \end{bmatrix}$$

3. Construction of the feedback control and properties

Proposition

Let $W^* = \{w_i^*\}_{i=1}^{2N}$ and (y^*, z^*) be optimal for problem (P) corresponding to $(y^0, z^0) \in D(A^{1/2}) \times D(A^{1/2})$. Then,

$$W^*(t) = -B^* R(y^*(t), z^*(t)), \text{ for all } t > 0,$$

and it satisfies the Riccati algebraic equation

$$2\left(R(y^0,z^0), \mathcal{A}(y^0,z^0)\right)_{H\times H} + \left\|B^* R(y^0,z^0)\right\|^2_{\mathbb{R}^{2N}} = \left\|Ay^0\right\|^2_H + \left\|Az^0\right\|^2_H,$$

$\forall\ (y^0, z^0) \in D(A) \times D(A).$

4. Feedback stabilization of the closed loop nonlinear system

- $$\begin{aligned} \frac{d}{dt}(y(t), z(t)) + \mathcal{A}(y(t), z(t)) &= \mathcal{G}(y(t)) - BB^* R(y(t, z(t)), \\ (y(0), z(0)) &= (y_0, z_0), \end{aligned} \qquad (NS)$$

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- $$F_r(y) = y^2 \int_0^1 (1-s)F'''(\varphi_\infty + sy)dy = y^3 + 3\varphi_\infty y^2.$$

4. Feedback stabilization of the closed loop nonlinear system

Theorem

Let $(y_0, z_0) \in D(A^{1/2}) \times D(A^{1/2})$. *There exists* ρ *such that if*

$$\|y_0\|_{D(A^{1/2})} + \|z_0\|_{D(A^{1/2})} \leq \rho$$

the closed loop system (NS) has a unique solution

$$\begin{aligned}(y, z) \;\; \in \;\; & C([0, \infty); H \times H) \cap L^2(0, \infty; D(A) \times D(A)) \\ & \cap W^{1,2}(0, \infty; (D(A^{1/2}) \times D(A^{1/2}))'),\end{aligned}$$

which is exponentially stable, namely

$$\|y(t)\|_{D(A^{1/2})} + \|z(t)\|_{D(A^{1/2})} \leq C_\infty e^{-k_\infty t}(\|y_0\|_{D(A^{1/2})} + \|z_0\|_{D(A^{1/2})}),$$

for some positive constants k_∞ *and* C_∞ *depending on the data and* $\|\varphi_\infty\|_\infty$.

4. Feedback stabilization of the closed loop nonlinear system

Proof.

Proof is organized in 3 steps: existence, uniqueness and stabilization. □

- Step 1. Existence is proved on every interval $[0, T]$ by the Schauder fixed point theorem.

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- Step 2. Uniqueness is proved on $[0, T]$ following by an usual method and using that BB^* is linear continuous from $V' \times V' \to V' \times V'$.
- Existence and uniqueness on $[0, \infty)$ follow by those above.
- Step 3. Estimates using Riccati eq. and the properties of R lead to

$$\|y(t)\|_{D(A^{1/2})} + \|z(t)\|_{D(A^{1/2})} \leq C_\infty e^{-k_\infty t}(\|y_0\|_{D(A^{1/2})} + \|z_0\|_{D(A^{1/2})}).$$

4. Feedback stabilization of the closed loop nonlinear system

Let $(y_0, z_0) \in D(A^{1/2}) \times D(A^{1/2})$

$$S_T = \Big\{ (y,z) \in L^2(0,T; H \times H); \sup_{t \in (0,T)} \left(\|y(t)\|^2_{D(A^{1/2})} + \|z(t)\|^2_{D(A^{1/2})} \right) + \int_0^T \left(\|Ay(t)\|_H^2 + \|Az(t)\|_H^2 \right) dt \leq r^2 \leq r_1^2 \Big\}$$

S_T is a convex closed subset of $L^2(0,T; D(A^{1/2}) \times D(A^{1/2}))$

4. Feedback stabilization of the closed loop nonlinear system

Fix $(\overline{y}, \overline{z}) \in S_T$ and consider the Cauchy problem

$$\begin{aligned} \frac{d}{dt}(y(t), z(t)) + \mathcal{A}(y(t), z(t)) + BB^* R(y(t), z(t)) &= \mathcal{G}(\overline{y}(t)), \text{ a.e. } t, \\ (y(0), z(0)) &= (y_0, z_0). \end{aligned}$$

Define

$$\Psi_T : S_T \to L^2(0, T; D(A^{1/2}) \times D(A^{1/2})), \quad \Psi_T(\overline{y}, \overline{z}) = (y, z)$$

i) $\Psi_T(S_T) \subset S_T$ provided that r is well chosen
ii) $\Psi_T(S_T)$ is relatively compact in $L^2(0, T; D(A^{1/2}) \times D(A^{1/2}))$
iii) Ψ_T is continuous in $L^2(0, T; D(A^{1/2}) \times D(A^{1/2}))$ norm.

4. Feedback stabilization of the closed loop nonlinear system

$$C_1\rho^2 + C_2(r^6 + \|\varphi_\infty\|_{2,\infty}^2 r^4) \leq r^2$$

$$\rho < r\sqrt{\frac{1}{C_1}}$$

$$C_2 r^4 + C_2 \|\varphi_\infty\|_{2,\infty}^2 r^2 - 1 \leq 0 \Longrightarrow r \in (0, r_1]$$

where C_i are constant dependent on the problem parameters and $\|\varphi_\infty\|_\infty$.

5. Stabilization in the case with a singular potential

$$F(r) = (1+r)\ln(1+r) + (1-r)\ln(1-r) - ar^2, \; r \in (-1,1), \; a > 0.$$

- Let ε be positive fixed, $\varepsilon \in (0,1)$ and assume that

$$\varphi_\infty \text{ is analytic}, \quad |\varphi_\infty(x)| \leq 1-\varepsilon \text{ for } x \in \overline{\Omega}.$$

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- Define $\chi_\varepsilon \in C_0^\infty(\mathbb{R})$ such that

$$\chi_\varepsilon(r) = \begin{cases} 1, & \text{for } |r| \leq 1-\varepsilon \\ 0, & \text{for } |r| \geq 1-\frac{\varepsilon}{2}, \end{cases}$$

and $0 < \chi_\varepsilon(r) \leq 1$ for $r \in (-1+\frac{\varepsilon}{2}, -1+\varepsilon] \cup [1-\varepsilon, 1-\frac{\varepsilon}{2})$.

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- Define the regularized potential $F_\varepsilon \in C_0^\infty(\mathbb{R})$,

$$F_\varepsilon(r) = \begin{cases} F(r), & \text{for } r \in [1-\varepsilon, 1+\varepsilon] \\ F(r)\kappa_\varepsilon(r), & \text{for } r \in (-1+\frac{\varepsilon}{2}, -1+\varepsilon] \cup [1-\varepsilon, 1-\frac{\varepsilon}{2}) \\ 0, & \text{for } |r| \geq 1-\frac{\varepsilon}{2}. \end{cases}$$

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- All results given in Section 4 remain true for the problem with F_ε.

5. Stabilization in the case with a singular potential

Theorem

Let $\varepsilon \in (0,1)$ be arbitrary, but fixed. For all pairs

$$(y_0, z_0) \in D(A^{1/2}) \times D(A^{1/2}) \text{ with } \|y_0\|_{D(A^{1/2})} + \|z_0\|_{D(A^{1/2})} \leq \rho,$$

the closed loop system corresponding to the logarithmic potential F has, in the one-dimensional case, a unique solution.
The solution is exponentially stable and satisfies

$$\|y(t)\|_{D(A^{1/2})} + \|z(t)\|_{D(A^{1/2})} \leq C_\infty e^{-k_\infty t}(\|y_0\|_{D(A^{1/2})} + \|z_0\|_{D(A^{1/2})}).$$

Proof

$$
\begin{aligned}
& y_t + \frac{\nu}{\tau^2}(A + A^{-1} - 2)y - \frac{1}{\tau}(A^{-1} - I)(F_\varepsilon'(y + \varphi_\infty) - F_\varepsilon{}'(\varphi_\infty)) \\
& + \frac{\gamma}{\tau}(A^{-1} - I)z - \frac{\gamma l}{\tau}(A^{-1} - I)y \\
= \quad & f_\omega v, \text{ in } (0, \infty) \times \Omega
\end{aligned}
$$

Proof

$$\begin{aligned} & y_t + \frac{\nu}{\tau^2}(A + A^{-1} - 2)y - \frac{1}{\tau}(A^{-1} - I)(F'_\varepsilon(y + \varphi_\infty) - F'(\varphi_\infty)) \\ & + \frac{\gamma}{\tau}(A^{-1} - I)z - \frac{\gamma l}{\tau}(A^{-1} - I)y \\ = \ & f_\omega v, \text{ in } (0, \infty) \times \Omega \end{aligned}$$

- By the previous result

$$\|(y(t), z(t))\|_{\mathcal{V}} \leq C_\infty e^{-k_\infty t} \|(y_0, z_0)\|_{\mathcal{V}} \leq C_\infty e^{-k_\infty t} \rho$$

5. Stabilization in the case with a singular potential

- By the previous result

$$\|(y(t), z(t))\|_{\mathcal{V}} \leq C_{\infty} e^{-k_{\infty} t} \|(y_0, z_0)\|_{\mathcal{V}} \leq C_{\infty} e^{-k_{\infty} t} \rho$$

- $H^1(\Omega)$ is compact in $C(\overline{\Omega})$ for $d = 1$

$$|y(t)| \leq \|y(t)\|_{C(\overline{\Omega})} \leq C_{\Omega} \|y(t)\|_{D(A^{1/2})} \leq C_{\Omega} C_{\infty} e^{-k_{\infty} t} \rho$$

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$$\|(y(t), z(t))\|_{\mathcal{V}} \leq C_\infty e^{-k_\infty t} \|(y_0, z_0)\|_{\mathcal{V}} \leq C_\infty e^{-k_\infty t} \rho$$

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- $|y(t)| \to 0$, as $t \to \infty$

$$|y(t)| \leq \|y(t)\|_{C(\overline{\Omega})} \leq C_\Omega \|y(t)\|_{D(A^{1/2})} \leq C_\Omega C_\infty e^{-k_\infty t} \rho < 1 - \varepsilon$$

5. Stabilization in the case with a singular potential

- By the previous result

$$\|(y(t), z(t))\|_{\mathcal{V}} \leq C_\infty e^{-k_\infty t} \|(y_0, z_0)\|_{\mathcal{V}} \leq C_\infty e^{-k_\infty t} \rho$$

- $H^1(\Omega)$ is compact in $C(\overline{\Omega})$ for $d = 1$

$$|y(t)| \leq \|y(t)\|_{C(\overline{\Omega})} \leq C_\Omega \|y(t)\|_{D(A^{1/2})} \leq C_\Omega C_\infty e^{-k_\infty t} \rho$$

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$$|y(t)| \leq \|y(t)\|_{C(\overline{\Omega})} \leq C_\Omega \|y(t)\|_{D(A^{1/2})} \leq C_\Omega C_\infty e^{-k_\infty t} \rho < 1 - \varepsilon$$

- $$t > \frac{1}{k_\infty} \ln \frac{\rho C_\infty C_\Omega}{1 - \varepsilon}$$

5. Stabilization in the case with a singular potential

- Set a new ρ such that

$$|y(t) + \varphi_\infty| < 1 - \varepsilon \text{ for all } t \geq 0,$$

and consequently $F'_\varepsilon(y + \varphi_\infty) = F'(y + \varphi_\infty)$, proving thus the result for the system with F.

6. Stabilization in the nonviscous case with a regular potential

- V. Barbu, P. Colli, G. Gilardi, G. M., J. Differential Equations, 262(2017).

6. Stabilization in the nonviscous case with a regular potential

- V. Barbu, P. Colli, G. Gilardi, G. M., J. Differential Equations, 262(2017).
- Some function transformations lead to a system with a self-adjoint operator $\mathcal{A} = \begin{bmatrix} \nu\Delta^2 - F_l\Delta & \gamma\Delta \\ \gamma\Delta & -\Delta \end{bmatrix}$

$$D(\mathcal{A}) = \Big\{ w = (y, z) \in H^2(\Omega) \times H^1(\Omega);\ \mathcal{A}w \in H \times H, \frac{\partial y}{\partial \nu} = \frac{\partial \Delta y}{\partial \nu} = \frac{\partial z}{\partial \nu} = 0 \text{ on } \Gamma \Big\}$$

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- $\mathcal{A}$ is self-adjoint $\Longrightarrow$ its eigenvalues are real and semi-simple.

5. Stabilization in the nonviscous case with a regular potential

Theorem

Let $\chi_\infty := \|\nabla\varphi_\infty\|_\infty + \|\Delta\varphi_\infty\|_\infty$. *There exists* $\chi_0 > 0$ *such that the following hold true. If* $\chi_\infty \leq \chi_0$ *there exists* ρ *such that for all pairs*

$$\|y_0\|_{D(A^{1/2})} + \|z_0\|_{D(A^{1/4})} \leq \rho,$$

the closed loop system has a unique solution

$$\begin{aligned}(y,z) \quad \in \quad & C(0,\infty; H \times H) \cap L^2(0,\infty; D(A^{3/2}) \times D(A^{3/4})) \\ & \cap W^{1,2}(0,\infty; (D(A^{1/2}) \times D(A^{1/4}))'),\end{aligned}$$

which is exponentially stable, that is

$$\|y(t)\|_{D(A^{1/2})} + \|z(t)\|_{D(A^{1/4})} \leq C_P e^{-kt}(\|y_0\|_{D(A^{1/2})} + \|z_0\|_{D(A^{1/4})}).$$

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Thank you for your attention