OPTIMAL CONTROL METHODS IN SHAPE OPTIMIZATION

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CONTENTS

Introduction

A typical example of shape optimization problems has the form:

$$\underset{\Omega\in\mathcal{O}}{\mathrm{Min}} \int_\Lambda j(x, y_\Omega(x), \nabla y_\Omega(x))dx,$$

$$-\Delta y_\Omega = f \quad in\ \Omega,$$

$$y_\Omega = 0 \quad on\ \partial\Omega$$

with other supplementary constraints (on y, Ω, etc.), if necessary. Here, $\Omega \subset D$ is an (unknown) domain, D is some given bounded Lipschitzian domain, $f \in L^2(D)$, $j(., ., .)$ is a Caratheodory mapping and Λ is either Ω or some fixed subdomain $E \subset D$.

Introduction

Let me also mention that many geometric optimization problems arising in mechanics (for plates, beams, arches, curved rods or shells), are expressed, as well, as optimal control problems by the coefficients, due to the special form of their models. This point is not discussed here.

The presentation will discuss in detail two cases: optimization of a plate with holes and a penalization approach to a general shape optimization problem. Boundary observation problems will also be presented if time allows.
An essential ingredient in these developments is the new implicit parametrization method that allows an advantageous description of implicitly defined manifolds via iterated Hamiltonian systems. We start with a short description in this respect.

Arbitrary Dimension

We impose the classical independence assumption on some family of C^1 mappings $F_1, F_2, \dots, F_l$, in some point $x^0 \in \Omega \subset R^d, l \leq d-1$. To fix ideas, we assume $F_j(x^0) = 0$ and

$$\frac{D(F_1, F_2, \dots, F_l)}{D(x_1, x_2, \dots, x_l)} \neq 0 \quad \text{in } x^0 = (x_1^0, x_2^0, \dots, x_d^0).$$

This remain valid in a neighborhood V and we introduce the undetermined linear algebraic system with unknowns $v(x) \in R^d, \; x \in V$:

$$\mathrm{v(x)} \cdot \nabla F_j(x) = 0, \quad j = \overline{1, l}.$$

Arbitrary Dimension

We denote by $A(x)$ the corresponding $l \times l$ nonsingular matrix and the vectors $\nabla F_1(x), \ldots, \nabla F_l(x)$ are independent, for $x \in V$. We shall use $d - l$ solutions obtained by fixing the last $d - l$ components of the vector $v(x) \in R^d$ to be the rows of the identity matrix in R^{d-l}, multiplied by $\det A(x)$. Then, the first l components are uniquely determined, by inverting $A(x)$.

In this way, the obtained $d - l$ solutions, denoted by $v_1(x), \ldots, v_{d-l}(x) \in R^d$ are linear independent, for any $x \in V$. Moreover, these vector fields are continuous in V.

Arbitrary Dimension

We introduce now $d-l$ nonlinear systems of first order partial differential equations associated to the vector fields $(v_j(x))_{j=\overline{1,d-l}}$, $x \in V \subset \Omega$.

We use here the order $v_1, v_2, \ldots, v_{d-l}$ to fix ideas. Moreover, we denote the sequence of independent variables by $t_1, t_2, \ldots, t_{d-l}$.

These systems have an iterated character in the sense that the solution of one of them is used as initial condition in the next one. Consequently, the independent variables in the "previous" systems enter as parameters in the next system just via the initial conditions.

Arbitrary Dimension

$$\frac{\partial y_1(t_1)}{\partial t_1} = v_1(y_1(t_1)), \quad t_1 \in I_1 \subset R,$$

$$y_1(0) = x^0;$$

$$\frac{\partial y_2(t_1, t_2)}{\partial t_2} = v_2(y_2(t_1, t_2)), \quad t_2 \in I_2(t_1) \subset R,$$

$$y_2(t_1, 0) = y_1(t_1);$$

$$\dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots$$

$$\frac{\partial y_{d-l}(t_1, t_2, \dots, t_{d-l})}{\partial t_{d-l}} = v_{d-l}(y_{d-l}(t_1, t_2, \dots, t_{d-l})),$$

$$t_{d-l} \in I_{d-l}(t_1, \dots, t_{d-l-1}),$$

$$y_{d-l}(t_1, \dots, t_{d-l-1}, 0) = y_{d-l-1}(t_1, t_2, \dots, t_{d-l-1}).$$

Arbitrary Dimension

Here, the notations $I_1, I_2(t_1), \ldots, I_{d-l}(t_1, \ldots, t_{d-l-1})$ are $d-l$ real intervals, containing 0 in interior and depending, in principle, on the "previous" parameters.

Due to their simple structure, we stress that each equation may be interpreted as an ordinary differential system with parameters, although partial differential notations are used.

The existence of the solutions $y_1, y_2, \ldots, y_{d-l}$ follows by the Peano theorem due to the continuity of the vector fields $(v_j)_{j=\overline{1,d-l}}$ on V.

Arbitrary Dimension

Proposition

For every $k = \overline{1, l}$, $j = \overline{1, d-l}$, we have

$$F_k(y_j(t_1, t_2, \ldots, t_j)) = 0, \quad \forall\, (t_1, t_2, \ldots, t_j) \in I_1 \times I_2 \times \ldots \times I_j.$$

Theorem

Under the above assumptions, then the differential system consists of $d-l$ subsystems of dimension d with the uniqueness property in V.

Importance of Hamiltonian type structure and references.

Arbitrary Dimension

Theorem

a) There are closed intervals $I_j \subset R$, $0 \in \mathrm{int} I_j$, independent of the parameters, such that $I_j \subset I_j(t_1, t_2, \ldots, t_{j-1})$, $j = \overline{1, d-l}$.

b) The unique solutions of the differential systems are of class C^1 in any existence point and we have:

$$\frac{\partial y_{d-l}}{\partial t_k}(t_1, \ldots, t_{d-l}) = v_k(y_{d-l}(t_1, \ldots, t_{d-l})), \quad k = \overline{1, d-l}.$$

Theorem

If $F_k \in C^1(\Omega)$, $k = \overline{1, l}$, with the independence property, and the I_j are sufficiently small, $j = \overline{1, d-l}$, then the mapping

$$y_{d-l} : I_1 \times I_2 \times \ldots \times I_{d-l} \to R^d$$

is regular and one-to-one on its image.

Arbitrary Dimension

The local solution of the initial system is a $d-l$ dimensional manifold around x^0 and $y_{d-l}(t_1, t_2, \dots, t_{d-l})$ is a local parametrization of this manifold on $I_1 \times I_2 \times \dots \times I_{d-l}$.

Geometric interpretation: basis in tangent space and integration.

Choose the last $d-l$ components of the solutions $v_j(x) \in R^d$ as the rows of the identity matrix in R^{d-l}. We obtain :

Proposition

The last $d-l$ components of y_{d-l} have the form $(t_1 + x^0_{l+1}, t_2 + x^0_{l+2}, \dots, t_j + x^0_{l+j}, x^0_{l+j+1}, \dots, t_{d-l} + x^0_d)$, that is the first l components of y_{d-l} give the unique solution of the implicit system on $x^0 + (I_1 \times I_2 \times \dots \times I_{d-l})$.

Arbitrary Dimension

Notice that the propositions are valid just for C^1 and independence assumptions. We also get an evaluation of the existence neighborhood in the implicit function theorem, from Peano's theorem.

By the relation

$$y_{d-l}(t_1, t_2, \ldots, t_j, 0, \ldots, 0) = y_j(t_1, t_2, \ldots, t_j)$$

we see that $y_1, y_2, \ldots, y_{d-l}$ have continuous partial derivatives with respect to their arguments.

IMPORTANT: the above parametrizations may be more advantageous in applications since we may use maximal solutions, as it will be shown in dimension three. They are not constrained by the function condition...

EXAMPLES IN DIMENSION THREE : SURFACES

- The hypothesis $\nabla f(x_0, y_0, z_0) \neq 0$ is specified in the form $f_x(x_0, y_0, z_0) \neq 0$.
- We associate to the equation $f(x, y, z) = 0$, $f(x_0, y_0, z_0) = 0$, two iterated Hamiltonian systems:

$$
\begin{aligned}
x' &= -f_y(x, y, z), && t \in I_1,\\
y' &= f_x(x, y, z), && t \in I_1,\\
z' &= 0, && t \in I_1,\\
x(0) &= x_0, \quad y(0) = y_0, \quad z(0) = z_0;
\end{aligned}
$$

EXAMPLES IN DIMENSION THREE : SURFACES

- The second system is:

$$
\begin{array}{rcll}
\dot{\varphi} & = & -f_z(\varphi,\psi,\xi), & s \in I_2(t), \\
\dot{\psi} & = & 0, & s \in I_2(t), \\
\dot{\xi} & = & f_x(\varphi,\psi,\xi), & s \in I_2(t), \\
\varphi(0) & = & x(t), \ \ \psi(0) = y(t), \ \ \xi(0) = z(t). &
\end{array}
$$

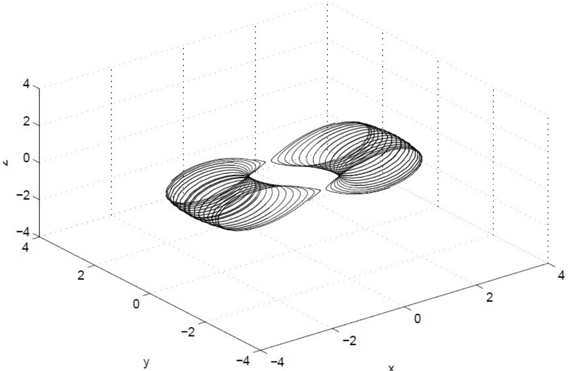
Comments: dimension two - geometrical interpretation, numerical solution with MatLab.

EXAMPLES IN DIMENSION THREE : SURFACES

- The next two examples are solved with MatLab.

3) $f(x,y,z) = (x^2 + y^2 + z^2 + R^2 - r^2)^2 - 4R^2(x^2 + y^2)$

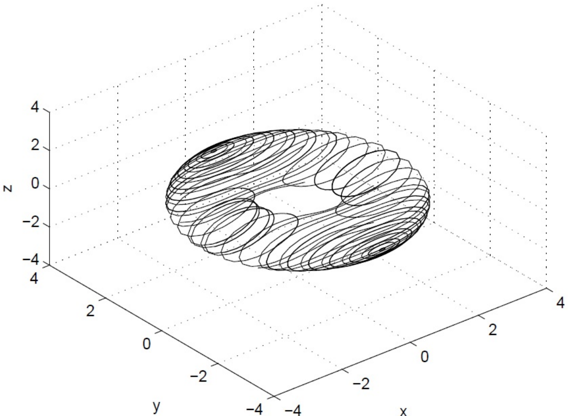
$R = 2,\ r = 1,\ (x_0, y_0, z_0) = (1, 0, 0)$



EXAMPLES IN DIMENSION THREE : SURFACES

4) $f(x,y,z) = (x^2 + y^2 + z^2 + R^2 - r^2)^2 - 4R^2(x^2 + y^2)$

$R = 2,\ r = 1,\ (x_0, y_0, z_0) = (3, 0, 0)$



Generalized Solutions in Arbitrary Dimension

We discuss now the nonlinear implicit system in the absence of the nondegeneracy hypothesis, i. e. for $\det A(x^0) = 0$ (in fact all the maximal order determinants are null). We consider instead that there is $\{x^n\} \subset \Omega$, such that:

$$x^n \to x^0, \quad \mathrm{rank} J(x^n) = l, \ n \in N,$$

where $J(x^n)$ denotes the Jacobian matrix of $F_1, F_2, \ldots, F_l \in C^1(\Omega)$, in x^n.

Notice that in case this is not fulfilled, it means that rank $J(x) < l$ in $x \in W$, where W is a neighbourhood of x^0. Then $F_1, F_2, \ldots, F_l$ are not functionally independent in W and the problem (1.1) can be reformulated by using less functionals. That is the above property is in fact always valid, except for not well formulated implicit systems.

Generalized Solutions in Arbitrary Dimension

In each x^n, one can solve the system $F_j(x) = F_j(x^n),\ j = \overline{1,l},\ x \in \overline{\Omega}_1$, where Ω_1 is open and bounded such that $x^0 \in \Omega_1 \subset\subset \Omega$ and one can find the solution via the differential system.
We denote by $T_n \subset R^d$ the above solution and it may be assumed a compact in R^d (it is clearly closed due to the continuity of $F_j, j = \overline{1,l}$). We may also assume that $\{T_n\}$ are uniformly bounded since Ω_1 is bounded and, on a subsequence α, we have $T_n \to T_\alpha,\ n \to \infty$, in the Hausdorff-Pompeiu metric, where T_α is some compact subset in R^d.

Definition

$T = \bigcup\limits_{\alpha} T_\alpha$ is the local generalized solution of the nonlinear system in x^0, in the critical case rank $J(x^0) < l$. The union is taken for all the sequences and subsequences satisfying the above conditions.

Generalized Solutions in Arbitrary Dimension

The above definitions cover all the possible critical or non critical cases. For instance, if we have just one equation and x^0 is an extremum for the respective function, then the generalized solution is just x^0. If the respective function is identically zero in $O \subset \Omega$ and x^0 is on the boundary of O, then the generalized solution is the boundary of O - see the Example below. A complete description of the level sets (even of positive measure) around x^0 may be obtained via the generalized solution. Generally speaking, the generalized solution is not a manifold and may be not a compact subset. The computed approximation may be not connected.

Generalized Solutions in Arbitrary Dimension

Proposition

We have $x^0 \in T_\alpha \subset T \subset \partial M_{x_0}, \forall \alpha$, *where* ∂M_{x_0} *is the connected component of* ∂M *containing* x_0 *and* M *is the solution. In particular*

$$F_j(x) = 0,\ j = \overline{1, l},\ \forall\ x \in T.$$

If x^0 is a regular point, then we denote by S the (local) solution obtained via the implicit function theorem around x^0. In the *Definition*, we choose $x^n \to x^0$, $x^n \in S$ and the uniqueness property from the implicit functions theorem gives that $T_n = S$, locally for n big enough. This choice of x^n satisfies the conditions since $J(x^n) \to J(x^0)$.

We see that in the classical case, one obtains $T = S$ (locally), that is the *Definition* gives indeed a generalization of the classical local solution of the implicit functions theorem.

Generalized Solutions in Arbitrary Dimension

In R^2, take $d = 2, l = 1$ and

$$f(x_1, x_2) = \begin{cases} x_1^2(x_2^2 - x_1^2)^2 & \textit{if } x_1 < 0, \quad |x_2| \le |x_1| \\ 0 & \textit{otherwise}. \end{cases}$$

Clearly f is in $C^1(R^2)$ and $\nabla f(x_1, x_2) = 0$, on the second line. Take $x^0 = (0, 0)$ and $x^n \to x^0$, $x^n = (x_1^n, x_2^n)$, $x_1^n < 0$, $|x_2^n| < |x_1^n|$. In such points x^n, one can use previous theorems and the differential system may be chosen of Hamiltonian type:

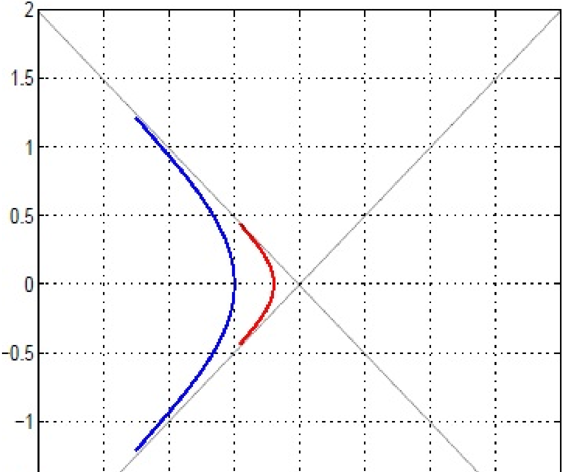
$$x_1'(t) = -4x_1^2 x_2(x_2^2 - x_1^2),$$

$$x_2'(t) = 2x_1(x_2^2 - x_1^2)(x_2^2 - 3x_1^2),$$

$$(x_1(0), x_2(0)) = x^n.$$

Generalized Solutions in Arbitrary Dimension

We represent the solution T_n obtained with Matlab, for $x^n = (-\frac{1}{n}, 0), n = 2, 5$



Generalized Solutions in Arbitrary Dimension

This generalized solution contains the essential information about the solution set of (1.1), since it gives its boundary (and in the proposition, the inclusion becomes equality). If we define

$$f_1(x_1, x_2) = x_1^2(x_1^2 + x_2^2 - 1)_-^2$$

and $x^0 = (0, 1)$, then ∂M is connected and the corresponding generalized solution is ∂M without the lower half of the unit circle. The inclusion is strict in this case. This is also related to the local character of our construction.

Proposition

Let x^0 be the unique critical point of (1.1) in the closed ball $B(x^0)$. Then, $T = M$ in $B(x^0)$.

Generalized Solutions in Arbitrary Dimension

Proposition

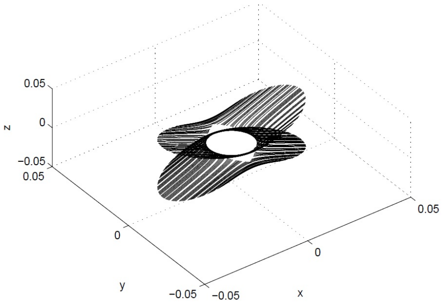
Let $F_j \in C^1(\Omega)$, $j = \overline{1, l}$ and $x^n \to x^0$, $x^n, x^0 \in \Omega$. Denote by $\widetilde{T}_n, \widetilde{T}_0$ the generalized solutions of (1.1) contained in the bounded domain Ω, corresponding to the initial conditions x^n, respectively x^0. Then

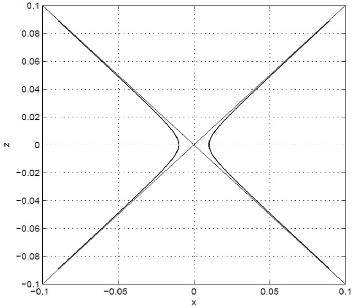
$$\limsup_{n\to\infty} \widetilde{T}_n \subset \widetilde{T}_0.$$

There are examples with the above inclusion strict.

DIMENSION THREE : EXAMPLES CRITICAL CASE

$f(x, y, z) = x^2 + y^2 - z^2$, $(x_0, y_0, z_0) = (0, 0, 0)$,
$\nabla f(x_0, y_0, z_0) = (0, 0, 0)$





The generalized solution around (x_0, y_0, z_0) and its section through a vertical plane. Here, we have $T = M$.

Penalization

We introduce now the basic assumptions on the family $\mathcal{F}$ of all admissible shape functions $g \in C^1(D)$ that will be used in the next pages, in the study of shape optimization problems, in R^2.

$$g(x, y) > 0 \quad \text{on} \quad \partial D,$$

$$|\nabla g(x, y)| > 0 \quad \text{on } G = \{(x, y) \in D;\ g(x, y) = 0\}.$$

Notice that the admissible family $\mathcal{O}$ of open sets is defined by

$$\Omega_g = \{(x, y) \in D;\ g(x, y) < 0\}.$$

Penalization

The family $\mathcal{O}$ of admissible open sets is rich and, for cost functionals defined on some given subset $E \subset D$, the natural constraint

E $\subset \Omega, \ \forall\, \Omega \in \mathcal{F}$,

can be expressed as

g $\leq 0 \text{ in } E, \ \forall\, g \in \mathcal{F}$.

Penalization

Proposition

Under above hypotheses, G is a finite union of disjoint closed curves of class C^1, without self intersections and not intersecting ∂D, parametrized by the solution of

$$x'(t) = -\frac{\partial g}{\partial y}(x(t), y(t)),\ t \in I,$$

$$y'(t) = \frac{\partial g}{\partial x}(x(t), y(t)),\ t \in I,$$

$$g(x(t), y(t)) = 0,\quad \forall\ t \in I.$$

when some initial point (x^0, y^0) is chosen on each component.

Penalization

The mappings $g \in \mathcal{F}$ will be interpreted as a control parameter and we introduce a supplementary control unknown $u \in L^2(D)$ and consider the perturbed state system defined in D:

$$-\Delta y = f + H(g)u \quad \text{in } D,$$

$$y = 0 \quad \text{on} \quad \partial D,$$

where $H : R \to R$ is the Heaviside function. Then, $H(g)$ is the characteristic function of $D \setminus \Omega_g$.

Penalization

We introduce the following state constrained optimal control problem, defined in the fixed domain D:

$$\operatorname*{Min}_{g,u} \int_D (1 - H(g)) j(x, y(x), \nabla y(x)) dx$$

$$\int_{\partial\Omega_g} |y(\sigma)|^2 d\sigma = 0,$$

for any $g \in \mathcal{F}$ and $u \in L^2(D)$.

Proposition

For any $g \in \mathcal{F}$, there is $u_g \in L^2(D)$ (not unique) such that the solution of above state system coincides with the solution of original state system in Ω_g and satisfies the state constraint. The two costs coincide.

Penalization

Corollary

The shape optimization problem is equivalent with the constrained optimal control problem, defined in D.

Remark

The state constraint has an implicit character and the unknown geometry $\partial\Omega_g$ is still fully present in it, that shows the difficulty of the problem.

We denote by $z_g(t) = (z_g^1(t), z_g^2(t))$, $t \in I_g$, the unique solution of the Hamiltonian system, where $I_g = [0, T_g]$ is the corresponding period and some initial condition has to be fixed on $\partial\Omega_g$. The penalized optimal control problem is:

Penalization

$$\underset{g,u}{\mathrm{Min}}[\int_D (1 - H(g))j(x, y(x), \nabla y(x))dx + \frac{1}{\varepsilon} \int_{I_g} |y(z_g(t))|^2 |z_g(t)| dt],$$

subject to the same state system and to $g \in \mathcal{F}$, $u \in L^2(D)$ and for $\varepsilon > 0$ given.
In case $\partial\Omega_g$ has several connected components, then the last integral has to be replaced by a finite sum of integrals.

Penalization

Lemma

Let $j(x, y)$ be a Caratheodory function on $D \times R$, bounded from below by a constant. Denote by $[y_n^\varepsilon, g_n^\varepsilon, u_n^\varepsilon]$ a minimizing sequence in the penalized problem. Then, on a subsequence denoted by $m(n)$, the pairs $[g_{m(n)}^\varepsilon, y_{m(n)}^\varepsilon]$ give a minimizing cost in the shape optimization problem, $y_{m(n)}^\varepsilon$ satisfies state equation in $\Omega_{g_{m(n)}^\varepsilon}$ and the boundary condition is fulfilled with a perturbation of order $\varepsilon^{\frac{1}{2}}$.

Penalization

Let $\varepsilon_k \to 0$ be some sequence of positive quantities. Taking into account the Lemma , with $\varepsilon = \varepsilon_k$, we denote shortly $y^k = y^{\varepsilon_k}_{m(n)_k}$, $g^k = g^{\varepsilon_k}_{m(n)_k}$, $\Omega^k = \Omega_{g^{\varepsilon_k}_{m(n)_k}}$. Then, Ω^k is a bounded sequence of open sets and we have $\Omega^k \to \Omega^*$ in the Hausdorff - Pompeiu complementary sense, on a subsequence denoted again by ε_k. We assume that $\bar{\Omega}_k \to \bar{\Omega}^*$ in the Hausdorff - Pompeiu metric too.

Penalization

Proposition

Assume that $j(x, \cdot)$ is coercive in $D \times R$:

$$j(x, y) \geq \alpha |y|^2 - \beta, \ \alpha > 0; \ \beta \in R. \tag{1}$$

Then, there is an extension $\hat{y}_k$ of $y^k|_{\Omega^k}$, bounded in $L^2(D)$. If y^* is its weak limit on a subsequence, in $L^2(D)$, then $y^*|_{\Omega^*}$ satisfies the state equation in the distributions sense. If Ω^* is of class C and $y^* \in H^1(D)$, the boundary condition is also satisfied.

Keldys-Hedberg stability

Penalization

In the next result, we bring more clarifications, when j depends on ∇y as well.

Corollary

Under the above conditions, assume that j satisfies the stronger coercivity assumption on $D \times R \times R^2$:

$$j(x, y, v) \geq \alpha_1 |v|^2 + \beta_1 |y|^2 - \gamma,\ \alpha_1 > 0,\ \beta_1 > 0,\ \gamma \in R, \qquad (2)$$

and $j(x, y, \cdot)$ is convex. Then, $[y^*, \Omega^*]$ is an optimal pair for the shape optimization problem (P).

Plate

Let $\Omega \subset \mathbb{R}^2$ be a bounded, smooth (multiply) connected open subset representing the shape of a plate of constant thickness (normalized to one). We consider the fourth order partial differential equation

$$\Delta\Delta y = f \text{ in } \Omega,$$
$$y = 0, \quad \Delta y = 0 \text{ on } \partial\Omega,$$

where $f \in L^2(\Omega)$ is the load and $y \in H^4(\Omega) \cap H_0^1(\Omega)$ is the vertical deflection of the plate. The existence, the regularity and the uniqueness of the strong solution of is well known, under $\mathcal{C}^{1,1}$ conditions for $\partial\Omega$, Grisvard 1985.
The difficulty in the numerical solution of is that the shape of Ω may be very complicated, if multiply connected. Moreover, in the corresponding shape optimization problems, the geometry may change in each iteration in a complex way.

Plate

We consider now another simply connected smooth bounded domain $D \subset \mathbb{R}^2$ such that $\Omega \subset D$ and define the following approximation for weak solutions, in a sense to be made precise below.

$$
\begin{aligned}
&-\Delta y_\epsilon + \frac{1}{\epsilon}(1 - H_\Omega) y_\epsilon = z_\epsilon \text{ in } D,\\
&y_\epsilon = 0 \text{ on } \partial D,\\
&-\Delta z_\epsilon + \frac{1}{\epsilon}(1 - H_\Omega) z_\epsilon = f \text{ in } D,\\
&z_\epsilon = 0 \text{ on } \partial D,
\end{aligned}
$$

where H_Ω is the characteristic function of Ω in D, $y_\epsilon, z_\epsilon \in H_0^1(D)$.

Proposition

If Ω is of class $\mathcal{C}$, then $y_\epsilon|_\Omega \to y$ weakly in $H_0^1(\Omega)$ and strongly in $L^2(\Omega)$, where $y \in H^2(\Omega) \cap H_0^1(\Omega)$ satisfies the plate equation as a weak solution.

Plate

We associate the following minimization problem

$\min_{\Omega \in \mathcal{O}} \int_\Lambda J(\mathbf{x}, y(\mathbf{x}))\, d\mathbf{x}$,

where $\mathcal{O}$ is the class of admissible domains to be defined below, $y \in H_0^1(\Omega)$ is the weak solution of the original equation, Λ may be Ω or $\partial\Omega$ or some part of Ω or $\partial\Omega$ and J is the performance index of Carathéodory type (measurable in $\mathbf{x}$ and continuous in y).
Any $\Omega \in \mathcal{O}$ is an open set of class $\mathcal{C}$, contained in some given bounded domain $D \subset \mathbb{R}^2$. We may add the constraint

$\mathrm{E} \subset \Omega, \quad \forall \Omega \in \mathcal{O}$
where $E \subset\subset D$ is some given not empty subset of $\mathbb{R}^2$.

Plate

Let $X(D)$ denote a subset of $\mathcal{C}(\overline{D})$. We associate with any $g \in X(D)$, the open set

$$\Omega_g = int\left\{\mathbf{x} \in D;\ g(\mathbf{x}) \geq 0\right\}.$$

In the absence of regularity assumptions and due to the possible presence of critical points of g, it is possible that g has level set $\{\mathbf{x} \in D;\ g(\mathbf{x}) = k\}$ of positive measure. For the constraint E, then $X(D)$ should include the condition:

g(x)$\geq$ 0 in E.

Notice that Ω_g is a Carathéodory open set, i.e. cracks or cuts are not allowed. However, high oscillations of the boundary are possible (and the segment property may not be always valid and has to be imposed separately).

Plate

If $H : \mathbb{R} \to \mathbb{R}$ denotes the maximal monotone extension of the Heaviside function then $H(g)$ is the characteristic function of $\overline{\Omega}_g$. An example of a regularization of the characteristic function is obtained with H^ϵ

$$\mathrm{H}^\epsilon(r) = \begin{cases} 1 - \frac{1}{2}e^{-\frac{r}{\epsilon}}, & r \geq 0, \\ \frac{1}{2}e^{\frac{r}{\epsilon}}, & r < 0 \end{cases}$$

but other choices are possible. The cost is approximated by

$$\int_E J\left(\mathbf{x}, y_\epsilon(\mathbf{x})\right) d\mathbf{x}, \quad \text{if } \Lambda = E,$$

$$\int_D H^\epsilon(g) J\left(\mathbf{x}, y_\epsilon(\mathbf{x})\right) d\mathbf{x}, \quad \text{if } \Lambda = \Omega.$$

Together with the approximating state system we get the approximation of the shape optimization problem as a control problem.

Plate

Proposition

The mappings $g \to y_\epsilon = y_\epsilon(g)$, $g \to z_\epsilon = z_\epsilon(g)$ defined by the approximating state system with H_Ω replaced by $H^\epsilon(g)$ are Gâteaux differentiable between $\mathcal{C}(D)$ and $H_0^1(\Omega)$ and $w = \nabla y_\epsilon(g)v$, $u = \nabla z_\epsilon(g)v$ for any v in $\mathcal{C}(D)$ satisfy the following system in variations:

$$\begin{aligned} -\Delta u + \frac{1}{\epsilon}(1 - H^\epsilon(g))u &= \frac{1}{\epsilon}(H^\epsilon)'(g)z_\epsilon v, \\ -\Delta w + \frac{1}{\epsilon}(1 - H^\epsilon(g))w &= u + \frac{1}{\epsilon}(H^\epsilon)'(g)y_\epsilon v, \end{aligned}$$

with $u, w \in H_0^1(\Omega)$.

Plate

We introduce now the adjoint system. To do this, we shall consider the two cases of cost functionals. First a special form of the cost on E :

1/2 $\int_E (y_\epsilon - y_d)^2 d\mathbf{x}$,

-$\Delta p + \frac{1}{\epsilon}(1 - H^\epsilon(g))p = \chi_E(y_\epsilon - y_d)$ in D,
-$\Delta q + \frac{1}{\epsilon}(1 - H^\epsilon(g))q = p$ in D,
p=0, q=0 on ∂D,

where χ_E is the characteristic function of E in D.
For the second cost, the adjoint equation for p becomes

-$\Delta p + \frac{1}{\epsilon}(1 - H^\epsilon(g))p = H^\epsilon(g) J_y'(\mathbf{x}, y_\epsilon) v$ in D,
under differentiability assumptions.

Plate

Proposition

The directional derivative of the cost functional is given by

$$\frac{1}{\epsilon}\int_D (H^\epsilon)'(g)v(y_\epsilon p + z_\epsilon q)d\mathbf{x},$$

for p, q satisfying the adjoint system and for any $v \in \mathcal{C}(D)$.

Corollary

The directional derivative of the second cost functional has the form:

$$\int_D (H^\epsilon)'(g)\left[J(\mathbf{x}, y_\epsilon(\mathbf{x})) + \frac{1}{\epsilon}(y_\epsilon(\mathbf{x})p(\mathbf{x}) + z_\epsilon(\mathbf{x})q(\mathbf{x}))\right] v(\mathbf{x})\, d\mathbf{x}.$$

Plate

Corollary

Let $g_\epsilon^* \in X(D)$ denote an approximating optimal solution. The optimality conditions for g_ϵ^* are given by the state system, the corresponding adjoint system and the maximum principle:
$\int_D (H^\epsilon)'(g_\epsilon^*)(y_\epsilon^* p_\epsilon^* + z_\epsilon^* q_\epsilon^*) v \, d\mathbf{x} \leq 0, \quad \forall v,$
respectively
$\int_D (H^\epsilon)'(g_\epsilon^*) \left[J(\mathbf{x}, y_\epsilon^*(\mathbf{x})) + \frac{1}{\epsilon}(y_\epsilon^*(\mathbf{x}) p_\epsilon^*(\mathbf{x}) + z_\epsilon^*(\mathbf{x}) q_\epsilon^*(\mathbf{x})) \right] v(\mathbf{x}) \, d\mathbf{x}$
$\leq 0, \quad \forall v,$
where $y_\epsilon^*, z_\epsilon^* \in H_0^1(D)$ denote the approximating optimal states, $p_\epsilon^*, q_\epsilon^*$ denote the corresponding adjoint states and $v \in \mathcal{C}(\overline{D})$ is any admissible variation such that $g_\epsilon^* + \lambda v \in X(D)$ for $\lambda > 0$, small.

Plate

Algorithm

Step 1 Start with $n = 0$, $\epsilon > 0$ given "small" and select some initial g_n.
Step 2 Compute $y_\epsilon^n,\ z_\epsilon^n$ the solution of the state system with H_{Ω_g} replaced by $H^\epsilon(g)$.
Step 3 Compute $p^n,\ q^n$ the solution of the adjoint sytem.
Step 4 Compute the gradient of the considered cost functional.
Step 5 Denote by w_n the chosen descent direction and define $\widetilde{g}_n = g_n + \lambda_n w_n$, where $\lambda_n > 0$ is obtained via some line search.
Step 6 Compute $g_{n+1} = Proj_{X(D)}(\widetilde{g}_n)$, if the constraint on g is imposed.
Step 7 If $|g_n - g_{n+1}|$ and/or $|\nabla j(g_n)|$ are below some prescribed tolerance parameter, then **Stop**. If not, update $n := n + 1$ and go to **Step 2**.
We underline the combination of both topological and boundary variations.

Plate

Ex.1
We have $D =]-1,1[\times]-1,1[$, 53360 triangles and 26981 vertices on it, the load $f = 3$, the cost function $j(g) = \frac{1}{2}\int_\Omega (y_\epsilon - y_d)^2 d\mathbf{x}$, where $y_d(x_1, x_2) = -(x_1 - 0.5)^2 - (x_2 - 0.5)^2 + \frac{1}{16}$. The initial geometric parametrization function is

$$g_0(x_1, x_2) = \min\left(x_1^2 + x_2^2 - \frac{1}{16};\ (x_1 - 0.5)^2 + x_2^2 - \frac{1}{64};\ 1 - x_1^2 - x_2^2\right)$$

which corresponds to a domain with two holes.
The penalization parameter is $\epsilon = 10^{-5}$.

Plate

We use in the iterations the descent direction

$$w_n = -\left[\frac{1}{2}(y_\epsilon^n - y_d)^2 + \frac{1}{\epsilon}(y_\epsilon^n p^n + z_\epsilon^n q^n)\right]. \tag{3}$$

The cost function decreases rapidly at the first iterations $j(g_0) = 2.29164$, $j(g_1) = 0.00083009$, $j(g_2) = 0.000510025$, $j(g_3) = 0.000379625$, but for $n \geq 4$, Ω_n is similar to Ω_3 and cost function decreases slowly $j(g_8) = 0.000171446$, $j(g_{11}) = 0.00012326$, $j(g_{14}) = 0.000100719$. The initial domain and some computed domains are presented in the Figures.

Plate

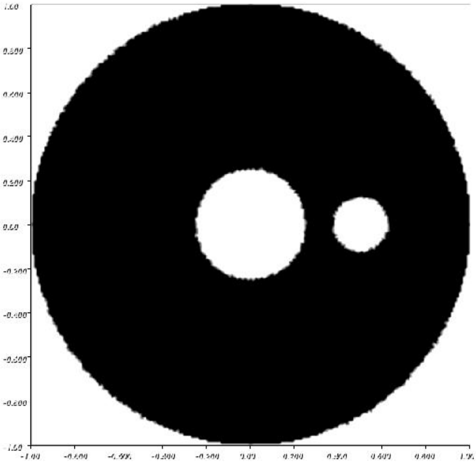


Figure: First iteration Ex 1

Plate

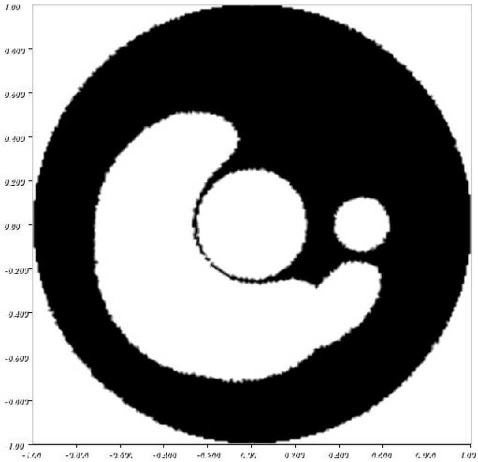


Figure: Second iteration Ex 1

Plate

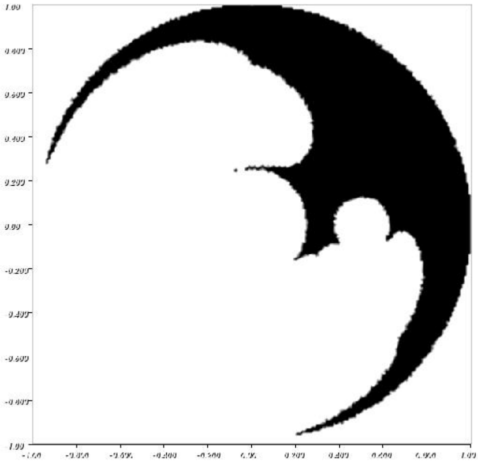


Figure: Third iteration Ex 1

Plate

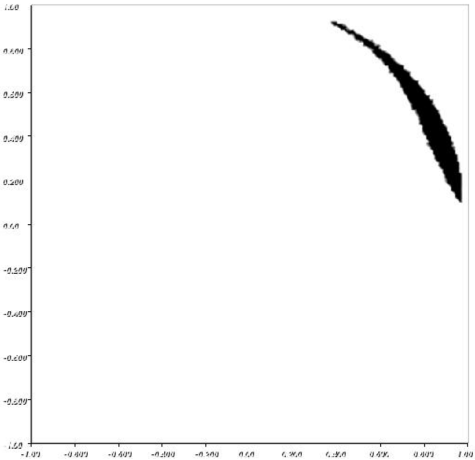


Figure: Fourth iteration Ex 1

Plate

Ex. 2

We have again $D =]-1,1[\times]-1,1[$. We use for D a mesh of 53360 triangles and 26981 vertices and for the approximation of g, y, z we use piecewise linear finite element, globally continuous. The load is $f = 1$, the cost function is $j(g) = \int_\Omega (y_\epsilon - y_d) d\mathbf{x}$ where y_d is given by

$$y_d(x_1, x_2) = \begin{cases} 1, & \text{if } \frac{1}{9} \leq x_1^2 + x_2^2 \leq \frac{1}{4} \\ -1, & \text{otherwise.} \end{cases}$$

The penalization parameter is $\epsilon = 10^{-3}$ and $J(\mathbf{x}, y_\epsilon(\mathbf{x})) = y_\epsilon - y_d$ (not positive).
We get the following descent direction

$$w_n = -\left[(y_\epsilon^n - y_d) + \frac{1}{\epsilon}(y_\epsilon^n p^n + z_\epsilon^n q^n) \right]. \tag{4}$$

Plate

The sequence $(j(g_n))_{n\in\mathbb{N}}$ is decreasing. For the stopping test, we use: `if` $j(g_{n+1}) > j(g_n) - tol$ `then` STOP, where $tol = 10^{-6}$. For the initial parametrization function $g_0(x_1, x_2) = -x_1^2 - x_2^2 + \frac{3}{4}$, that corresponds to a simply connected domain, the stopping test is obtained for $n = 3$, the values of the cost function are: $j(g_0) = 1.51761$, $j(g_1) = -0.417807$, $j(g_2) = -0.421269$, $j(g_3) = -0.423723$. Some computed domains are presented in the following Figures.

Plate

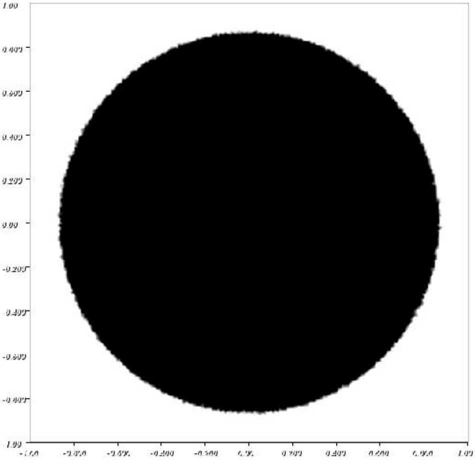


Figure: First iteration Ex 2

Plate

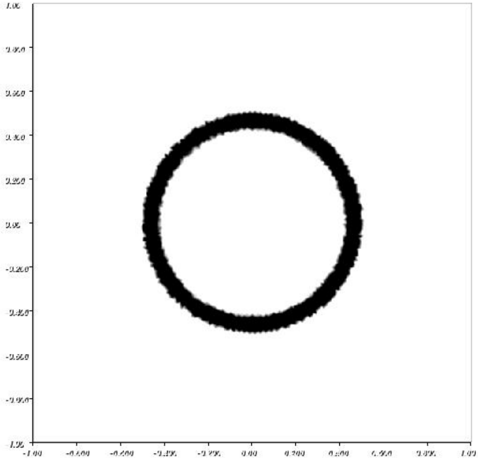


Figure: Second iteration Ex 2

Plate

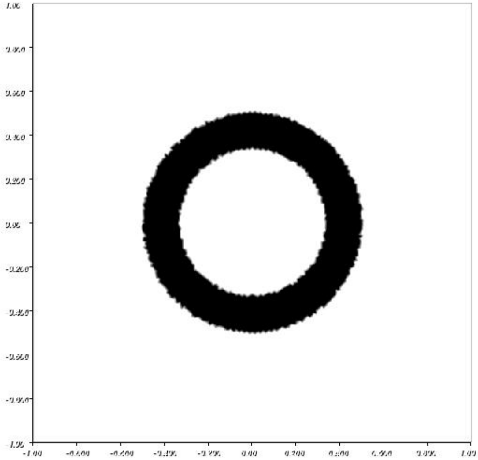


Figure: Third iteration Ex 2

Applications

One motivating application for the above discussion is in shape optimization problems. A typical example has the form:

$$\operatorname*{Min}_{\Omega \in \mathcal{O}} \int_{\Lambda} j(x, y_\Omega(x))dx,$$

$$-\Delta y_\Omega = f \quad in\, \Omega,$$

$$y_\Omega = 0 \quad on\, \partial\Omega$$

with other supplementary constraints, if necessary. Other differential operators or other boundary conditions may be studied as well.

Applications

Here, $\mathcal{O}$ is a family of admissible domains in R^3, satisfying certain regularity hypotheses and conditions like

$$E \subset \Omega \subset D, \; \forall \, \Omega \in \mathcal{O}$$

where E, D are given (bounded) domains, E may be even void, etc.
Function $f \in L^2(D)$ and Λ may be either E (if nonvoid) or Ω or $\partial\Omega$, etc. The integrand $j(\cdot,\cdot) : D \times R \to R$ is of Carathéodory type.
The problem has a similar structure with an optimal control problem, the main difference and difficulty being that the minimization parameter is the domain Ω itself.

Applications

The idea is to represent the unknown domains as level sets $F(x_1, x_2, x_3) \leq 0$ in D. We assume as usual $F_{,1}(x^0) \neq 0$.

Now, to compute the gradient of such a cost functional, we can use functional variations that are perturbations of the form

$$F(x_1, x_2, x_3) + \lambda h(x_1, x_2, x_3) = 0,$$

Such perturbations of the geometry may be very complex, including topological and boundary perturbations simultaneously. This is important in applications to shape optimization .

Applications

We approximate the equation in D

$$-\Delta y_\varepsilon + \frac{1}{\varepsilon}(1 - H^\varepsilon(g))y_\varepsilon = f \quad \text{in } D,$$

$$y_\varepsilon = 0 \quad \text{on } \partial D.$$

Proposition *If $\Omega = \Omega_g$ is of class C, then $y_\varepsilon|_{\Omega_g} \to y$ (the solution of the original state system) weakly in $H^1(\Omega_g)$ and strongly in $L^2(\Omega_g)$.*

Applications

An example in dimension two is the following

$$\operatorname*{Min}_{\Omega\in\mathcal{O}} \int_{\Lambda} \left(\frac{\partial y_\Omega}{\partial n}\right)^2 d\sigma,$$

$$-\Delta y_\Omega = f \quad \text{in } \Omega,$$

$$y_\Omega = 0 \quad \text{on } \partial\Omega,$$

$$E \subset \Omega \subset D, \ \forall\, \Omega \in \mathcal{O}.$$

We assume that the admissible domains in $\mathcal{O}$ are defined as level sets of functions $g \in G_{ad}$.

Applications

Proposition *If $f \in L^p(D), p > 2$, the directional derivative of the cost is given by*

$L = \int_I [A + B + C],$

where

$$A = 2\frac{\partial y_\varepsilon}{\partial n_g}(x_g(t), y_g(t))\{\frac{\partial}{\partial n_g}[\nabla y_\varepsilon(x_g(t), y_g(t))].(z(t), w(t)) + [\nabla y_\varepsilon(x_g(t), y_g(t)$$

$$[\nabla n_g(x_g(t), y_g(t)).(z(t), w(t))]\}\sqrt{(\dot{x}_g(t))^2 + (\dot{y}_g(t))^2},$$

Applications

$$B = 2\frac{\partial y_\varepsilon}{\partial n_g}(x_g(t), y_g(t))\{[\nabla y_\varepsilon.\nabla h/|\nabla g|](x_g(t), y_g(t)) - [\frac{\partial y_\varepsilon}{\partial n_g}\nabla g.\nabla h/|\nabla g|^2]$$

$$(x_g(t), y_g(t))\}\sqrt{(\dot{x}_g(t))^2 + (\dot{y}_g(t))^2},$$

$$C = 2\{[\frac{\partial y_\varepsilon}{\partial n_g}^2 |\nabla g|^{-2}](x_g(t), y_g(t))\}(\dot{x}_g(t), \dot{y}_g(t)).(z(t), w(t)).$$

where (z, w) solve the system in variations associated with the Hamiltonian system describing the geometry and I is its existence interval.

THANK YOU!