One-dimensional symmetry for the Euler equations and related semilinear elliptic equations / Propriétés de symétrie pour les équations d'Euler et certaines équations elliptiques semi-linéaires

François HAMEL

Aix-Marseille University, France

In collaboration with Nikolai NADIRASHVILI

Atelier de travail Equations aux Dérivées Partielles, IMAR

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I. Shear flows of an ideal fluid

Euler equations

$$\begin{cases} v \cdot \nabla v + \nabla p = 0 & \text{in } \Omega \\ \mathrm{div}\ v = 0 & \text{in } \Omega \end{cases}$$

with $v \in C^2(\overline{\Omega})$

Shear flow

$$v = (v_1, 0, \cdots, 0)$$

(up to rotation) and

$$v_1 = v_1(x_2, \cdots, x_N)$$

Shear flow $\Longleftrightarrow$ the pressure p is constant

Shear flows in two-dimensional domains $\Omega \subset \mathbb{R}^2$?

Two-dimensional strip

$$\Omega_2 = \mathbb{R} \times (0,1) = \big\{x = (x_1, x_2) \in \mathbb{R}^2,\ 0 < x_2 < 1\big\}$$

Theorem

Assume that $v_2 = 0$ on $\partial\Omega_2$ ($v \cdot n = 0$ on $\partial\Omega_2$) and

$$\inf_{\Omega_2} |v| > 0.$$

Then v is a shear flow:

$$v(x_1, x_2) = (v_1(x_2), 0) \ \text{ in } \overline{\Omega_2}.$$

Remark: The flow v is not assumed to be *a priori* bounded in Ω_2. But it is *a posteriori* bounded from the conclusion, since $v = (v_1(x_2), 0)$ and the cross section $[0, 1]$ is bounded.

Sufficient condition $\inf_{\Omega_2}|v|>0$*: no stagnation point in* $\overline{\Omega_2}$ *nor at infinity*

- Theorem: any non-shear flow which is tangential on $\partial\Omega_2$ must have a stagnation point in $\overline{\Omega_2}$ or at infinity.

- Example 1: cellular flow (for $\alpha\neq 0$)

$$\begin{aligned} v(x_1,x_2) &= \nabla^\perp\big(\sin(\alpha x_1)\sin(\pi x_2)\big)\\ &= \big(-\pi\sin(\alpha x_1)\cos(\pi x_2),\alpha\cos(\alpha x_1)\sin(\pi x_2)\big)\end{aligned}$$

  with $p(x_1,x_2)=(\pi^2/4)\cos(2\alpha x_1)+(\alpha^2/4)\cos(2\pi x_2)$.

  Stationary points in $\overline{\Omega_2}$.

- Example 2:

$$v(x_1,x_2)=\nabla^\perp\big(\sin(\pi x_2)\,e^{x_1}\big)=\big(-\pi\cos(\pi x_2)\,e^{x_1},\sin(\pi x_2)\,e^{x_1}\big)$$

  with $p(x_1,x_2)=-(\pi^2/2)e^{2x_1}$.

  No stagnation point in $\overline{\Omega_2}$, but $\inf_{\Omega_2}|v|=0$.

But shear flows $v=(v_1(x_2),0)$ *do not necessarily satisfy* $\inf_{\Omega_2}|v|>0$ *!*

Theorem does not hold in dimension 3

$$\Omega = \big\{x = (x_1, x_2, x_3) \in \mathbb{R}^3;\ x_2^2 + x_3^2 < 1\big\}$$

Flow

$$v(x) = (1, -x_3, x_2)$$

tangential on the boundary $\partial\Omega$, and

$$1 \le |v| \le \sqrt{2} \text{ in } \Omega.$$

Pressure $p(x) = \dfrac{x_2^2 + x_3^2}{2}$.

The flow is not a shear flow !

Half-plane

$$\mathbb{R}^2_+ = \mathbb{R} \times (0, +\infty) = \{x = (x_1, x_2) \in \mathbb{R}^2,\ x_2 > 0\}$$

Theorem

Assume that $v_2 = 0$ on $\partial\mathbb{R}^2_+$ ($v \cdot n = 0$ on $\partial\mathbb{R}^2_+$) and

$$0 < \inf_{\mathbb{R}^2_+} |v| \leq \sup_{\mathbb{R}^2_+} |v| < +\infty.$$

Then v is a shear flow:

$$v(x_1, x_2) = (v_1(x_2), 0) \ \text{ in } \overline{\mathbb{R}^2_+}.$$

The strict inequalities $0 < \inf_{\mathbb{R}^2_+} |v| \le \sup_{\mathbb{R}^2_+} |v| < +\infty$ *cannot be dropped in general*

- Example 1: cellular flow

$$\begin{aligned} v(x_1, x_2) &= \nabla^\perp\big(\sin(\alpha x_1)\sin(\pi x_2)\big) \\ &= \big(-\pi\sin(\alpha x_1)\cos(\pi x_2), \alpha\cos(\alpha x_1)\sin(\pi x_2)\big) \end{aligned}$$

  It is bounded in $\mathbb{R}^2_+$, tangential on $\partial\mathbb{R}^2_+$.

  But $\inf_{\mathbb{R}^2_+} |v| = \min_{\mathbb{R}^2_+} |v| = 0$, and v is not a shear flow.

- Example 2:

$$v(x_1, x_2) = \nabla^\perp\big(x_2\cosh(x_1)\big) = \big(-\cosh(x_1), x_2\sinh(x_1)\big)$$

  with $p(x_1, x_2) = -\cosh(2x_1)/4 + x_2^2/2$.

  The flow v is tangential on $\partial\mathbb{R}^2_+$ and $\inf_{\mathbb{R}^2_+} |v| > 0$.

  But $\sup_{\mathbb{R}^2_+} |v| = +\infty$, and v is not a shear flow.

Open question:

Can the assumption $0 < \inf_{\mathbb{R}^2_+} |v| \le \sup_{\mathbb{R}^2_+} |v| < +\infty$ be replaced with

$$\forall A > 0, \quad 0 < \inf_{\mathbb{R}\times(0,A)} |v| \le \sup_{\mathbb{R}\times(0,A)} |v| < +\infty \ ?$$

Case of the plane $\mathbb{R}^2$

Theorem

Assume that v is uniformly continuous and

$$0 < \inf_{\mathbb{R}^2} |v| \le \sup_{\mathbb{R}^2} |v| < +\infty.$$

Then v is a shear flow: there exist a unit vector e and $V : \mathbb{R} \to \mathbb{R}$ s.t.

$$v(x) = V(x \cdot e^{\perp})\, e \ \text{ in } \mathbb{R}^2$$

(hence, $v \cdot e$ has a constant sign).

Corollary

Let v be a $C^2(\mathbb{R}^2)$ periodic flow.

If $|v(x)| \neq 0$ for every $x \in \mathbb{R}^2$, then v is a shear flow.

Corollary

Let v be a shear flow such that $0 < \inf_{\mathbb{R}^2} |v| \leq \sup_{\mathbb{R}^2} |v| < +\infty$.

If $\|v' - v\|_{L^\infty(\mathbb{R}^2)} \ll 1$ and v' is $C^2(\mathbb{R}^2)$ and uniformly continuous, then v' is a shear flow.

Remark: in the theorem, if one also assumes that $v \cdot e > 0$ in $\mathbb{R}^2$ for some unit vector e, then the proof is much easier!

Scheme of the proofs

- Stream function u of the flow v:

$$\nabla^{\perp} u = v$$

- Geometric properties of the streamlines of the flow
- Strip Ω_2: the streamlines go from $-\infty$ to $+\infty$ in the x_1 direction
  Half-plane $\mathbb{R}^2_+$: they are also bounded in x_2
- The streamlines foliate the domain
- Vorticity

$$\frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2} = \Delta u$$

  is constant along the streamlines
- Semilinear elliptic equation

$$\Delta u + f(u) = 0$$

Two-dimensional strip

$$\Omega_2 = \mathbb{R} \times (0,1) = \{x = (x_1, x_2) \in \mathbb{R}^2,\ 0 < x_2 < 1\}$$

Theorem

Let $f : \mathbb{R} \to \mathbb{R}$ be locally Lipschitz continuous and let $u \in C^2(\overline{\Omega_2})$ solve

$$\begin{cases} \Delta u + f(u) = 0 & \text{in } \Omega_2 \\ 0 < u < c & \text{in } \Omega_2 \end{cases}$$

with $u = 0$ on $\{x_2 = 0\}$ and $u = c$ on $\{x_2 = 1\}$.

Then u is one-dimensional:

$$u(x_1, x_2) = U(x_2) \ \text{ in } \overline{\Omega_2}$$

and $U'(x_2) > 0$ for all $0 < x_2 < 1$.

Remark: $U'(0)$ and $U'(1)$ may vanish. Example:

$$\begin{cases} u(x_1, x_2) = U(x_2) := 1 - \cos(\pi x_2) \ \text{ in } \Omega_2, \\ 0 < u < 2 \ \text{ in } \Omega_2, \quad U(0) = 0, \quad U(1) = 2, \end{cases}$$

with $f(s) = \pi^2(s-1)$.

One has $U' > 0$ in $(0,1)$, but $U'(0) = U'(1) = 0$.

Two main steps in the proof of the theorem

- $\dfrac{\partial u}{\partial x_2} \geq 0$ in $\overline{\Omega_2}$

  [Berestycki, Caffarelli, Nirenberg]

  (true in dimension $N = 2$)

- Monotonicity in $x_2 \Longrightarrow$ one-dimensional symmetry

Theorem

N-dimensional slab (with $N \geq 2$):

$$\Omega_N = \mathbb{R}^{N-1} \times (0,1) = \{x = (x_1, \dots, x_N) \in \mathbb{R}^N,\ 0 < x_N < 1\}$$

Let $f : \mathbb{R} \to \mathbb{R}$ be locally Lipschitz continuous and let $u \in C^2(\overline{\Omega_N})$ solve

$$\begin{cases} \Delta u + f(u) = 0 & \text{in } \Omega_N \\ \dfrac{\partial u}{\partial x_N} \geq 0 & \text{in } \Omega_N \end{cases}$$

with $u = 0$ on $\{x_N = 0\}$ and $u = c > 0$ on $\{x_N = 1\}$.

Then u is one-dimensional:

$$u(x_1, \cdots, x_N) = U(x_N) \quad \text{in } \overline{\Omega_N}$$

and $U'(x_N) > 0$ for all $0 < x_N < 1$.

Strategy: sliding method in any direction $\tau = (\tau', \tau_N)$ with $\tau_N > 0$, and strong maximum principle

Proposition

Let $f:\mathbb{R}\to\mathbb{R}$ be locally Lipschitz continuous and let $u\in C^2(\overline{\Omega_N})$ solve

$$\begin{cases} \Delta u + f(u) = 0 & \text{in } \Omega_N \\ 0 < u < c & \text{in } \Omega_N \end{cases}$$

with $u=0$ on $\{x_N=0\}$ and $u=c$ on $\{x_N=1\}$.

If

$$\forall\, 0<x_N<1,\quad 0<\inf_{x'\in\mathbb{R}^{N-1}} u(x',x_N) \le \sup_{x'\in\mathbb{R}^{N-1}} u(x',x_N) < c$$

or if

$$f(c)\le 0\le f(0),$$

then u is one-dimensional:

$$u(x_1,\cdots,x_N)=U(x_N)\ \text{ in } \overline{\Omega_N}$$

and $U'(x_N)>0$ for all $0<x_N<1$.

Case of the two-dimensional strip $\Omega_2 = \mathbb{R} \times (0,1)$

How to reduce the Euler equations

$$\begin{cases} v \cdot \nabla v + \nabla p = 0 & \text{in } \Omega_2 \\ \mathrm{div}\ v = 0 & \text{in } \Omega_2 \end{cases}$$

with $v \cdot n = 0$ on $\partial\Omega_2$ and $\inf_{\Omega_2} |v| > 0$,

to the elliptic equation

$$\begin{cases} \Delta u + f(u) = 0 & \text{in } \Omega_2 \\ 0 < u < c & \text{in } \Omega_2 \end{cases}$$

with $u = 0$ on $\{x_2 = 0\}$ and $u = c$ on $\{x_2 = 1\}$?

- Stream function $u \in C^3(\overline{\Omega_2})$ defined by

$$\frac{\partial u}{\partial x_1} = v_2, \quad \frac{\partial u}{\partial x_2} = -v_1$$

$$|\nabla u| = |v| \geq \eta > 0$$

Normalization $u(0,0) = 0$

$v_2 = 0$ on $\partial\Omega_2 \Longrightarrow$

$$u = 0 \text{ on } \{x_2 = 0\} \text{ and } u = c \text{ on } \{x_2 = 1\} \ (c \in \mathbb{R})$$

Each level curve Γ_z of u (connected component of the level set of u containing z) is the streamline of the flow v containing z

- The streamlines are unbounded:

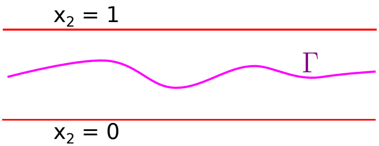
Lemma

Let $\Gamma \subset \overline{\Omega_2}$ be a streamline.

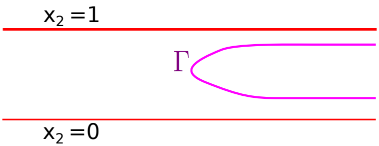
Let $\gamma : \mathbb{R} \to \Gamma$ be a parametrization of Γ.

Then

$$|\gamma(t)| \to +\infty \text{ as } t \to \pm\infty.$$



(a) A possible streamline Γ



(b) Another possible streamline Γ

- Gradient flow

$$\dot{y} = \nabla u$$

Lemma

Let $G \subset \Omega_2$ be open. Assume u is bounded in G.

Let Σ be a trajectory of the gradient flow in G.

Let $g : \mathbb{R} \to \Sigma$ be a parametrization of Σ.

Then

$$\mathrm{length}(\Sigma) \leq \frac{\sup_G u - \inf_G u}{\eta}$$

and $\mathrm{dist}(g(t), \partial G) \to 0$ as $t \to \pm\infty$.

Proof: $\dfrac{d}{dt}\, u(y(t)) \geq \eta\, |\dot{y}(t)|$

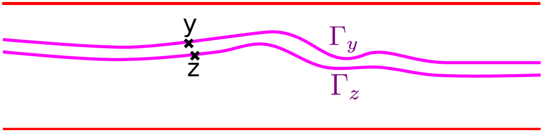
- $B(x,r) = \{y \in \overline{\Omega_2},\ |y-x| < r\}$

Lemma

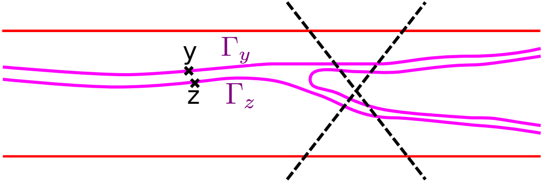
Given point $x \in \overline{\Omega_2}$ and given $\varepsilon > 0$.

Then, there is $r > 0$ such that:

$$\forall\, y, z \in B(x,r), \quad \mathrm{dist}_{\mathcal{H}}(\Gamma_y, \Gamma_z) \le \varepsilon.$$



(c) Two streamlines with $y \simeq z$



(d) Impossible

- Streamlines go from $-\infty$ to $+\infty$ in the direction x_1

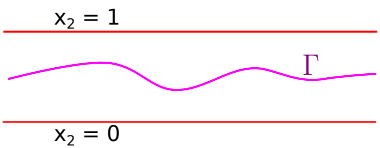
Lemma

Let $\Gamma \subset \overline{\Omega_2}$ be a streamline.

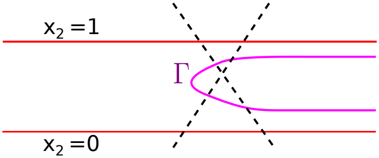
Then Γ has a parametrization $\gamma : \mathbb{R} \to \Gamma$, $t \mapsto \gamma(t) = (\gamma_1(t), \gamma_2(t))$ such that

$$\gamma_1(t) \to \pm\infty \text{ as } t \to \pm\infty.$$

Proof: continuation argument



(e) Any streamline Γ



(f) Impossible

- $v_2 = 0$ on $\partial\Omega_2 \Longrightarrow u = 0$ on $\{x_2 = 0\}$ and $u = c$ on $\{x_2 = 1\}$

  Assume $\dfrac{\partial u}{\partial x_2}(0,0) = -v_1(0,0) > 0$ (wlog)

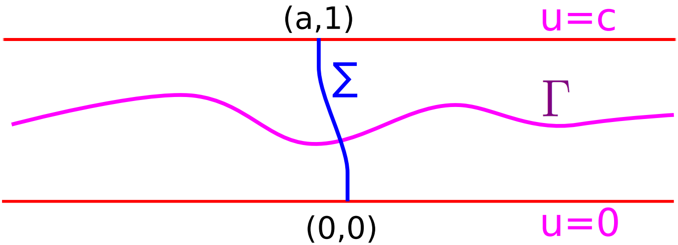
Lemma

The function u is bounded in Ω_2.

Furthermore, $c > 0$ and

$$0 < u < c \text{ in } \Omega_2$$

Trajectory Σ of the gradient flow $\dot\sigma(t) = \nabla u(\sigma(t))$ with $\sigma(0)=(0,0)$ and $t \in [0,\tau]$.



- Vorticity

$$\frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2} = \Delta u$$

is constant along the streamlines:

$$v \cdot \nabla(\Delta u) = 0 \text{ in } \overline{\Omega_2}$$

- Semilinear elliptic equation

$$\Delta u + f(u) = 0 \text{ in } \overline{\Omega_2}$$

with $f(s) = -\Delta u\big(\sigma(\theta^{-1}(s))\big)$ for $s \in [0, c]$ and $\theta(t) = u(\sigma(t))$
$(\Delta u(\sigma(t)) + f(u(\sigma(t)) = 0)$

- Liouville-type theorem in the strip $\Longrightarrow$

$u(x_1, x_2) = U(x_2)$ and $v(x_1, x_2) = (-U'(x_2), 0)$ is a shear flow.

Case of the half-plane $\mathbb{R}^2_+ = \mathbb{R} \times (0, +\infty)$

- Potential function $u \in C^3(\overline{\mathbb{R}^2_+})$ defined by

$$\frac{\partial u}{\partial x_1} = v_2, \quad \frac{\partial u}{\partial x_2} = -v_1$$

  Normalization $u(0,0) = 0$

  $v_2 = 0$ on $\partial\mathbb{R}^2_+ \Longrightarrow u = 0$ on $\partial\mathbb{R}^2_+ = \{x_2 = 0\}$

- The streamlines are unbounded.

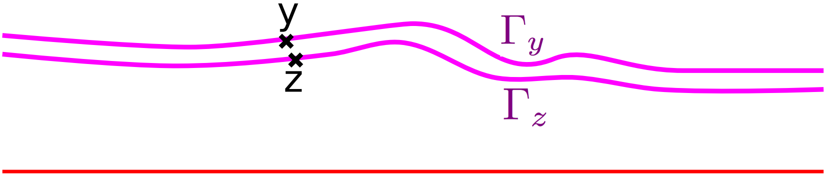
  Let $\gamma : \mathbb{R} \to \Gamma$ be a parametrization of a streamline $\Gamma \subset \overline{\mathbb{R}^2_+}$. Then

$$|\gamma(t)| \to +\infty \text{ as } t \to \pm\infty.$$

- $B(x,r)=\{y\in\overline{\mathbb{R}^2_+},\ |y-x|<r\}$

  For any point $x\in\overline{\mathbb{R}^2_+}$ and any $\varepsilon>0$, there is $r>0$ such that:

$$\forall\, y,z\in B(x,r),\quad \mathrm{dist}_{\mathcal{H}}(\Gamma_y,\Gamma_z)\le\varepsilon.$$



- All streamlines are bounded in the direction x_2:

Lemma

Let $\Gamma \subset \overline{\mathbb{R}^2_+}$ be a streamline.

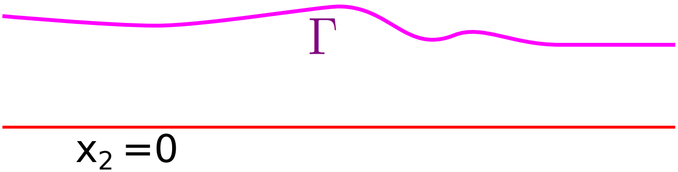
Then there is $A > 0$ such that

$$\Gamma \subset \mathbb{R} \times [0, A]$$

and Γ has a parametrization $\gamma : \mathbb{R} \to \Gamma$, $t \mapsto \gamma(t) = (\gamma_1(t), \gamma_2(t))$ such that

$$\gamma_1(t) \to \pm\infty \text{ as } t \to \pm\infty.$$

Proof: continuation argument



- $v_2 = 0$ on $\partial\mathbb{R}^2_+ \Longrightarrow u = 0$ on $\{x_2 = 0\}$

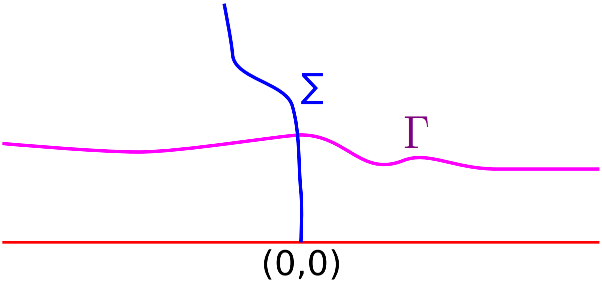
  Assume $\dfrac{\partial u}{\partial x_2}(0,0) = -v_1(0,0) > 0$ (wlog)

Lemma

Then

$$u > 0 \text{ in } \mathbb{R}^2_+$$

Trajectory Σ of the gradient flow $\dot\sigma(t) = \nabla u(\sigma(t))$ with $\sigma(0)=(0,0)$ and $t \in [0,+\infty)$.



- Vorticity $\frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2} = \Delta u$ constant along the streamlines:

- Semilinear elliptic equation

$$\Delta u + f(u) = 0 \text{ in } \overline{\mathbb{R}^2_+}$$

with $f(s) = -\Delta u\big(\sigma(\theta^{-1}(s))\big)$ for $s \in [0, +\infty)$ and $\theta(t) = u(\sigma(t))$ $(\Delta u(\sigma(t)) + f(u(\sigma(t)) = 0)$

- $u = 0$ on $\partial\mathbb{R}^2_+$ and $u > 0$ in $\mathbb{R}^2_+ \Longrightarrow$

$$\frac{\partial u}{\partial x_2} > 0 \text{ in } \mathbb{R}^2_+$$

[Berestycki, Caffarelli, Nirenberg], [Dancer], [Farina, Sciunzi]

- $|\nabla u| = |v|$ bounded $\Longrightarrow$

$$u(x_1, x_2) = U(x_2)$$

[Farina, Valdinoci]

Conclusion: $v(x_1, x_2) = (-U'(x_2), 0)$ is a shear flow.

Case of the plane $\mathbb{R}^2$

- Stream function $u \in C^3(\mathbb{R}^2)$ defined by

$$\frac{\partial u}{\partial x_1} = v_2, \quad \frac{\partial u}{\partial x_2} = -v_1$$

- The streamlines are unbounded.
- The trajectories of the gradient flow are unbounded.
- Each level set of u has only one connected component.
- Equation for the stream function:

$$\Delta u + f(u) = 0 \text{ in } \mathbb{R}^2.$$

- Argument ϕ of v:

$$\frac{v(x)}{|v(x)|} = (\cos\phi(x), \sin\phi(x)).$$

- Uniformly elliptic equation

$$\boxed{\mathrm{div}(|v|^2\nabla\phi) = 0}$$

Key-estimate

$$|\phi(x)| = O(\ln|x|) \ \text{ as } |x| \to +\infty.$$

- [Moser] $\Longrightarrow$ ϕ is constant $\Longrightarrow$ v is a shear flow.