A characterization related to Schrödinger equations

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[1] Joint work with F. Faraci

Consider the following nonlinear Schrödinger equation

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\Delta\psi + V(x)\psi - f(x,|\psi|), \quad \text{in } \mathbb{R}^N \times \mathbb{R}_+ \setminus \{0\},$$

which plays a central role in quantum mechanic as it predicts the future behavior of a dynamic system.

$$\psi(x,t) = u(x)e^{-i\frac{E}{\hbar}t}$$

$$-\Delta u + V(x)u = f(x,u), \quad \text{in } \mathbb{R}^N. \tag{1}$$

Introduction

With variational methods, the existence and multiplicity of nontrivial solutions for such problem have been extensively studied in the literature over the last decades:

Theorem (Rabinowitz, 92)

If the potential V is coercive and f satisfies standard mountain pass assumptions then the problem (1) *has a positive solution.*

- In the class of bounded from below potentials, several attempts have been made to find general assumptions on V in order to obtain existence and multiplicity results: Bartsch, Pankov and Wang (CCM 2001), Bartch and Wand (CPDE 95), Benci and Fortunato (JMAA 1978) etc.

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- In these works, the nonlinearity f is required to satisfy the well-know Ambrosetti-Rabinowitz condition, thus it is superlinear at infinity.

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If the potential V is coercive and f satisfies standard mountain pass assumptions then the problem (1) *has a positive solution.*

- For a sublinear growth of f we mention, Kristály (NODEA 2007) $\rightarrow$ the existence of certain multiple weak solutions.

Motivation

Most of the aforementioned papers provide *sufficient* conditions on the nonlinear term f in order to prove existence/multiplicity type results.

- Ricceri proved a characterization theorem which is stated for one-dimensional Dirichlet problem; more precisely, the following condition

for each $b > 0$, the function $\xi \to \dfrac{F(\xi)}{\xi^2}$ is not constant in $]0, b]$;

characterizes the existence of the solutions for the following problem

$$\begin{cases} -u'' = \lambda\alpha(x)f(u), & \text{in }]0,1[\\ u > 0, & \text{in }]0,1[\\ u(1) = u(0) = 0. \end{cases}$$

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One dimensional problem

Theorem (Ricceri, 2015)

Assume that there exists $a > 0$, the function $\xi \to \dfrac{F(\xi)}{\xi^2}$ is non-increasing in $]0, a]$.
Then, the following conditions are equivalent:

(i) *for each $b > 0$, the function $\xi \to \dfrac{F(\xi)}{\xi^2}$ is not constant in $]0, b]$;*

(ii) *for each $r > 0$, there exists an open interval $I \subseteq]0, +\infty[$ such that for every $\lambda \in I$, the problem has a solution u_λ satisfying*

$$\int_0^1 |u'_\lambda|^2 dt < r.$$

Question

- Can we prove a similar characterization result on the whole $\mathbb{R}^N$?
  - Recently, this result has been extended by Anello to higher dimension, i.e. when the interval $(0, 1)$ is replaced by a bounded domain $\Omega \subset \mathbb{R}^N$ ($N \in \mathbb{N}$) with smooth boundary.
- We will establish a *characterization* result for stationary Schrödinger equations on unbounded domains; even more, our arguments work on not necessarily linear structures.

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The variational framework

The subspace of $H^1(\mathbb{R}^N)$,

$$H_V = \left\{ u \in H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(x) u^2 dx < +\infty \right\},$$

equipped with the norm

$$\|u\| = \left(\int_{\mathbb{R}^N} (|\nabla u|^2 + V(x) u^2) dx \right)^{\frac{1}{2}},$$

turns out to be a Hilbert space compactly embedded in $L^2(\mathbb{R}^N)$. Thus, by interpolation inequality, H_V is compactly embedded in $L^q(\mathbb{R}^N)$ for every $q \in [2, 2^\star[$.

The variational framework

The energy functional associated to problem $(\mathcal{P}_\lambda)$ is the functional $\mathcal{E} : H_V \to \mathbb{R}$ defined by

$$\mathcal{E}(u) = \frac{1}{2}\|u\|^2 - \lambda \int_{\mathbb{R}^N} \alpha(x)F(u)dx,$$

which is differentiable in H_V with derivative, at any $u \in H_V$, given by

$$\mathcal{E}'(u)(v) = \int_{\mathbb{R}^N} (\nabla u \nabla v + V(x)uv)dx - \lambda \int_{\mathbb{R}^N} \alpha(x)f(u)vdx \qquad \text{for all } v \in H_V.$$

Critical points of $\mathcal{E}$ are weak solutions of $(\mathcal{P}_\lambda)$.

Our tools

Theorem A [Ricceri, Fixed Point Theory Appl. (2014)]

Let $(X, \langle \cdot, \cdot \rangle)$ be a real Hilbert space, $J : X \to \mathbb{R}$ a sequentially weakly upper semicontinuous and Gâteaux differentiable functional, with $J(0) = 0$. Assume that, for some $r > 0$, there exists a global maximum $\hat{x}$ of the restriction of J to $B_r = \{x \in X : \|x\|^2 \leq r\}$ such that

$$\langle J'(\hat{x}), \hat{x} \rangle < 2J(\hat{x}). \tag{2}$$

Then, there exists an open interval $I \subseteq]0, +\infty[$ such that, for each $\lambda \in I$, the equation $x = \lambda J'(x)$ has a non-zero solution lying in $\mathrm{int}(B_r)$.

Determining the interval

Set

$$\beta_r = \sup_{B_r} J, \qquad \delta_r = \sup_{x \in B_r \setminus \{0\}} \frac{J(x)}{\|x\|^2}$$

and

$$\eta(s) = \sup_{y \in B_r} \frac{r - \|y\|^2}{s - J(y)} \qquad \text{for all } s \in]\beta_r, +\infty[.$$

Then, η is convex and decreasing in $]\beta_r, +\infty[$. Moreover,

$$I = \frac{1}{2}\eta(]\beta_r, r\delta_r[).$$

A regularity result

Theorem (Faraci, F., 2018)

Assume $N \geq 3$ and let $g : \mathbb{R}^N \times [0, +\infty[\to \mathbb{R}$ be a Carathéodory function with primitive $G(x,t) = \int_0^t g(x,\xi)d\xi$ such that, for some constants $k > 0$ and $2 < q < 2^\star$ one has

$$g(x,\xi) \leq k(\xi + \xi^{q-1}) \text{ for all } \xi \geq 0, \text{uniformly in } x \in \mathbb{R}^N.$$

Let $u \in H_V$ be a non negative critical point of the functional $\mathcal{G} : H_V \to \mathbb{R}$

$$\mathcal{G}(u) = \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}^N} G(x,u)dx.$$

Then,

i) *for every $\rho > 0$, $u \in L^\infty(B_\rho)$;*

ii) *$u \in L^\infty(\mathbb{R}^N)$ and $\lim_{|x|\to\infty} u(x) = 0$;*

iii) $\|u\|_\infty \leq (C_1 + C_2\|u\|_{2^\star}^{q-2})^{\frac{1}{2^\star - q}} \|u\|_{2^\star}$.

Remark

- The proof of the above result is based in the Nash-Moser iteration technique;
- For nonlinear structures it is important to control the volume of the balls;

A key lemma

Consider now the following minimization problem

$$\text{(M)}\quad \min\{\|u\|^2 : \; u \in H_V, \|\alpha^{\frac{1}{2}} u\|_2 = 1\}.$$

Lemma

Problem **(M)** *has a solution* λ_α*, i.e. there exists a positive function* φ_α *such that*

$$\lambda_\alpha = \min\{\|u\|^2 : \; u \in H_V, \|\alpha^{\frac{1}{2}} u\|_2 = 1\} = \|\varphi_\alpha\|^2,$$

with $\|\alpha^{\frac{1}{2}}\varphi_\alpha\|_2 = 1$, $\varphi_\alpha \in L^\infty(\mathbb{R}^N)$, *and* $\lim_{|x|\to\infty} \varphi_\alpha(x) = 0$.

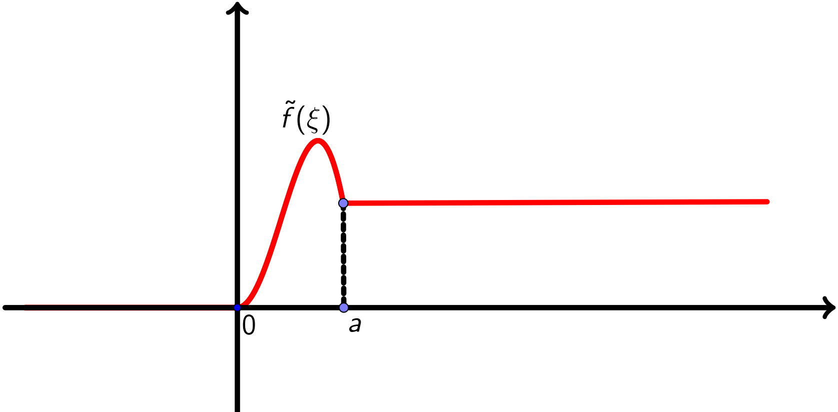
Moreover, φ_α is an eigenfunction of the equation

$$-\Delta u + V(x)u = \lambda\alpha(x)u, \qquad u \in H_V.$$

Proof the Theorem 1

Proof: $(i) \Rightarrow (ii)$

$$\tilde{f}(\xi) = \begin{cases} 0, & \text{if } \xi \in]-\infty, 0] \\ f(\xi), & \text{if } \xi \in]0, a] \\ f(a), & \text{if } \xi \in]a, +\infty[\end{cases} \quad \text{and} \quad \tilde{F}(\xi) = \int_0^{\xi} \tilde{f}(t)dt$$



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By assumption the function $\xi \to \frac{F(\xi)}{\xi^2}$ is non-increasing in $]0, a]$

$$\tilde{f}(\xi)\xi \leq 2\tilde{F}(\xi) \qquad \text{for all } \xi \in [0, +\infty[,$$

$$\Downarrow$$

$$\xi \to \frac{\tilde{F}(\xi)}{\xi^2} \text{ is non-increasing in }]0, +\infty[$$

Let $\sigma \in]0, +\infty]$ defined as

$$\sigma \equiv \lim_{\xi \to 0} \frac{\tilde{F}(\xi)}{\xi^2} = \sup_{\xi > 0} \frac{\tilde{F}(\xi)}{\xi^2}$$

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$\sigma = \infty$

Case 1: $\sigma < +\infty$

Define the functional

$$J : H_V \to \mathbb{R}, \qquad J(u) = \int_{\mathbb{R}^N} \alpha(x)\tilde{F}(u)dx$$

- J is sequentially weakly continuous
- J is Gâteaux differentiable with derivative given by

$$\langle J'(u), v\rangle = \int_{\mathbb{R}^N} \alpha(x)\tilde{f}(u)vdx, \qquad \text{for all } v \in H_V.$$

- $\displaystyle\sup_{u\in H_V\setminus\{0\}} \frac{J(u)}{\|u\|^2} = \frac{\sigma}{\lambda_\alpha}$

Case 1: $\sigma < +\infty$

Let $r > 0$ and $\hat{u}$ the global maximum of $J_{|_{B_r}}$. Notice that $\hat{u} \neq 0$ as $J(t\varphi_\alpha) > 0$ for every t small enough.

In order to apply Theorem A we need $\langle J'(\hat{u}), \hat{u}\rangle < 2J(\hat{u})$.

1^o : $\hat{u} \in \mathrm{int}(B_r) \Rightarrow J'(\hat{u}) = 0$. Thus

$$\langle J'(\hat{u}), \hat{u}\rangle < 2J(\hat{u}).$$

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In order to apply Theorem A we need $\langle J'(\hat{u}), \hat{u} \rangle < 2J(\hat{u})$.

2^o : $\|\hat{u}\|^2 = r \Rightarrow$ there exists $\mu > 0$ (Lagrange multiplier) such that

$$-\Delta \hat{u} + V(x)\hat{u} = \frac{1}{\mu}\alpha(x)\tilde{f}(\hat{u}), \text{ in } \mathbb{R}^N.$$

Thus, from the regularity result: $\hat{u} \in C^0(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ and $\lim_{|x|\to\infty} \hat{u}(x) = 0$.

$$\langle J'(\hat{u}), \hat{u} \rangle - 2J(\hat{u}) = \int_{\mathbb{R}^N} \alpha(x)[\tilde{f}(\hat{u})\hat{u} - 2\tilde{F}(\hat{u})] \leq 0.$$

If $\langle J'(\hat{u}), \hat{u} \rangle - 2J(\hat{u}) = 0$, then

$$\tilde{f}(\xi)\xi - 2\tilde{F}(\xi) = 0 \text{ for all } \xi \in [0, \|\hat{u}\|_\infty] \textit{ contradiction}.$$

Case 1: $\sigma < +\infty$

From Theorem A $\Rightarrow$ there exists $I \subseteq]0, +\infty[$ such that for every $\lambda \in I$,

$$u \to \frac{\|u\|^2}{2} - \lambda J(u)$$

has a non-zero critical point u_λ with $\int_{\mathbb{R}^N} (|\nabla u_\lambda|^2 + V(x)u_\lambda^2)dx < r$.
Thus, u_λ turns out to be a solution of the problem

$$\begin{cases} -\Delta u + V(x)u = \lambda\alpha(x)\tilde{f}(u), & \text{in } \mathbb{R}^N \\ u > 0, & \text{in } \mathbb{R}^N \\ u \to 0, & \text{as } |x| \to \infty. \end{cases} \qquad (\tilde{\mathcal{P}}_\lambda)$$

Claim: u_λ is a solution of $(\mathcal{P}_\lambda)$.

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Case 1: $\sigma < +\infty$

One can see that

$$I = \frac{1}{2}\left]\frac{\lambda_\alpha}{\sigma}, \delta\right[\qquad \text{for some} \qquad \delta > \frac{\lambda_\alpha}{\sigma}$$

It's enough to prove that

$$\lim_{\lambda \to \frac{\lambda_\alpha}{2\sigma}} \|u_\lambda\|_\infty = 0.$$

This implies that there exists a number $\varepsilon_r > 0$ such that for every

$$\lambda \in \left]\frac{\lambda_\alpha}{2\sigma}, \frac{\lambda_\alpha}{2\sigma} + \varepsilon_r\right[, \; \|u_\lambda\|_\infty \leq a \Rightarrow \tilde{f}(u_\lambda) = f(u_\lambda).$$

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One can see that

$$I = \frac{1}{2}\left]\frac{\lambda_\alpha}{\sigma}, \delta\right[\qquad \text{for some} \qquad \delta > \frac{\lambda_\alpha}{\sigma}$$

Fix a sequence $\lambda_n \to \frac{\lambda_\alpha}{2\sigma}^+$. Since $\|u_{\lambda_n}\|^2 \le r$, one has

- $u_{\lambda_n} \rightharpoonup u_0 \in B_r$,
- $u_{\lambda_n} \to u_0$ in $L^2(\mathbb{R}^N)$.

Thus, being u_{λ_n} a solution of $(\mathcal{P}_{\lambda_n})$,

$$\int_{\mathbb{R}^N} (\nabla u_{\lambda_n} \nabla v + V(x) u_{\lambda_n} v) dx = \lambda_n \int_{\mathbb{R}^N} \alpha(x) \tilde{f}(u_{\lambda_n}) v dx \qquad \text{for all} \ \ v \in H_V,$$

passing to the limit

$$-\Delta u_0 + V(x) u_0 = \frac{\lambda_\alpha}{2\sigma} \alpha(x) \tilde{f}(u_0) \qquad \text{in } \mathbb{R}^N.$$

Case 1: $\sigma < +\infty$

Assume $u_0 \neq 0$. Thus, using as test function $v = u_{\lambda_n}$, in

$$\int_{\mathbb{R}^N} (\nabla u_{\lambda_n} \nabla v + V(x) u_{\lambda_n} v) dx = \lambda_n \int_{\mathbb{R}^N} \alpha(x) \tilde{f}(u_{\lambda_n}) v dx \qquad \text{for all} \quad v \in H_V,$$

Case 1: $\sigma < +\infty$

Assume $u_0 \neq 0$. Thus, using as test function $v = u_{\lambda_n}$,

$$\|u_{\lambda_n}\|^2 = \lambda_n \int_{\mathbb{R}^N} \alpha(x)\tilde{f}(u_{\lambda_n})u_{\lambda_n}dx$$

and passing to the limit,

$$\begin{aligned} \|u_0\|^2 &\leq \liminf_{n\to\infty} \|u_{\lambda_n}\|^2 = \frac{\lambda_\alpha}{2\sigma}\int_{\mathbb{R}^N} \alpha(x)\tilde{f}(u_0)u_0 dx \\ &< \frac{\lambda_\alpha}{\sigma}\int_{\mathbb{R}^N} \alpha(x)\tilde{F}(u_0)dx \leq \lambda_\alpha \int_{\mathbb{R}^N} \alpha(x)(u_0)^2 dx \leq \|u_0\|^2. \end{aligned}$$

Thus, $u_0 = 0$, and $\lim_{n\to\infty} \|u_{\lambda_n}\| = 0$. Hence $\lim_{n\to\infty} \|u_{\lambda_n}\|_{2^\star} = 0$ and $\lim_{n\to\infty} \|u_{\lambda_n}\|_\infty = 0$. Therefore,

$$\lim_{\lambda \to \frac{\lambda_\alpha}{2\sigma}^+} \|u_\lambda\|_\infty = 0.$$

Hence, u_λ turns out to be a solution of the original problem $(\mathcal{P}_\lambda)$.

$(ii) \Rightarrow (i)$

Assume there exist two positive constants b, c such that

$$\frac{F(\xi)}{\xi^2} = c \qquad \text{for all} \;\; \xi \in]0, b].$$

Thus,

$$f(\xi) = 2c\xi \qquad \text{for all} \;\; \xi \in [0, b]. \tag{3}$$

- Let $\{r_n\}$ be a sequence of positive numbers such that $r_n \to 0^+$.
- Then, for every $n \in \mathbb{N}$ there exists an interval $I_n =]\frac{\lambda_\alpha}{2\sigma}, \frac{\lambda_\alpha}{2\sigma} + \varepsilon_n[$ such that for every $\lambda \in I_n$, $(\mathcal{P}_\lambda)$ has a solution $u_{\lambda,n}$ with $\|u_{\lambda,n}\|^2 < r_n$.
- Thus,

$$\lim_n \sup_{\lambda \in I_n} \|u_{\lambda,n}\| = 0.$$

$(ii) \Rightarrow (i)$

From the regularity result we obtain that

$$\lim_n \sup_{\lambda \in I_n} \|u_{\lambda,n}\|_\infty = 0.$$

Let us fix n_0 big enough such that $\sup_{\lambda \in I_{n_0}} \|u_{\lambda,n_0}\|_\infty < b$. We deduce that for every $\lambda \in I_{n_0}$, u_{λ,n_0} is a solution of the equation

$$-\Delta u + V(x)u = 2\lambda c\alpha(x)u, \quad \text{in } \mathbb{R}^N,$$

against the discreteness of the spectrum of the Schrödinger operator $-\Delta + V$.

Corollary

Let also $V : \mathbb{R}^N \to \mathbb{R}$ be in $L^\infty_{\rm loc}(\mathbb{R}^N)$, such that $\mathrm{essinf}_{\mathbb{R}^N} V \equiv V_0 > 0$ and

$$\int_{B(x)} \frac{1}{V(y)} dy \to 0 \qquad \text{as } |x| \to \infty,$$

Corollary

Assume that for some $a > 0$ the function $\xi \to \frac{F(\xi)}{\xi^2}$ is non-increasing in $(0, a]$. Then, the following conditions are equivalent:

- (i) *for each $b > 0$, the function $\xi \to \frac{F(\xi)}{\xi^2}$ is not constant in $(0, b]$;*
- (ii) *for each $r > 0$, there exists an open interval $I \subseteq (0, +\infty)$ such that for every $\lambda \in I$, problem $\mathscr{P}_\lambda$ has a nontrivial solution $u_\lambda \in H^1(\mathbb{R}^N)$ satisfying $\int_{\mathbb{R}^N} \left(|\nabla u_\lambda|^2 + V(x) u_\lambda^2\right) dx < r$.*

This set of assumption on V implies both the compactness of the embedding of $H^1_V(\mathbb{R}^N)$ into and the discreteness of the spectrum of the Schrödinger operator, due to Benci and Fortunato.

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Schrödinger equation on noncompact Riemannian manifold

$$\begin{cases} -\Delta_g u + V(x)u = \lambda\alpha(x)f(u), & \text{in } M \\ u > 0, & \text{in } M \\ u \to 0, & \text{as } d_g(x_0, x) \to \infty. \end{cases} \qquad (P_\lambda)$$

- (M, g) be a complete, non-compact N-dimensional $(N > 2)$ Riemannian manifold satisfying
  1) the curvature condition (**C**):

  $$\mathrm{Ric}_{(M,g)}(x) \geq -(N-1)H(d_g(\tilde{x}_0, x)) \qquad x \in M$$

  where $H \in C^1([0,\infty))$ is non-negative, bounded satisfying $\int_0^\infty tH(t)\mathrm{d}t = b_0 < +\infty$,
  2) $\inf_{x \in M} \mathrm{Vol_g(B_x(1))} > 0.$

Remark

- 1) $\Rightarrow$ volume growth property, i.e. $\mathrm{Vol}_g(B_x(\rho)) \leq e^{(N-1)b_0}\omega_N \rho^N, \ \rho > 0$;
- 1)+2) $\Rightarrow$ $H^1_g(M) \hookrightarrow L^p(M)$ is continuous for $p \in [2, 2^*]$ [Hebey (1999)].

It is well-known that

$$\mathrm{Vol}_{\mathbb{H}^N}(t) = N\omega_N \int_0^t \sinh^{N-1} s ds \leq \omega_N t^n e^{(N-1)t}$$

Thus we can't prove the same regularity result!

Remark

- 1) $\Rightarrow$ volume growth property, i.e. $\mathrm{Vol}_g(B_x(\rho)) \leq e^{(N-1)b_0}\omega_N\rho^N, \ \rho > 0$;
- 1)+2) $\Rightarrow H_g^1(M) \hookrightarrow L^p(M)$ is continuous for $p \in [2, 2^*]$ [Hebey (1999)].

- $\alpha : M \to]0, +\infty[$ be in $L^\infty(M)$
- $f : [0, \infty[\to [0, +\infty[$ continuous, subcritical, with $f(0) = 0$
- $V : M \to \mathbb{R}$ such that $\mathrm{essinf}_{x \in M} V(x) > 0$; and $\lim_{d_g(x_0,x)\to\infty} V(x) = +\infty$, for some $x_0 \in M$.

Remark

The above conditions on V implies

- the embedding $H_V^1(M) \hookrightarrow L^p(M)$ is compact for all $p \in [2, 2^*)$;
- the spectrum of the operator $-\Delta_g + V(x)$ is discrete [Poupaud (2005)].

Our result

Our result reads as follows:

Theorem

Assume that for some $a > 0$, the function $\xi \to \dfrac{F(\xi)}{\xi^2}$ is non-increasing in $]0, a]$.

Then, the following conditions are equivalent:

- (i) *for each $b > 0$, the function $\xi \to \dfrac{F(\xi)}{\xi^2}$ is not constant in $(0, b]$;*
- (ii) *for each $r > 0$, there exists an open interval $I \subseteq]0, +\infty[$ such that for every $\lambda \in I$, problem $(\mathcal{P}_\lambda)$ has a solution $u_\lambda \in H^1_g(M)$ satisfying*

$$\int_M \left(|\nabla u_\lambda|^2 + V(x)u_\lambda^2\right) dv_g < r.$$

Bibliography

- G. Anello, *A characterization related to the Dirichlet problem for an elliptic equation*, Funkcial. Ekvac., 59, 113–122, 2016.
- V. Benci, D. Fortunato, *Discreteness conditions of the spectrum of Schrödinger operators*, J. Math. Anal. Appl. 64 (1978), 695–700.
- F.Faraci, Cs. Farkas, *A characterization related to Schrödinger equations on Riemannian manifolds*, Comm. Contemporary Math., 2018.
- B. Ricceri, *A note on spherical maxima sharing the same Lagrange multiplier*, Fixed Point Theory Appl., 2014:25, 9 pages (2014).
- B. Ricceri, *A characterization related to a two-point boundary value problem.* J. Nonlinear Convex Anal., 16, 79–82, 2015.

Thank you for your attention!