Singularities of Steady Free Surface Water Flows under Gravity

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AIM OF THE TALK: **characterization**, by means of **geometric methods**, of all possible **singularities** in the problems of

- $2D$ **steady/travelling free-surface gravity water waves** (with or without vorticity)
- $3D$ **axisymmetric steady/travelling free-surface water flows with gravity** (without vorticity)

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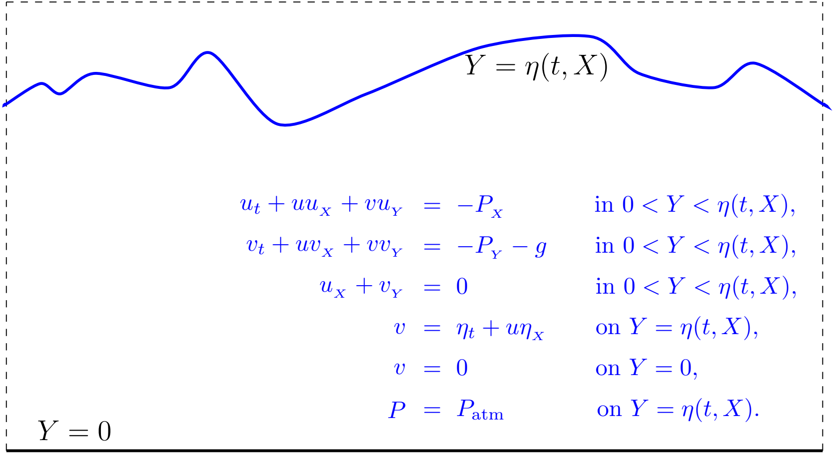
- Varvaruca, E.; Weiss, G. S. A geometric approach to generalized Stokes conjectures. *Acta Math.* **206** (2011), no. 2, 363–403.
- Varvaruca, E.; Weiss, G. S. The Stokes conjecture for waves with vorticity. *Ann. Inst. H. Poincaré Anal. Non Linéaire* **29** (2012), no. 6, 861–885.
- Varvaruca, E.; Weiss, G. S. Singularities of steady axisymmetric free surface flows with gravity. *Comm. Pure Appl. Math.* **67** (2014), no. 8, 1263–1306.

The governing equations for 2D free-surface water waves under gravity

Water: inviscid, incompressible fluid



$$
\begin{aligned}
u_t + uu_X + vu_Y &= -P_X && \text{in } 0 < Y < \eta(t,X), \\
v_t + uv_X + vv_Y &= -P_Y - g && \text{in } 0 < Y < \eta(t,X), \\
u_X + v_Y &= 0 && \text{in } 0 < Y < \eta(t,X), \\
v &= \eta_t + u\eta_X && \text{on } Y = \eta(t,X), \\
v &= 0 && \text{on } Y = 0, \\
P &= P_{\text{atm}} && \text{on } Y = \eta(t,X).
\end{aligned}
$$

Travelling waves

$\eta = \eta(X-ct), u = u(X-ct, Y), v = v(X-ct, Y), P = P(X-ct, Y)$, $(X-ct, Y) \mapsto (X, Y)$.

Incompressibility implies the existence of a **stream function** ψ:

$$\psi_X = -v, \quad \psi_Y = u - c.$$

Eliminate the pressure P:

$$(\Delta\psi)_X \psi_Y = (\Delta\psi)_Y \psi_X.$$

The above holds whenever the **vorticity** $\omega := v_X - u_Y = -\Delta\psi$ satisfies:

$$-\Delta\psi = \gamma(\psi),$$

where γ can be any function ("**vorticity function**").

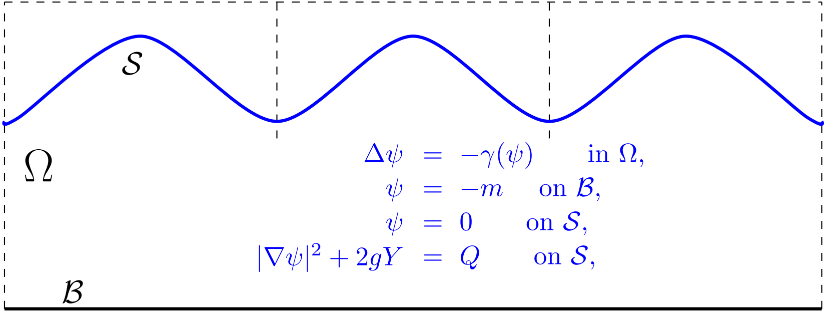
Recover the pressure P (**Bernoulli's Theorem**):

$$P + \frac{1}{2}|\nabla\psi|^2 + gY + \Gamma(\psi) = \text{constant},$$

where $\Gamma(r) = \int_0^r \gamma(t)dt$.

The stream function formulation of the 2D steady water wave problem

Find a L-**periodic** curve $\mathcal{S}$, a line $\mathcal{B}_D = \{(X, -D) : X \in \mathbb{R}\}$ in the (X, Y)-plane, and a L-**periodic** function ψ in the domain Ω between $\mathcal{B}_D$ and $\mathcal{S}$, such that

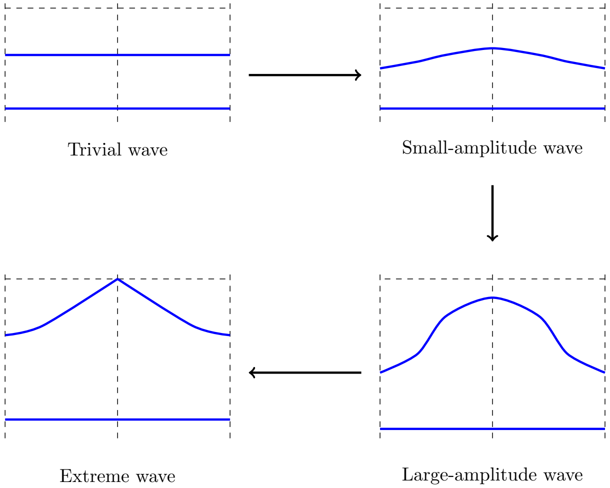


$$
\begin{aligned}
\Delta\psi &= -\gamma(\psi) && \text{in } \Omega, \\
\psi &= -m && \text{on } \mathcal{B}, \\
\psi &= 0 && \text{on } \mathcal{S}, \\
|\nabla\psi|^2 + 2gY &= Q && \text{on } \mathcal{S},
\end{aligned}
$$

where g is a given positive constant, L, M, Q, D are constants $\gamma : I \subset \mathbb{R} \to \mathbb{R}$ is a function.

The 'big picture' as envisioned by Stokes

Stokes(1847, 1880) envisioned that, in the **irrotational case** ($\gamma \equiv 0$), there exists a **family of regular (i.e., smooth) waves parametrized by amplitude**, **connecting** a wave with a flat free surface to a **wave of extreme form** (and also of **greatest height**), which has a **stagnation point** ($\nabla\psi = (0,0)$) at the crest and a **corner of** 120° there.



Three different issues:

- existence of smooth waves;
- existence of extreme waves (waves with stagnation points at the crests);
- regularity of extreme waves: **Stokes Conjecture**: any extreme wave necessarily has corners of 120° at stagnation points (some ambiguity about the initially-assumed regularity, the sense in which the equations are satisfied, ...)

The Stokes Conjecture: formal argument vs. rigorous result

Formal argument: Stokes (1880) **assumed** that the free surface **has a corner** (of unspecified size) at the stagnation point, while the stream function, when written in polar coordinates, **has an asymptotic expansion**, and he deduced that the corner must be symmetric and of size 120°, while the leading order term in the asymptotic expansion of the stream function must be $\frac{2\sqrt{g}}{3}\rho^{3/2}\cos(3(\theta+\pi/2)/2)$.

In fact, one can immediately check that the **only homogeneous solution** of the problem is the **Stokes corner flow**:

$$\psi_0 = \frac{2\sqrt{g}}{3}\rho^{3/2}\cos(3(\theta+\pi/2)/2)\chi_{\{(\rho\cos\theta,\rho\sin\theta):-5\pi/6<\theta<-\pi/6\}}.$$

Requirements of a rigorous result: consider suitable **weak solutions**, in which **no assumptions** are made either on the existence **of a corner** or on the **asymptotic behaviour** of the stream function.

A digression on Bernoulli free-boundary problems

in the sense of Shargorodsky & Toland (2004):

$\gamma \equiv 0$, but the Bernoulli condition

$$|\nabla\psi| = (-2gY)^{1/2} \quad \text{on } \mathcal{S},$$

is replaced by

$$|\nabla\psi| = f(Y) \quad \text{on } \mathcal{S},$$

where $f : (-\infty, 0] \to [0, \infty)$, $f(0) = 0$, $f(r) > 0 \, \forall \, r < 0$.

Varvaruca (2006):

- **proof** of a **generalization of the Stokes conjecture** (along the same lines, but simpler and more transparent than that for water waves of AF&T(1982)): **if** $f(r) \sim c(-r)^{\alpha}$ as $r \nearrow 0$, **then the free boundary must have a corner of size** $\pi/(\alpha + 1)$.
- an **explicit example** of a singular free boundary which **does not have a corner**.

Steady $3D$ axisymmetric free-surface water flows with gravity

(X, Y, Z) space: fluid domain Ω is rotationally symmetric about the vertical axis $\{(Z, 0, 0) : Z \in \mathbf{R}\}$, with free surface $\mathcal{S}$.
cylindrical coordinates $(X, Y, Z) = (R\cos\phi, R\sin\phi, Z)$;
axisymmetric velocity field:

$$\mathbf{V}(X, Y, Z) = (V_R(R, Z)\cos\phi, V_R(R, Z)\sin\phi, V_Z(R, Z))\,.$$

Incompressibility implies existence of a **Stokes stream function** $\Psi = \Psi(R, Z)$ such that

$$V_R = -\frac{1}{R}\frac{\partial\Psi}{\partial Z}, \quad V_Z = \frac{1}{R}\frac{\partial\Psi}{\partial R}.$$

Steady $3D$ axisymmetric free-surface water flows with gravity – continued

Eliminating the pressure, and assuming for simplicity that the flow is **irrotational**:

$$\frac{\partial}{\partial R}\left(\frac{1}{R}\frac{\partial\Psi}{\partial R}(R,Z)\right)+\frac{\partial}{\partial Z}\left(\frac{1}{R}\frac{\partial\Psi}{\partial Z}(R,Z)\right)=0.$$

On the free surface $\mathcal{S}$: $\Psi = 0$ and

$$\frac{1}{R^2}|\nabla\Psi(R,Z)|^2+gZ=Q.$$

Alt, Caffarelli, Friedman (1982): regularity **away from the axis of symmetry and the set of stagnation points** for minimizers of a certain energy

Garabedian (1985): an **explicit example** of a **homogeneous** solution of degree $5/2$, in which **the water domain is above the air domain** and occupies, in the half-plane $\{(R,Z) : R \geq 0\}$ a **cone** with vertex at the origin and opening angle approximately $114.799°$ – **Garabedian pointed bubble solution**

$2D$ steady irrotational free-surface gravity waves

Change notation: $(X, Y) \longrightarrow (x_1, x_2)$; **normalize** $Q = 0$ (by vertical translation) and $g = 1/2$ (by scaling).

Local version of the problem: $\mathcal{D} \subset \mathbf{R}^2 = \{(x_1, x_2) : x_1, x_2 \in \mathbf{R}\}$ bounded open connected set, **stream function** $\psi : \mathcal{D} \to \mathbf{R}$, $\psi \geq 0$ in $\mathcal{D}$, **water phase** $\equiv \{\psi > 0\}$, **air phase** $\equiv \{\psi = 0\}$.

$$\Delta\psi(x_1, x_2) = 0 \text{ in the water phase } \{\psi > 0\},$$
$$|\nabla\psi(x_1, x_2)|^2 = -x_2 \text{ on the free surface } \partial\{\psi > 0\}.$$

$3D$ axisymmetric steady irrotational free-surface flows under gravity

Change notation: $(R, Z) \longrightarrow (x_1, x_2)$; **normalize** $Q = 0$ (by vertical translation) and $g = 1/2$ (by scaling).

Local version of the problem: $\mathcal{D} \subset \mathbf{R}^2_+ = \{(x_1, x_2) : x_1 \in \mathbf{R}_+, x_2 \in \mathbf{R}\}$ bounded open connected set, **Stokes stream function** $\Psi : \mathcal{D} \to \mathbf{R}$, $\Psi \geq 0$ in $\mathcal{D}$, **water phase** $\equiv \{\Psi > 0\}$, **air phase** $\equiv \{\Psi = 0\}$.

$$\operatorname{div}\left(\frac{1}{x_1}\nabla\Psi(x_1, x_2)\right) = 0 \text{ in the water phase } \{\Psi > 0\},$$
$$\frac{1}{x_1^2}|\nabla\Psi(x_1, x_2)|^2 = -x_2 \text{ on the free surface } \partial\{\Psi > 0\}.$$

CONSIDER IN WHAT FOLLOWS ONLY THE AXISYMMETRIC $3D$ PROBLEM

BLOW-UP ANALYSIS: Around any point x^0 with $\Psi(x^0) = 0$ study, for suitable α,

$$\lim_{r \to 0+} \frac{\Psi(x^0 + rx)}{r^\alpha} \qquad \textbf{(blow-up sequence)}.$$

- a (unique) limit function exists;
- the limit function satisfies a natural PDE;
- the limit function is homogeneous of the expected degree;
- the limit function is not identically zero;
- infer information about the solution of the original problem.

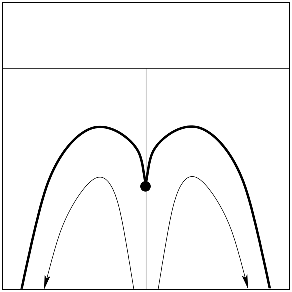
The most natural α?

(Almost) invariant scalings for the problem:

$$\frac{\Psi(x^0 + rx)}{r} \text{ in the case } x_1^0 \neq 0, x_2^0 \neq 0,$$
$$\frac{\Psi(x^0 + rx)}{r^{3/2}} \text{ in the case } x_1^0 \neq 0, x_2^0 = 0,$$
$$\frac{\Psi(x^0 + rx)}{r^2} \text{ in the case } x_1^0 = 0, x_2^0 \neq 0,$$
$$\frac{\Psi(x^0 + rx)}{r^{5/2}} \text{ in the case } x_1^0 = x_2^0 = 0.$$

Informal overview of results

Points on the horizontal axis different from the origin
$(x_1^0 \neq 0, x_2^0 = 0)$: **Stokes corner flow** with corner of $120°$.
Points on the vertical axis different from the origin $(x_1^0 = 0, x_2^0 \neq 0)$:
three types of **air cusps** pointing in the direction of the axis of symmetry.
No flat horizontal points are possible!



Informal overview of results - continued

Origin ($x_1^0 = x_2^0 = 0$): **either Garabedian pointed bubble** solution (**an angle of opening** $\approx 114.799°$ with **water above air**)

No scaling invariant solutions exist with air above water!

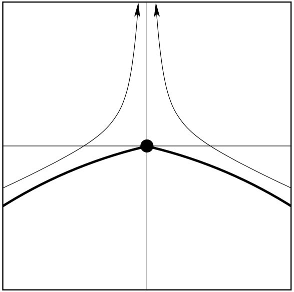
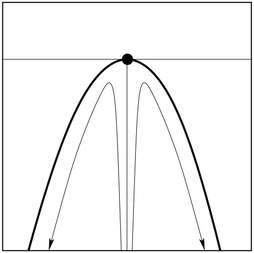


Figure: Garabedian pointed bubble asymptotics

Informal overview of results - continued

Origin ($x_1^0 = x_2^0 = 0$): **or degenerate** ($\Psi(rx)/r^{5/2} \to 0$) **flat horizontal point**



In the case of **air above water**, we **continue the analysis** and prove that

$$\frac{\Psi(rx)}{\sqrt{\int_{\partial B_r^+(0)} \frac{1}{x_1} \Psi^2 \, d\mathcal{H}^1}} \to \frac{x_1^2 x_2^-}{\sqrt{\int_{\partial B_1^+(0)} \frac{1}{x_1} x_1^4 x_2^2 \, d\mathcal{H}^1}} \quad \text{as } r \to 0+,$$

using a new **frequency formula**, and a **concentration compactness** result of Delort (1992).

METHODS ILLUSTRATED on the (very simple) problem of HARMONIC FUNCTIONS

$$\Delta u = 0 \quad \text{in } B_R(0) \subset \mathbb{R}^2.$$

Unique solution with $u(0) = 0$ is:

$$u(x_1, x_2) = u(\rho\cos\theta, \rho\sin\theta) = \sum_{k=N}^{\infty} \rho^k \left(a_k \cos k\theta + b_k \sin k\theta\right), \qquad |a_N| + |b_N| \neq 0.$$

Leading-order behaviour:

$$\frac{u(rx)}{r^N} \to \rho^N (a_N \cos N\theta + b_N \sin N\theta) \qquad \text{as } r \to 0.$$

ALMGREN's Frequency Formula:

$$r \mapsto \frac{r \int_{B_r(0)} |\nabla u|^2 \, dx}{\int_{\partial B_r(0)} u^2 d\mathcal{H}^1} \text{ is nondecreasing} \qquad \text{and} \qquad \lim_{r \to 0} \frac{r \int_{B_r(0)} |\nabla u|^2 \, dx}{\int_{\partial B_r(0)} u^2 d\mathcal{H}^1} = N.$$

Family of Monotonicity Formulas: for any $\alpha > 0$, if

$$M^\alpha(r) = r^{-2\alpha} \int_{B_r(0)} |\nabla u|^2 \, dx - \alpha r^{-1-2\alpha} \int_{\partial B_r(0)} u^2 \, d\mathcal{H}^1,$$

then

$$(M^\alpha(r))' = 2r^{-2\alpha} \int_{\partial B_r(0)} \left(\nabla u \cdot \nu - \alpha \frac{u}{r}\right)^2 \, d\mathcal{H}^1.$$

Switch notation from Ψ to u, and **"reflect"** the problem at the hyperplane $\{x_2 = 0\}$:
$\mathcal{D}$ bounded open connected set contained in $\{(x_1, x_2) : x_1 \geq 0\}$,

$$\begin{aligned} \operatorname{div}\left(\frac{1}{x_1}\nabla u\right) &= 0 \quad \text{in } \mathcal{D} \cap \{u > 0\}, \\ \frac{1}{x_1^2}|\nabla u|^2 &= x_2 \quad \text{on } \mathcal{D} \cap \partial\{u > 0\}. \end{aligned} \qquad (\mathcal{P})$$

Definition (Variational solution)

We define $u \in W^{1,2}_{\mathrm{w,loc}}(\mathcal{D})$ to be a **variational solution** of $(\mathcal{P})$ if $u \in C^0(\mathcal{D}) \cap C^2(\mathcal{D} \cap \{u > 0\})$, $u \geq 0$ in $\mathcal{D}$, $u = 0$ on $\{x_1 = 0\}$,

$$\lim_{x \to x^0,\, x \in \mathcal{D} \cap \{u>0\}} \frac{\partial_2 u(x)}{x_1} = 0 \quad \text{and} \quad \lim_{x \to x^0,\, x \in \mathcal{D} \cap \{u>0\}} \frac{\partial_1 u(x)}{x_1} \quad \text{exists}$$

for any $x^0 \in \mathcal{D} \cap \{x_1 = 0\}$, and the first variation with respect to **domain variations** of the functional

$$E(v) := \int_{\mathcal{D}} \left(\frac{1}{x_1^2}|\nabla v|^2 + x_2 \chi_{\{v>0\}}\right) x_1\, dx$$

vanishes at $v = u$, i.e.

$$\begin{aligned} 0 &= -\frac{d}{d\epsilon} E(u(x + \epsilon\phi(x)))|_{\epsilon=0} \\ &= \int_{\mathcal{D}} \Bigg[\left(\frac{1}{x_1}|\nabla u|^2 + x_1 x_2 \chi_{\{u>0\}}\right) \operatorname{div} \phi - 2\frac{1}{x_1}\nabla u D\phi \nabla u \\ &\quad + \left(-\frac{1}{x_1^2}|\nabla u|^2 + x_2\chi_{\{u>0\}}\right)\phi_1 + x_1\chi_{\{u>0\}}\phi_2 \Bigg] dx \end{aligned}$$

for any $\phi = (\phi_1, \phi_2) \in C^1_0(\mathcal{D}; \mathbf{R}^2)$ such that $\phi_1 = 0$ on $\{x_1 = 0\}$.

Theorem (Homogeneity of blow-ups)

Let u be a variational solution of $(\mathcal{P})$ satisfying a suitable growth condition, and let $x^0 \in \Omega$ be such that $u(x^0) = 0$. Then the following hold.

In the case $x_1^0 > 0$, $x_2^0 > 0$, let $0 < r_m \to 0+$ as $m \to \infty$ be a sequence such that

$$\frac{u(x^0 + r_m x)}{r_m} \to u_0 \quad \textit{weakly in } W^{1,2}_{\mathrm{loc}}(\mathbf{R}^2).$$

Then the **blow-up limit** u_0 *is a* **homogeneous** *function* **of degree** 1 *on* $\mathbf{R}^2$.

In the case $x_2^0 = 0$, let $0 < r_m \to 0+$ as $m \to \infty$ be a sequence such that

$$\frac{u(x^0 + r_m x)}{r_m{}^{3/2}} \to u_0 \quad \textit{weakly in } W^{1,2}_{\mathrm{loc}}(\mathbf{R}^2).$$

Then the blow-up limit u_0 is a **homogeneous** *function* **of degree** $3/2$ *on* $\mathbf{R}^2$.

In the case $x_1^0 = 0$, let $0 < r_m \to 0+$ as $m \to \infty$ be a sequence such that

$$\frac{u(x^0 + r_m x)}{r_m{}^2} \to u_0 \quad \textit{weakly in } W^{1,2}_{w,\mathrm{loc}}(\mathbf{R}^2_+).$$

Then the blow-up limit u_0 is a **homogeneous** *function* **of degree** 2 *on* $\mathbf{R}^2_+$.

In the case $x_1^0 = x_2^0 = 0$, let $0 < r_m \to 0+$ as $m \to \infty$ be a sequence such that

$$\frac{u(x^0 + r_m x)}{r_m{}^{5/2}} \to u_0 \quad \textit{weakly in } W^{1,2}_{w,\mathrm{loc}}(\mathbf{R}^2_+).$$

Then the blow-up limit u_0 is a **homogeneous** *function* **of degree** $5/2$ *on* $\mathbf{R}^2_+$.

Theorem (Monotonicity Formula)

Let u be a variational solution of $(\mathcal{P})$, let $x^0 \in \Omega$, and let $\delta := \operatorname{dist}(x^0, \partial\Omega)/2$. Let, for any $r \in (0, \delta)$,

$$I(r) = \int_{B_r^+(x^0)} \left(\frac{1}{x_1} |\nabla u|^2 + x_1 x_2 \chi_{\{u>0\}} \right) dx, \qquad J(r) = \int_{\partial B_r^+(x^0)} \frac{1}{x_1} u^2 \, d\mathcal{H}^1.$$

In the case $x_1^0 = x_2^0 = 0$, let, for any $r \in (0, \delta)$,

$$M^{x_1 x_2}(r) = r^{-4} I(r) - \frac{5}{2} r^{-5} J(r).$$

Then, for a.e. $r \in (0, \delta)$,

$$(M^{x_1 x_2}(r))' = 2r^{-4} \int_{\partial B_r^+(0)} \frac{1}{x_1} \left(\nabla u \cdot \nu - \frac{5}{2} \frac{u}{r} \right)^2 d\mathcal{H}^1.$$

We denote $M^{x_1 x_2}(0+) = \lim_{r \to 0+} M^{x_1 x_2}(r)$.

Similar identities (involving additional perturbative terms, due to the almost invariant, rather than fully invariant, scaling of the problem) hold,

- in the case $x_1^0 > 0$, $x_2^0 > 0$, for $M^{int}(r) = r^{-2} I(r) - r^{-3} J(r)$,
- in the case $x_1^0 > 0$, $x_2^0 = 0$, for $M^{x_2}(r) = r^{-3} I(r) - \frac{3}{2} r^{-4} J(r)$,
- in the case $x_1^0 = 0$, $x_2^0 > 0$, for $M^{x_1}(r) = r^{-3} I(r) - 2r^{-4} J(r)$.

Proof of Monotonicity Formula is based on:

- a **Pohozaev-type identity**, obtained by taking a suitable sequence of test functions in the definition of a variational solution:

$$
\begin{aligned}
0 = & 2\int_{B_r^+(x^0)} x_1 x_2 \chi_{\{u>0\}}\,dx - r\int_{\partial B_r^+(x^0)} \left(\frac{1}{x_1}|\nabla u|^2 + x_1 x_2 \chi_{\{u>0\}}\right) d\mathcal{H}^1 \\
& + 2r\int_{\partial B_r^+(x^0)} \frac{1}{x_1}(\nabla u\cdot\nu)^2\, d\mathcal{H}^1 \\
& + \int_{B_r^+(x^0)} \left(-\frac{x_1 - x_1^0}{x_1^2}|\nabla u|^2 + \left[(x_1 - x_1^0)x_2 + (x_2 - x_2^0)x_1\right]\chi_{\{u>0\}}\right) dx.
\end{aligned}
$$

- an integration by parts formula:

$$
\int_{B_r^+(x^0)} \frac{1}{x_1}|\nabla u|^2\,dx = \int_{\partial B_r^+(x^0)} \frac{1}{x_1} u\nabla u\cdot\nu\, d\mathcal{H}^1.
$$

Monotonicity Formula implies **homogeneity of blow-ups**:

$$
\begin{aligned}
2\int_{B_\sigma^+(0)\setminus B_\tau^+(0)} |x|^{-6}\frac{1}{x_1}\left(\nabla u_m(x)\cdot x - \frac{5}{2}u_m(x)\right)^2 dx \\
= M^{x_1x_2}(r_m\sigma) - M^{x_1x_2}(r_m\tau) \to 0 \quad \text{as } m\to\infty.
\end{aligned}
$$

Proposition

Under the assumptions of the previous result, the following also hold.

- *Let* u_m *a* **blow-up sequence** *which converges* **weakly** *in* $W^{1,2}_{\mathrm{w,loc}}(\mathbf{R}^2_+)$ *(in* $W^{1,2}_{\mathrm{loc}}(\mathbf{R}^2)$ *in the cases where* $x_1^0 > 0$*) to a* **blow-up limit** u_0*. Then* u_m *converges to* u_0 **strongly** *in* $W^{1,2}_{\mathrm{w,loc}}(\mathbf{R}^2_+)$ *(***strongly** *in* $W^{1,2}_{\mathrm{loc}}(\mathbf{R}^2)$ *in the cases where* $x_1^0 > 0$*).*
- *If* $x_1^0 > 0$ *and* $x_2^0 \neq 0$*, then*

$$M^{int}(0+) = x_1^0 x_2^0 \lim_{r\to 0+} r^{-2} \int_{B_r^+(x^0)} \chi_{\{u>0\}}\, dx.$$

If $x_1^0 > 0$ *and* $x_2^0 = 0$*, then*

$$M^{x_2}(0+) = x_1^0 \lim_{r\to 0+} r^{-3} \int_{B_r^+(x^0)} x_2 \chi_{\{u>0\}}\, dx.$$

If $x_1^0 = 0$ *and* $x_2^0 \neq 0$*, then*

$$M^{x_1}(0+) = x_2^0 \lim_{r\to 0+} r^{-3} \int_{B_r^+(x^0)} x_1 \chi_{\{u>0\}}\, dx.$$

If $x_1^0 = x_2^0 = 0$*, then*

$$M^{x_1 x_2}(0+) = \lim_{r\to 0+} r^{-4} \int_{B_r^+(x^0)} x_1 x_2 \chi_{\{u>0\}}\, dx.$$

We shall use the name **density** for each of the quantities

$$M^{int}(0+), \quad M^{x_2}(0+), \quad M^{x_1}(0+), \quad M^{x_1 x_2}(0+).$$

Theorem (Characterization of blow-up limits)

Let u be a variational solution of $(\mathcal{P})$, let $x^0 \in \mathcal{D}$ with $x_2^0 \geq 0$ be such that $u(x^0) = 0$, and suppose that u satisfies some suitable growth conditions. Then the following hold:

(i) *In the case $x_1^0 > 0$, $x_2^0 = 0$, the only possible* **blow-up limits** *are*

$$u_0(\rho \sin\theta, \rho\cos\theta) = \frac{\sqrt{2}x_1^0}{3}\rho^{3/2}\cos\left(\frac{3}{2}\theta\right)\chi_{\{(\rho\sin\theta,\rho\cos\theta):-\pi/3<\theta<\pi/3\}},$$

with corresponding **density**

$$M^{x_2}(0+) = x_1^0 \int_{B_1 \cap \{(\rho\sin\theta,\rho\cos\theta):-\pi/3<\theta<\pi/3\}} x_2 \, dx,$$

and $u_0(x) = 0$, with possible values of the **density**

$$M^{x_2}(0+) \in \left\{ x_1^0 \int_{B_1} x_2^+ \, dx, x_1^0 \int_{B_1} x_2^- \, dx, 0 \right\}.$$

Theorem (Characterization of blow-up limits – continued)

If $M^{x_2}(0+) = x_1^0 \int_{B_1 \cap \{(\rho \sin\theta, \rho\cos\theta): -\pi/3 < \theta < \pi/3\}} x_2 \, dx$, *then*

$$\frac{u(x^0 + rx)}{r^{3/2}} \to \frac{\sqrt{2} x_1^0}{3} \rho^{3/2} \cos\left(\frac{3}{2}\theta\right) \chi_{\{(\rho\sin\theta, \rho\cos\theta): -\pi/3 < \theta < \pi/3\}}$$

as $r \to 0+$, *strongly in* $W^{1,2}_{\mathrm{loc}}(\mathbf{R}^2)$ *and locally uniformly,*
while if

$$M^{x_2}(0+) \in \left\{ x_1^0 \int_{B_1} x_2^+ \, dx, x_1^0 \int_{B_1} x_2^- \, dx, 0 \right\},$$

then

$$\frac{u(x^0 + rx)}{r^{3/2}} \to 0 \quad \text{as } r \to 0+,$$

strongly in $W^{1,2}_{\mathrm{loc}}(\mathbf{R}^2)$ *and locally uniformly.*

Theorem (Characterization of blow-up limits – continued)

(ii) *In the case* $x_1^0 = 0$, $x_2^0 > 0$, *the only possible* **blow-up limits** *are*

$$u_0(x) = \alpha x_1^2$$

with α *a positive constant and corresponding* **density**

$$M^{x_1}(0+) = x_2^0 \int_{B_1^+} x_1 \, dx,$$

and $u_0(x) = 0$, *with possible values of the* **density**

$$M^{x_1}(0+) \in \left\{ x_2^0 \int_{B_1^+} x_1 \, dx, 0 \right\}.$$

Theorem (Characterization of blow-up limits – continued)

(iii) *In the case* $x_1^0 = x_2^0 = 0$, *the only possible* **blow-up limits** *are*

$$u_0(\rho \sin\theta, \rho\cos\theta) = \beta_0 \rho^{5/2} \sin^2\theta P'_{3/2}(-\cos\theta) \chi_{\{(\rho\sin\theta,\rho\cos\theta):\pi-\theta^*<\theta<\pi\}},$$

with corresponding **density**

$$M^{x_1 x_2}(0+) = \int_{B_1^+ \cap \{(\rho\sin\theta,\rho\cos\theta):\pi-\theta^*<\theta<\pi\}} x_1 x_2 \, dx,$$

where $P_{3/2}$ *is the* **Legendre function of the first kind**, $\theta^* := \arccos s^*$, *where* $s^* \in (-1, 0)$ *is the unique solution* $s \in (-1, 1)$ *of* $P'_{3/2}(s) = 0$ *(so that* θ^* **corresponds to an angle of** $\approx 114.799°$*), while* β_0 *is a unique positive constant,*

and $u_0(x) = 0$, *with possible values of the* **density**

$$M^{x_1 x_2}(0+) \in \left\{ \int_{B_1^+} x_1 x_2^+ \, dx, \int_{B_1^+} x_1 x_2^- \, dx, 0 \right\}.$$

Theorem (Characterization of blow-up limits – continued)

If $M^{x_1x_2}(0+) = \int_{B_1^+ \cap \{(\rho\sin\theta,\rho\cos\theta):\pi-\theta^*<\theta<\pi\}} x_1 x_2\,dx$, *then*

$$\frac{u(rx)}{r^{5/2}} \to \beta_0 \rho^{5/2} \sin^2\theta P'_{3/2}(-\cos\theta)\chi_{\{(\rho\sin\theta,\rho\cos\theta):\pi-\theta^*<\theta<\pi\}}$$

as $r \to 0+$, *strongly in* $W^{1,2}_{\mathrm{w,\ loc}}(\mathbf{R}^2_+)$ *and locally uniformly on* $\mathbf{R}^2_+$, *while if*

$$M^{x_1x_2}(0+) \in \left\{ \int_{B_1^+} x_1 x_2^+\,dx, \int_{B_1^+} x_1 x_2^-\,dx, 0 \right\},$$

then

$$\frac{u(rx)}{r^{5/2}} \to 0 \quad \textit{as } r \to 0+,$$

strongly in $W^{1,2}_{\mathrm{w,\ loc}}(\mathbf{R}^2_+)$ *and locally uniformly on* $\mathbf{R}^2_+$.

Theorem (Curve case)

Let u be a weak solution of $(\mathcal{P})$, let $x^0 \in \Omega$ be such that $x_1^0 x_2^0 = 0$, $x_2^0 \geq 0$ and $u(x^0) = 0$, and suppose that u satisfies a suitable growth condition. Suppose in addition that $\partial\{u > 0\} \cap B_1^+(x^0)$ is in a neighborhood of x^0 a **continuous injective curve** *$\sigma : I \to \mathbf{R}^2$, where I is an interval of $\mathbf{R}$ containing 0, such that $\sigma = (\sigma_1, \sigma_2)$ and $\sigma(0) = x^0$.*

In the case $x_1^0 > 0$, $x_2^0 = 0$, exactly one of the following holds:

(i_1) **Stokes corner:** $M^{x_2}(0+) = x_1^0 \int_{B_1} x_2 \chi_{\{(\rho \sin\theta, \rho\cos\theta): -\pi/3 < \theta < \pi/3\}}\, dx$, *in which case $\sigma_1(t) \neq x_1^0$ in $(-t_1, t_1) \setminus \{0\}$ and, depending on the parametrization, either*

$$\lim_{t \to 0+} \frac{\sigma_2(t)}{\sigma_1(t) - x_1^0} = \frac{1}{\sqrt{3}} \quad \textit{and} \quad \lim_{t \to 0-} \frac{\sigma_2(t)}{\sigma_1(t) - x_1^0} = -\frac{1}{\sqrt{3}},$$

or

$$\lim_{t \to 0+} \frac{\sigma_2(t)}{\sigma_1(t) - x_1^0} = -\frac{1}{\sqrt{3}} \quad \textit{and} \quad \lim_{t \to 0-} \frac{\sigma_2(t)}{\sigma_1(t) - x_1^0} = \frac{1}{\sqrt{3}}.$$

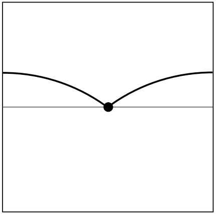


Figure: Stokes corner $(x_1^0 > 0, x_2^0 = 0)$

Theorem (Curve case – continued)

(i_2) $M^{x_2}(0+) = x_1^0 \int_{B_1} x_2^+ \, dx$ *or* $M^{x_2}(0+) = x_1^0 \int_{B_1} x_2^- \, dx$, *in which case* $\sigma_1(t) \neq x_1^0$ *in* $(-t_1, t_1) \setminus \{0\}$, $\sigma_1 - x_1^0$ *changes sign at* $t = 0$, *and*

$$\lim_{t \to 0} \frac{\sigma_2(t)}{\sigma_1(t) - x_1^0} = 0.$$

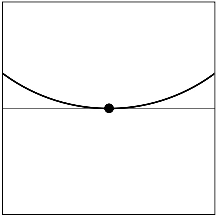


Figure: Horizontal point $(x_1^0 > 0, x_2^0 = 0)$

(i_3) $M^{x_2}(0+) = 0$, *– not possible if* $u = 0$ *in* $\{x_2 \leq 0\}$ *and a strong Bernstein inequality holds –, in which case* $\sigma_1(t) \neq x_1^0$ *in* $(-t_1, t_1) \setminus \{0\}$, $\sigma_1 - x_1^0$ *does not change its sign at* $t = 0$, *and*

$$\lim_{t \to 0} \frac{\sigma_2(t)}{\sigma_1(t) - x_1^0} = 0.$$

Theorem (Curve case – continued)

In the case $x_2^0 > 0$, $x_1^0 = 0$, *it is not possible that* $M^{x_1}(0+) = 0$, *and the following holds:*

(ii) $M^{x_1}(0+) = x_2^0 \int_{B_1^+} x_1\, dx$, *in which case either* $\sigma_2(t) \neq x_2^0$ *in* $(0, t_1)$ *and*

$$\lim_{t \to 0} \frac{\sigma_1(t)}{\sigma_2(t) - x_2^0} = 0,$$

(i.e., either a **downward** *or* **upward vertical cusp***)*

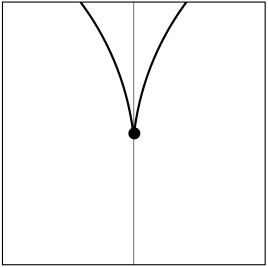


Figure: Downward Vertical Cusp ($x_1^0 = 0, x_2^0 > 0$)

Theorem (Curve case – continued)

or $\sigma_2(t) \neq x_2^0$ *in* $(-t_1, t_1) \setminus \{0\}$, $\sigma_2 - x_2^0$ *changes sign at* $t = 0$, *and*

$$\lim_{t \to 0} \frac{\sigma_1(t)}{\sigma_2(t) - x_2^0} = 0.$$

(i.e., a **double vertical cusp***)*

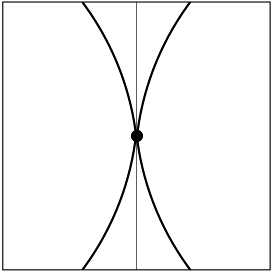


Figure: Double Vertical Cusp ($x_1^0 = 0, x_2^0 > 0$)

Theorem (Curve case – continued)

In the case $x_1^0 = x_2^0 = 0$, exactly one of the following holds:

(iii_1) **Garabedian corner:** $M^{x_1x_2}(0+) = \int_{B_1^+ \cap \{(\rho\sin\theta,\rho\cos\theta):\pi-\theta^*<\theta<\pi\}} x_1x_2\,dx$, *in which case* $\sigma_1(t) \neq 0$ *in* $(0,t_1)$ *and*

$$\lim_{t\to 0+} \frac{\sigma_2(t)}{\sigma_1(t)} = \cot(\pi - \theta^*),$$

where $\theta^* := \arccos s^*$ *and* $s^* \in (-1,0)$ *is such that* $P'_{3/2}(s^*) = 0$.

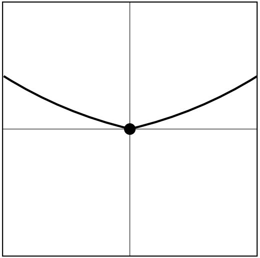


Figure: Garabedian corner ($x_1^0 = x_2^0 = 0$)

Theorem (Curve case – continued)

(iii_2) $M^{x_1x_2}(0+) = \int_{B_1^+} x_1 x_2^+ \, dx$ *or* $M^{x_1x_2}(0+) = \int_{B_1^+} x_1 x_2^- \, dx$, *in which case* $\sigma_1(t) \neq 0$ *in* $(0, t_1)$ *and*

$$\lim_{t\to 0} \frac{\sigma_2(t)}{\sigma_1(t)} = 0.$$

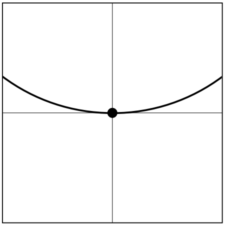


Figure: Horizontal point ($x_1^0 = x_2^0 = 0$)

(iii_3) $M^{x_1x_2}(0+) = 0$, – *not possible if* $u = 0$ *in* $\{x_2 \leq 0\}$ *and a strong Bernstein inequality holds* –, *in which case* $\sigma_1(t) \neq 0$ *in* $(-t_1, t_1) \setminus \{0\}$ *and*

$$\lim_{t\to 0} \frac{\sigma_2(t)}{\sigma_1(t)} = 0.$$

The case $x_1^0 = x_2^0 = 0$, $u = 0$ in $\{x_2 \le 0\}$ and $M^{x_1 x_2}(0+) = \int_{B_1^+} x_1 x_2^+ \, dx$ **(flat horizontal points, air above water)**.

Theorem (Degenerate points)

Let u be a weak solution of $(\mathcal{P})$ such that $u = 0$ in $\{x_2 \le 0\}$, let $x^0 = 0$, suppose that u satisfies a certain growth condition, that $M^{x_1 x_2}(0+) = \int_{B_1^+} x_1 x_2^+ \, dx$, and in addition that the free boundary $\partial\{u > 0\} \cap B_1^+$ is a **continuous injective curve** *$\sigma = (\sigma_1, \sigma_2)$ such that $\sigma(0) = 0$. Then $\sigma_1(t) \neq 0$ in $[0, t_1) \setminus \{0\}$,*

$$\lim_{t \to 0} \frac{\sigma_2(t)}{\sigma_1(t)} = 0$$

and

$$\frac{u(rx)}{\sqrt{\int_{\partial B_r^+} \frac{1}{x_1} u^2 \, d\mathcal{H}^1}} \to \frac{x_1^2 x_2^+}{\sqrt{\int_{\partial B_1^+} x_1^3 (x_2^+)^2 \, d\mathcal{H}^1}} \quad \text{as } r \to 0+,$$

strongly *in $W^{1,2}_{\mathrm{w,loc}}(B_1^+ \setminus \{0\})$ and weakly in $W^{1,2}_{\mathrm{w}}(B_1^+)$. Moreover,*

$$\frac{u(rx)}{r^\alpha} \to 0 \quad \text{in } L^2_{\mathrm{w}}(\partial B_1^+) \quad \text{for any } \alpha \in (0,3)$$

and

$$\frac{u(rx)}{r^\alpha} \quad \text{is unbounded in } L^2_{\mathrm{w}}(\partial B_1^+) \quad \text{for any } \alpha > 3.$$

Theorem (Frequency formula)

Let u be a variational solution of $(\mathcal{P})$ such that $u = 0$ in $\{x_2 \le 0\}$, let $x^0 = 0$, and let $\delta := \mathrm{dist}(0, \partial\mathcal{D})/2$. Let, for any $r \in (0, \delta)$,

$$D(r) = \frac{r \int_{B_r^+(0)} \frac{1}{x_1} |\nabla u|^2 \, dx}{\int_{\partial B_r^+(0)} \frac{1}{x_1} u^2 \, d\mathcal{H}^1} \quad \text{and} \quad V(r) = \frac{r \int_{B_r^+(0)} x_1 x_2^+ (1 - \chi_{\{u>0\}}) \, dx}{\int_{\partial B_r^+(0)} \frac{1}{x_1} u^2 \, d\mathcal{H}^1}.$$

Then the **"frequency" function**

$$H(r) = D(r) - V(r) = \frac{r \int_{B_r^+(0)} \left(\frac{1}{x_1} |\nabla u|^2 + x_1 x_2^+ (\chi_{\{u>0\}} - 1) \right) dx}{\int_{\partial B_r^+(0)} \frac{1}{x_1} u^2 \, d\mathcal{H}^1}$$

satisfies for a.e. $r \in (0, \delta)$ the identities

$$H'(r) = \frac{2}{r} \int_{\partial B_r^+(0)} \frac{1}{x_1} \left[\frac{r (\nabla u \cdot \nu)}{\left(\int_{\partial B_r^+(0)} \frac{1}{x_1} u^2 \, d\mathcal{H}^1 \right)^{1/2}} - D(r) \frac{u}{\left(\int_{\partial B_r^+(0)} \frac{1}{x_1} u^2 \, d\mathcal{H}^1 \right)^{1/2}} \right]^2 d\mathcal{H}^1 + \frac{2}{r} V^2(r) + \frac{2}{r} V(r) \left(H(r) - \frac{5}{2} \right)$$

and

$$H'(r) = \frac{2}{r} \int_{\partial B_r^+(0)} \frac{1}{x_1} \left[\frac{r (\nabla u \cdot \nu)}{\left(\int_{\partial B_r^+(0)} \frac{1}{x_1} u^2 \, d\mathcal{H}^1 \right)^{1/2}} - H(r) \frac{u}{\left(\int_{\partial B_r^+(0)} \frac{1}{x_1} u^2 \, d\mathcal{H}^1 \right)^{1/2}} \right]^2 d\mathcal{H}^1 + \frac{2}{r} V(r) \left(H(r) - \frac{5}{2} \right).$$

IDEAS of PROOF of the theorem on asymptotics at **degenerate points**:

- $H(r) \geq \frac{5}{2}$ for all $r \in (0, \delta)$.
- The function H is **nondecreasing** on $(0, \delta)$, and has a **right limit** $H(0+)$, where $H(0+) \geq 5/2$.
- There exist $\lim_{r \to 0+} V(r) = 0$ and $\lim_{r \to 0+} D(r) = H(0+)$.
- For any sequence $r_m \to 0+$ as $m \to \infty$, the sequence

$$v_m(x) := \frac{u(r_m x)}{\sqrt{\int_{\partial B_{r_m}^+} \frac{1}{x_1} u^2 \, d\mathcal{H}^1}}$$

  is **bounded** in $W^{1,2}_{\mathrm{w}}(B_1^+)$.
- For any sequence $r_m \to 0+$ as $m \to \infty$ such that the above sequence v_m converges weakly in $W^{1,2}_{\mathrm{w}}(B_1^+)$ to a **blow-up limit** v_0, the function v_0 is **homogeneous of degree** $H(0+)$ in B_1^+, and satisfies

$$v_0 \geq 0 \text{ in } B_1, \; v_0 \equiv 0 \text{ in } B_1^+ \cap \{x_2 \leq 0\} \text{ and } \int_{\partial B_1^+} \frac{1}{x_1} v_0^2 \, d\mathcal{H}^1 = 1.$$

IDEAS of PROOF – continued

- Moreover, v_m converges to v_0 **strongly** in $W^{1,2}_{w,\mathrm{loc}}(B_1^+ \setminus \{0\})$, v_0 is **continuous** on B_1^+ and $\mathrm{div}\,(\frac{1}{x_1}\nabla v_0)$ is a **nonnegative Radon measure** satisfying

$$v_0 \;\mathrm{div}\,\left(\frac{1}{x_1}\nabla v_0\right) = 0 \quad \text{in } B_1^+ \text{ (in the sense of Radon measures).}$$

- The proof of the strong convergence makes use of **concentration compactness** results of Delort (1992).
- Analysis of all possible **homogeneous** solutions of the above equation leads to the conclusion that necessarily $H(0+) = 3$ and that there is a **unique blow-up limit**

$$v_0(x) = \frac{x_1^2 x_2^+}{\sqrt{\int_{\partial B_1^+} x_1^3 (x_2^+)^2 \, d\mathcal{H}^1}}.$$

Thank you!