Sharp uncertainty principles on Riemannian manifolds: the effect of curvature

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- Quantum mechanics (E. H. Kennard and H. Weyl): more precisely the position σ_x of some particle is determined, the less precisely its momentum σ_p can be known, and vice versa:

$$\sigma_x \sigma_p \geq \frac{\hbar}{2}.$$

- Harmonic analysis (Fourier transform):

$$\left(\int_{\mathbb{R}} x^2 f(x)^2 dx \right) \left(\int_{\mathbb{R}} \xi^2 \hat{f}(\xi)^2 d\xi \right) \geq \frac{\|f\|_2^4}{16\pi^2}, \; f \in L^2(\mathbb{R}).$$

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  W. Heisenberg, E. H. Kennard,
  W. Pauli,
  H. Weyl,
  Ch. Fefferman.

Euclidean case: Heisenberg-Pauli-Weyl UP (shortly, **(HPW)**)

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The constant $\frac{n^2}{4}$ is **sharp** and the **extremals** are given (up to a constant/translation) by the family of Gaussians $u_\lambda(x) = e^{-\lambda|x|^2}$, $\lambda > 0$.

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What happens on curved settings?
Carron (JMPA, 1997),
Andersen (PAMS, 2002), Erb (PhD Thesis, 2009);
Kombe and Özaydin (TAMS, 2009&2013):
Let (M, g) be a complete, non-compact $n-$dimensional Riemannian manifold, and $\rho : M \to \mathbb{R}$ s.t. $|\nabla \rho| = 1$ and $\rho \Delta \rho \geq C$. Then,

$$\left(\int_M |\nabla u(x)|^2 dv \right) \left(\int_M \rho^2 u^2 dv \right) \geq \frac{(C+1)^2}{4} \|u\|_2^4, \ \forall u \in C_0^\infty(M).$$

- **Euclidean space:** $\Delta d = \frac{n-1}{d}$, $C = n - 1$. Sharp UP holds!
- **Hyperbolic space:** UP holds! Kombe and Özaydin claimed that $\frac{n^2}{4}$ is also sharp and $u(x) = e^{-\lambda d^2}$ is an extremal, where d is the hyperbolic distance. FALSE!

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Heisenberg-Pauli-Weyl inequality on Riemannian manifolds

Let (M,g) be an $n(\geq 3)-$dimensional, non-compact, complete Riemannian manifold, and for some $x_0 \in M$, consider

$$\left(\int_M |\nabla_g u|^2 \mathrm{d}V_g\right)\left(\int_M d_{x_0}^2 u^2 \mathrm{d}V_g\right) \geq \frac{n^2}{4}\left(\int_M u^2 \mathrm{d}V_g\right)^2, \qquad \textbf{(HPW)}_{x_0}$$

where

- $d_{x_0} = d_g(\cdot, x_0)$;
- $\nabla_g u(x) \in T_x M$ is the gradient of $u : M \to \mathbb{R}$ at $x \in M$;
- $\mathrm{d}V_g$ is the canonical volume element on (M,g).

Heisenberg-Pauli-Weyl inequality on Riemannian manifolds

Main result I: Non-positively curved case

Theorem (K., JMPA '18)

Let (M, g) be an $n-$dimensional **Cartan-Hadamard manifold** (simply connected, complete Riemannian manifold with $K_{\rm sect} \leq 0$).

(i) [Sharpness] The Heisenberg-Pauli-Weyl principle $(\mathbf{HPW})_{x_0}$ holds for every $x_0 \in M$; moreover, $\frac{n^2}{4}$ is sharp, i.e.,

$$\frac{n^2}{4} = \inf_{u \in C_0^\infty(M)\setminus\{0\}} \frac{\left(\int_M |\nabla_g u|^2 {\rm d}V_g\right)\left(\int_M d_{x_0}^2 u^2 {\rm d}V_g\right)}{\left(\int_M u^2 {\rm d}V_g\right)^2}.$$

(ii) [Extremals] The following statements are equivalent:

(a) $\frac{n^2}{4}$ is achieved by a positive extremal in $(\mathbf{HPW})_{x_0}$ for some $x_0 \in M$;

(b) (M, g) is isometric to $\mathbb{R}^n$.

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Remark:

- The validity of the Cartan-Hadamard conjecture is **not** needed to prove sharpness in HPW (as in generic L^p-Sobolev inequalities on manifolds)!

Cartan–Hadamard conjecture: *for every bounded open set $D \subset M$ with smooth boundary, we have*

$$\mathrm{Area}_g(\partial D) \geq n\omega_n^{\frac{1}{n}} \mathrm{Vol}_g(D)^{\frac{n-1}{n}},$$

and equality holds if and only if D is isometric to the $n-$dimensional euclidean ball with volume $\mathrm{Vol}_g(D)$.

- $n = 2$, Beckenbach and Radó [TAMS '33], and Weil [CRAS '26];
- $n = 3$, Kleiner [Invent. Math. '92];
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Heisenberg-Pauli-Weyl inequality on Riemannian manifolds

Proof; Non-positively curved case

Claim: *If $K_{\rm sect} \leq 0$, then $(\mathbf{HPW})_{x_0}$ holds for every $x_0 \in M$.*

- *Laplace comparison*: if $K_{\rm sect} \leq c \leq 0$, and $x_0 \in M$, then

$$\Delta_g d(x_0, x) \geq (n-1)\mathbf{ct}_c(d(x_0, x)) \text{ for a.e. } x \in M.$$

- Here, $\mathbf{ct}_c : (0, \infty) \to \mathbb{R}$ is defined by

$$\mathbf{ct}_c(\rho) = \begin{cases} \frac{1}{\rho} & \text{if} \quad c = 0, \\ \sqrt{|c|}\coth(\sqrt{|c|}\rho) & \text{if} \quad c < 0. \end{cases}$$

- Two-sided estimate of $\displaystyle\int_M \Delta_g d^2(x_0, x) u^2 \mathrm{d}V_g$ gives
  Quantitative HPW:

$$\int_M |\nabla_g u|^2 \mathrm{d}V_g \int_M d_{x_0}^2 u^2 \mathrm{d}V_g \geq \frac{n^2}{4}\left(\int_M \left(1 + \frac{n-1}{n}\mathbf{D}_c(d_{x_0})\right) u^2 \mathrm{d}V_g\right)^2,$$

where $\mathbf{D}_c(0) = 0$ and $\mathbf{D}_c(\rho) = \rho\mathbf{ct}_c(\rho) - 1 \geq 0$ for every $\rho > 0$.

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Claim: $\mathsf{C}_{\mathsf{HPW}} = \frac{n^2}{4}$, where

$$\mathsf{C}_{\mathsf{HPW}} = \inf_{u \in C_0^\infty(M)\setminus\{0\}} \frac{\left(\int_M |\nabla_g u|^2 \mathrm{d}V_g\right)\left(\int_M d_{x_0}^2 u^2 \mathrm{d}V_g\right)}{\left(\int_M u^2 \mathrm{d}V_g\right)^2}.$$

- $(\mathbf{HPW})_{x_0}$ implies $\mathsf{C}_{\mathsf{HPW}} \geq \frac{n^2}{4}$.
- Assume $\mathsf{C}_{\mathsf{HPW}} > \frac{n^2}{4}$. For small $\varepsilon > 0$ one has $(1-\varepsilon)\delta_{ij} \leq g_{ij} \leq (1+\varepsilon)\delta_{ij}$ (local charts around $x_0 \in M$); for some $\mathsf{C}'_{\mathsf{HPW}} > \frac{n^2}{4}$, we have

$$\left(\int_{\mathbb{R}^n} |\nabla u|^2 \mathrm{d}x\right)\left(\int_{\mathbb{R}^n} |x|^2 u^2 \mathrm{d}x\right) \geq \mathsf{C}'_{\mathsf{HPW}} \left(\int_{\mathbb{R}^n} u^2 \mathrm{d}x\right)^2.$$

Put $u(x) = e^{-|x|^2}$ as a test function; a contradiction.

Claim: *If $(M,g)=(\mathbb{R}^n,e)$, then $\frac{n^2}{4}$ is sharp and $u_\lambda(x)=e^{-\lambda|x-x_0|^2}$ $(\lambda>0)$ is a class of extremal functions in $(\mathbf{HPW})_{x_0}$ for every $x_0\in M$.*

- Direct computation;
- Polar coordinates, i.e., $(\rho,\theta)=\left(|x-x_0|,\frac{x-x_0}{|x-x_0|}\right)$, by using the identity

$$\int_0^\infty \rho^{n+1}e^{-\rho^2}d\rho=\frac{n}{2}\int_0^\infty \rho^{n-1}e^{-\rho^2}d\rho.$$

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(a) $\frac{n^2}{4}$ *is achieved by a positive extremal in* $(\mathbf{HPW})_{x_0}$ *for some* $x_0 \in M$;

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- (c)⇒(b)⇒(a) are trivial;
- If $\frac{n^2}{4}$ is sharp and $u_0 > 0$ is an extremal in $(\mathbf{HPW})_{x_0}$ for some $x_0 \in M$ (by the **Quantitative HPW**) ⇒

$$\Delta_g(d_g(x_0, x)^2) = 2n, \ \ x \in M \setminus \{x_0\}.$$

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- (proof continues) Stokes' theorem on Riemannian manifolds $\Rightarrow$

$$\frac{\frac{d}{d\rho}\mathrm{Vol}_F(B(x_0,\rho))}{\mathrm{Vol}_g(B(x_0,\rho))} = \frac{n}{\rho}.$$

- $\mathrm{Vol}_g(B(x_0,\rho)) = \omega_n \rho^n$ for all $\rho > 0$.
- Bishop-Gromov, i.e., $\rho \mapsto \frac{\mathrm{Vol}_g(B(x,\rho))}{\rho^n}$ is non-decreasing $\Rightarrow$

$$\mathrm{Vol}_g(B(x,\rho)) = \omega_n \rho^n \text{ for all } x \in M \text{ and } \rho > 0.$$

- Equality in volume comparison: $K_{\mathrm{sect}} \equiv 0$, i.e., (M,g) is flat.
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- By

$$\Delta_g(d_g(x_0,x)^2) = 2n,\ x \in M \setminus \{x_0\} \Rightarrow$$

for *every* geodesic segment $\gamma : [0,1] \to M$ we have

$$\begin{aligned} d_g^2(x_0,\gamma(s)) &= (1-s)d_g^2(x_0,\gamma(0)) + sd_g^2(x_0,\gamma(1)) \\ &\quad -s(1-s)d_g^2(\gamma(0),\gamma(1)),\ \forall s \in [0,1]. \end{aligned}$$

- Alexandrov's rigidity result $\Rightarrow$ geodesic triangle formed by the points x_0, $\gamma(0)$ and $\gamma(1)$ is *flat*.
- (M,g) is isometric to $\mathbb{R}^n$.

Heisenberg-Pauli-Weyl inequality on hyperbolic spaces

Kombe and Özaydin; TAMS, 2013

- Poincaré ball model $\mathbb{H}^n = \{x \in \mathbb{R}^n : |x| < 1\}$ endowed with the Riemannian metric $g(x) = (g_{ij}(x))_{i,j=1,\dots,n} = p(x)^2\delta_{ij}$, where $p(x) = \frac{2}{1-|x|^2}$;
- $(\mathbb{H}^n, g)$ is a Cartan-Hadamard manifold with $K_{\rm sect} = -1$;
- Hyperbolic distance between the origin and $x \in \mathbb{H}^n$ is

$$d_{\mathbb{H}^n}(0, x) = \ln\left(\frac{1+|x|}{1-|x|}\right).$$

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Theorem (Kombe and Özaydin, 2013)

Let $u \in C_0^\infty(\mathbb{H}^n)$, $d = d(x) = d_{\mathbb{H}^n}(0,x)$ and $n > 2$. Then

$$\left(\int_{\mathbb{H}^n} |\nabla_{\mathbb{H}^n} u|^2 dV_{\mathbb{H}^n}\right)\left(\int_{\mathbb{H}^n} d^2 u^2 dV_{\mathbb{H}^n}\right) \geq \frac{n^2}{4}\left(\int_{\mathbb{H}^n} u^2 dV_{\mathbb{H}^n}\right)^2. \quad (1)$$

Moreover, equality holds in (1) if $u(x) = Ae^{-\alpha d^2}$, where $A \in \mathbb{R}$, $\alpha = \frac{n-1}{n-2}\left(n - 1 + 2\pi\frac{C_{n-2}}{C_n}\right)$ and $C_n = \int_{\mathbb{H}^n} e^{-\alpha d^2} dV_{\mathbb{H}^n}$.

(1) is true, but the equality statement is FALSE:

(a) If equality holds, from our rigidity follows that $(\mathbb{H}^n, g)$ should be isometric to the Euclidean space $\mathbb{R}^n$;

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Theorem (K., JMPA '18)

Let (M, g) be a complete, $n-$dimensional Riemannian manifold with **non-negative Ricci curvature**. The following statements are equivalent:

(a) $\textbf{(HPW)}_{x_0}$ holds for some/every $x_0 \in M$;

(b) (M, g) is isometric to $\mathbb{R}^n$.

Heisenberg-Pauli-Weyl inequality on Riemannian manifolds

Proof; Non-negatively curved case I

Claim: *If $Ric_M \geq 0$ and $(\mathbf{HPW})_{x_0}$ holds for some $x_0 \in M$, then $K_{\rm sect} \equiv 0$.*

- *Auxiliary step*: $T : (0, \infty) \to \mathbb{R}$ defined by

$$T(\lambda) = \int_{\mathbb{R}^n} e^{-2\lambda |x-x_0|^2} dx, \ \lambda > 0,$$

the HPW identity in the Euclidean case becomes an ODE:

$$-\lambda T'(\lambda) = \frac{n}{2} T(\lambda), \ \lambda > 0.$$

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- On the manifold...

- (proof continues) We insert the function $\tilde{u}_\lambda(x) = e^{-\lambda d_g^2(x_0,x)}$ into $(\mathbf{HPW})_{x_0}$:

$$2\lambda \int_M d_g^2(x_0,x) e^{-2\lambda d_g^2(x_0,x)} dV_g(x) \geq \frac{n}{2} \int_M e^{-2\lambda d_g^2(x_0,x)} dV_g(x).$$

- Let $\mathscr{T}(\lambda) = \int_M e^{-2\lambda d_g^2(x_0,x)} dV_g(x), \ \lambda > 0.$
- On M :

$$-\lambda \mathscr{T}'(\lambda) \geq \frac{n}{2} \mathscr{T}(\lambda), \ \lambda > 0.$$

- On $\mathbb{R}^n$:

$$-\lambda T'(\lambda) = \frac{n}{2} T(\lambda), \ \lambda > 0.$$

- Comparison arguments, local analysis ('near infinity'):

$$\mathscr{T}(\lambda) \geq T(\lambda) \text{ for all } \lambda > 0.$$

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Heisenberg-Pauli-Weyl inequality on Riemannian manifolds

Proof; Non-negatively curved case II

- (proof continues) We insert the function $\tilde{u}_\lambda(x) = e^{-\lambda d_g^2(x_0,x)}$ into $(\textbf{HPW})_{x_0}$:

$$2\lambda \int_M d_g^2(x_0,x) e^{-2\lambda d_g^2(x_0,x)} dV_g(x) \geq \frac{n}{2} \int_M e^{-2\lambda d_g^2(x_0,x)} dV_g(x).$$

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- (proof continues)

$$\int_0^\infty \left(\mathrm{Vol}_g(B(x_0,\rho)) - \omega_n \rho^n\right) \rho e^{-2\lambda\rho^2} d\rho \geq 0 \text{ for all } \lambda > 0.$$

- Bishop-Gromov, i.e., $\rho \mapsto \frac{\mathrm{Vol}_g(B(x,\rho))}{\rho^n}$ is non-increasing $\Rightarrow$

$$\mathrm{Vol}_g(B(x_0,\rho)) = \omega_n \rho^n, \ \ \rho \geq 0;$$

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Heisenberg-Pauli-Weyl inequality on Riemannian manifolds

Proof; Non-negatively curved case III

- (proof continues)

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Let $p, q \in \mathbb{R}$ and $n \in \mathbb{N}$ be such that

$$0 < q < 2 < p \text{ and } 2 < n < \frac{2(p-q)}{p-2}. \tag{2}$$

For a fixed $x_0 \in M$, we consider the *Caffarelli-Kohn-Nirenberg inequality* on (M, g): for all $u \in C_0^\infty(M)$,

$$\left(\int_M |\nabla_g u|^2 \mathrm{d}V_g\right)\left(\int_M \frac{|u|^{2p-2}}{d_{x_0}^{2q-2}} \mathrm{d}V_g\right) \geq \frac{(n-q)^2}{p^2}\left(\int_M \frac{|u|^p}{d_{x_0}^q} \mathrm{d}V_g\right)^2.$$

- **Endpoint case I**: $(\mathbf{HPW})_{x_0}$ whenever $p \to 2$ and $q \to 0$:
  - Positively curved case: Xia (2004), and Kristály&Ohta (2013).
  - Negatively curved case: similar to the $(\mathbf{HPW})_{x_0}$; the extremals are not Gaussians but $u_\lambda(x) = \left(\lambda + |x - x_0|^{2-q}\right)^{\frac{1}{2-p}}$.
- **Endpoint case II**: Hardy-Poincaré inequality whenever $p \to 2$ and $q \to 2$, i.e.,

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For all $u \in C_0^\infty(M)$,

$$\int_M |\nabla_g u|^2 \mathrm{d}V_g \geq \frac{(n-2)^2}{4} \int_M \frac{u^2}{d_{x_0}^2} \mathrm{d}V_g. \qquad (\mathbf{HP})_{x_0}$$

- Positively curved case: NO rigidity is available;
- Negatively curved case:

Theorem (K., JMPA '18)

Let (M, g) be an $n-$dimensional Cartan-Hadamard manifold such that the sectional curvature is bounded from above by $c \leq 0$. Then for every $x_0 \in M$ and $u \in C_0^\infty(M)$, we have

$$\int_M |\nabla_g u|^2 \mathrm{d}V_g \geq \frac{(n-2)^2}{4} \int_M \frac{u^2}{d_{x_0}^2} \mathrm{d}V_g + \frac{3|c|(n-1)(n-2)}{2} \int_M \frac{u^2}{\pi^2 + |c| d_{x_0}^2}.$$

In addition, the constant $\frac{(n-2)^2}{4}$ is sharp.

Concluding questions/remarks

- If $Ric_M \geq 0$ and $\mathcal{C} \sim \frac{n^2}{4}$, is it true that the validity of

$$\left(\int_M |\nabla_g u|^2 \mathrm{d}V_g\right)\left(\int_M d_{x_0}^2 u^2 \mathrm{d}V_g\right) \geq \mathcal{C}^2 \left(\int_M u^2 \mathrm{d}V_g\right)^2, \ u \in C_0^\infty(M),$$

implies that M is 'close' to $\mathbb{R}^n$?

- Rigidity result for the Hardy-Poincaré inequality

$$\int_M |\nabla_g u|^2 \mathrm{d}V_g \geq \frac{(n-2)^2}{4} \int_M \frac{u^2}{d_{x_0}^2} \mathrm{d}V_g, \ u \in C_0^\infty(M),$$

on a Riemannian manifold (M, g) with $Ric_M \geq 0$ (lack of extremals)! No ODE can be associated in $\mathbb{R}^n$.

- The validity of the Cartan-Hadamard conjecture is not needed in HPW!

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