Dirichlet systems with mean curvature operator in Minkowski space

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Contents

INTRODUCTION

THE RADIAL CASE
- Some fixed point index estimations
- Non-existence, existence and multiplicity

THE NON-RADIAL CASE
- Lower and upper solutions
- Non-existence, existence and multiplicity

INTRODUCTION

- **mean curvature operator in Minkowski space**:

$$\mathcal{M}(u) = \text{div}\left(\frac{\nabla u}{\sqrt{1 - |\nabla u|^2}}\right)$$

R. Bartnik and L. Simon, *Comm. Math. Phys.* (1982-83) [*BS*]
C. Gerhardt, *Comm. Math. Phys.* (1983) [*Ge*]
C. Corsato, F. Obersnel, P. Omari and S. Rivetti, *J. Math. Anal. Appl.* (2013) [$COOR_2$]
C. Bereanu, P.J. and J. Mawhin, *Adv. Nonlinear Stud.* (2014) [BJM_3, BJM_4]
C. Corsato, F. Obersnel and P. Omari, *Georgian Math. J.* (2017) [*COO*]

C. Bereanu, P.J. and J. Mawhin, *Proc. Amer. Math. Soc.* (2008) [BJM_1]
C. Bereanu, P.J. and P.J. Torres, *J. Funct. Anal.* (2013) [*BJT*]
I. Coelho, C. Corsato and S. Rivetti, *Topol. Methods Nonlinear Anal.* (2014) [*CCR*]
R. Ma, H. Gao and Y. Lu, *J. Funct. Anal.* (2016) [*MGL*]

C. Bereanu, P.J. and J. Mawhin, *AIP Conf. Proc.* (2009) [BJM_2]
D. Gurban, P.J. and C. Şerban., *Adv. Nonlinear Stud.* (2017) [GJS_1]
D. Gurban and P.J., *Rend. Istit. Mat. Univ. Trieste* (2018) [GJ_1]

Lane-Emden nonlinearities:

- $k_1 u^p + k_2 v^q$:

R. Dalmasso, *Nonlinear Anal.* (2007) [*Da*]
J. Serrin and H. Zou, *Differential Integral Equations* (1996) [*SZ*]
H. Zou, *Math. Ann.* (2002) [*Zo*]

or

- $k_3 u^\alpha v^\beta$:

D. G. de Figueiredo, *An. Acad. Bras. Ci.* (2000) [*dF*]
J. García-Melián and J. D. Rossi, *J. Differential Equations* (2004) [*GR*]
P. Korman and J. Shi, *Discrete Contin. Dyn. Syst.* (2005) [*KS*]

♦ A single equation with $\mathcal{M}$, such nonlinearities (now, having the form $k_4 u^\gamma$):

C. Bereanu, P.J. and P.J. Torres, *J. Funct. Anal.* (2013) [*BJT*]
I. Coelho, C. Corsato, F. Obersnel and P. Omari, *Adv. Nonlinear Stud.* (2012) [*CCOO*]
I. Coelho, C. Corsato and S. Rivetti, *Topol. Methods Nonlinear Anal.* (2014) [*CCR*]
C. Corsato, F. Obersnel, P. Omari and S. Rivetti, *Discrete Contin. Dyn. Syst.* (2013) [$COOR_1$]
R. Ma, H. Gao and Y. Lu, *J. Funct. Anal.* (2016) [*MGL*]

▶ C. Bereanu, P.J. and P.J. Torres, *J. Funct. Anal.* (2013) [*BJT*]:

$$\mathcal{M}(u) + \lambda\mu(|x|)u^{\gamma} = 0 \ \text{ in } \mathcal{B}_R, \quad u|_{\partial\mathcal{B}_R} = 0 \tag{1.1}$$

where $\lambda > 0$ is a parameter, $\gamma > 1$, $R > 0$, μ is strictly positive on $(0, R]$ and $\mathcal{B}_R = \{x \in \mathbb{R}^N : |x| < R\}$.

- it was shown that there exists $\Lambda > 0$ such that (1.1) has zero, at least one or at least two positive solutions according to $\lambda \in (0, \Lambda)$, $\lambda = \Lambda$ or $\lambda > \Lambda$.

▶ D. Gurban and P.J., *Rend. Istit. Mat. Univ. Trieste* (2018) [*GJ*$_1$]:

- the same holds true for the Henon-Lane-Emden system depending on a parameter:

$$\begin{cases} \mathcal{M}(u) + \lambda\mu(|x|)(p+1)u^p v^{q+1} = 0 & \text{in } \mathcal{B}_R, \\ \mathcal{M}(v) + \lambda\mu(|x|)(q+1)u^{p+1} v^q = 0 & \text{in } \mathcal{B}_R, \\ u|_{\partial\mathcal{B}_R} = 0 = v|_{\partial\mathcal{B}_R} \end{cases} \tag{1.2}$$

with positive exponents p, q satisfying $\max\{p, q\} > 1$ and the function $\mu : [0, R] \to [0, \infty)$ being continuous with $\mu(r) > 0$ for all $r \in (0, R]$.

▶ ▶ **First goal (radial case)** [*GJ*$_2$]: to extend the study to:

$$\begin{cases} \mathcal{M}(u) + \lambda_1\mu_1(|x|)u^{p_1} v^{q_1} = 0 & \text{in } \mathcal{B}_R, \\ \mathcal{M}(v) + \lambda_2\mu_2(|x|)u^{p_2} v^{q_2} = 0 & \text{in } \mathcal{B}_R, \\ u|_{\partial\mathcal{B}_R} = 0 = |_{\partial\mathcal{B}_R}, \end{cases} \tag{1.3}$$

▶ ▶ **Second goal (non-radial case)** [*GJS*$_2$]: to consider Lane-Emden systems:

$$\begin{cases} \mathcal{M}(u) + \lambda_1 u^{p_1} v^{q_1} = 0, & \text{in } \Omega, \\ \mathcal{M}(v) + \lambda_2 u^{p_2} v^{q_2} = 0, & \text{in } \Omega, \\ u|_{\partial\Omega} = 0 = v|_{\partial\Omega}, \end{cases} \tag{1.4}$$

with Ω a **general** smooth bounded **domain** in $\mathbb{R}^N$ $(N \geq 2)$.

THE RADIAL CASE

Let the functions $g_1, g_2 : [0, R] \times [0, \infty)^2 \to [0, \infty)$ be continuous and consider the system:

$$\begin{cases} \mathcal{M}(u) + g_1(|x|, u, v) = 0 & \text{in } \mathcal{B}(R), \\ \mathcal{M}(v) + g_2(|x|, u, v) = 0 & \text{in } \mathcal{B}(R), \\ u|_{\partial\mathcal{B}(R)} = 0 = v|_{\partial\mathcal{B}(R)}. \end{cases} \quad (2.1)$$

Setting $r = |x|$ and $u(x) = u(r)$, $v(x) = v(r)$, the Dirichlet problem (2.1) reduces to the mixed boundary value problem:

$$\begin{cases} [r^{N-1}\varphi(u')]' + r^{N-1}g_1(r, u, v) = 0, \\ [r^{N-1}\varphi(v')]' + r^{N-1}g_2(r, u, v) = 0, \\ u'(0) = u(R) = 0 = v(R) = v'(0), \end{cases} \quad (2.2)$$

where

$$\varphi(y) = \frac{y}{\sqrt{1-y^2}} \quad (y \in \mathbb{R}, \ |y| < 1).$$

By a *solution* of (2.2) we mean a couple of nonnegative functions $(u, v) \in C^1[0, R] \times C^1[0, R]$ with $||u'||_\infty < 1$, $||v'||_\infty < 1$ and $r \mapsto r^{N-1}\varphi(u'(r))$, $r \mapsto r^{N-1}\varphi(v'(r))$ of class C^1 on $[0, R]$, which satisfies problem (2.2). Here and below, we denote by $\|\cdot\|_\infty$ the usual sup-norm on $C := C[0, R]$. We say that $u \in C$ is *positive* if $u > 0$ on $[0, R)$. By a *positive solution* of (2.2) we understand a solution (u, v) with both u and v positive.

$C^1 := C^1[0, R]$ with $\|u\|_1 = \|u\|_\infty + \|u'\|_\infty$
$C^1 \times C^1$ with $\|(u, v)\| = \max\{\|u\|_\infty, \|v\|_\infty\} + \max\{\|u'\|_\infty, \|v'\|_\infty\}$
$C^1_M := \{(u, v) \in C^1 \times C^1 : u'(0) = u(R) = 0 = v(R) = v'(0)\}$
$K := \{(u, v) \in C^1_M : u \geq 0 \leq v \text{ on } [0, R]\}$
$B(\rho) := \{(u, v) \in K : \|(u, v)\| < \rho\}$

$$S : C \to C,\ Su(r) = \frac{1}{r^{N-1}} \int_0^r t^{N-1} u(t)dt \quad (r \in (0, R]),\ Su(0) = 0$$

$$P : C \to C^1,\ Pu(r) = \int_r^R u(t)dt \quad (r \in [0, R])$$

$$N_{g_i} : C \times C \to C,\ N_{g_i}(u, v) = g_i(\cdot, u^+(\cdot), v^+(\cdot)) \quad (u, v \in C) \quad (i = 1, 2)$$

Proposition 2.1. *A couple of functions $(u, v) \in K$ is a solution of (2.2) if and only if it is a fixed point of the compact nonlinear operator*

$$\mathcal{D}_g : K \to K, \quad \mathcal{D}_g = \left(P \circ \varphi^{-1} \circ S \circ N_{g_1}, P \circ \varphi^{-1} \circ S \circ N_{g_2} \right).$$

In addition, the following hold true:

$$\|\mathcal{D}_g(u, v)\| < R + 1 \quad \forall\ (u, v) \in K \quad \text{and} \quad i(\mathcal{D}_g, B(d)) = 1 \quad \forall\ d \geq R + 1.$$

In particular, problem (2.2) always has a solution.

We fix the constants

$$b \in (0, R),\ 0 < \alpha < R - b,\ d \geq R + \alpha + 1 \tag{2.3}$$

- $\phi : C^1_M \to \mathbb{R}$ be defined by

$$\phi(u, v) = \min \left\{ \min_{[0,b]} u, \min_{[0,b]} v \right\}$$

and

$$U_\alpha := \{(u, v) \in B(d) : \phi(u, v) < \alpha\}$$

Notice that U_α is a bounded open, nonempty ($(0, 0) \in U_\alpha$) subset of K.

Proposition 2.2. *If*

$$\mathcal{D}_g(u,v) + t(\alpha,\alpha) \neq (u,v), \quad \text{for all } t \in [0,1] \text{ and } (u,v) \in K \text{ with } \phi(u,v) = \alpha, \tag{2.4}$$

then $i(\mathcal{D}_g, U_\alpha) = 0$.

Theorem 2.3. *If (2.4) is satisfied, then (2.2) has a nontrivial solution.*

• A function $g = g(r,s,t) : [0,R] \times [0,\infty)^2 \to [0,\infty)$ is said to be *quasi-monotone nondecreasing* with respect to t (resp. s) if for fixed r, s (resp. r, t) one has

$$g(r,s,t_1) \leq g(r,s,t_2) \text{ as } t_1 \leq t_2 \quad (\text{resp. } g(r,s_1,t) \leq g(r,s_2,t) \text{ as } s_1 \leq s_2).$$

Theorem 2.4. *Assume that each $g_i(r,s,t)$ is quasi-monotone nondecreasing with respect to both s, t, together with*

$$\varphi\left(\frac{\alpha}{R-b}\right) < \frac{1}{R^{N-1}} \int_0^b \tau^{N-1} g_i(\tau,\alpha,\alpha)\,d\tau \qquad (i=1,2). \tag{2.5}$$

Then (2.2) has a nontrivial solution.

Lemma 2.5. *Assume that*

(H_g) (i) $g_1(r,s,t) > 0 < g_2(r,s,t), \ \forall\, s,t > 0, \ \forall\, r \in (0,R]$;

(ii) $g_1(r,\xi,0) = g_2(r,0,\xi) = 0, \ \forall\, \xi > 0, \ \forall\, r \in (0,R]$.

If (u,v) is a nontrivial solution of problem (2.2), then (u,v) is a positive solution with both u and v strictly decreasing.

Some fixed point index estimations

- A *lower solution* of (2.2) is a couple of nonnegative functions $(\alpha_u, \alpha_v) \in C^1 \times C^1$, such that $\|\alpha_u'\|_\infty < 1$, $\|\alpha_v'\|_\infty < 1$, the mappings $r \mapsto r^{N-1}\varphi(\alpha_u'(r))$, $r \mapsto r^{N-1}\varphi(\alpha_v'(r))$ are of class C^1 on $[0, R]$ and satisfies

$$\begin{cases} [r^{N-1}\varphi(\alpha_u')]' + r^{N-1}g_1(r, \alpha_u, \alpha_v) \geq 0, \\ [r^{N-1}\varphi(\alpha_v')]' + r^{N-1}g_2(r, \alpha_u, \alpha_v) \geq 0, \\ \alpha_u(R) = 0, \quad \alpha_v(R) = 0. \end{cases} \tag{3.1}$$

- An *upper solution* $(\beta_u, \beta_v) \in C^1 \times C^1$ is defined by reversing the first two inequalities in (3.1) and asking $\beta_u(R) \geq 0,\ \beta_v(R) \geq 0$ instead of $\alpha_u(R) = 0,\ \alpha_v(R) = 0$.

Lemma 3.1. *Assume that (2.2) has a lower solution (α_u, α_v) and $g_1(r, s, t)$ (resp. $g_2(r, s, t)$) is quasi-monotone nondecreasing with respect to t (resp. s) and let*

$$\mathcal{A}_\alpha = \mathcal{A}_{(\alpha_u, \alpha_v)} := \{(u, v) \in K : \alpha_u \leq u,\ \alpha_v \leq v\}.$$

The following hold true:

(*i*) *problem (2.2) has always a solution in $\mathcal{A}_\alpha$;*

(*ii*) *if (2.2) has an unique solution (u_0, v_0) in $\mathcal{A}_\alpha$ and there exists $\rho_0 > 0$ such that $\overline{B}((u_0, v_0), \rho_0) := \{(u, v) \in K \ :\ \|(u - u_0, v - v_0)\| \leq \rho_0\} \subset \mathcal{A}_\alpha$, then*

$$i(\mathcal{D}_g, B((u_0, v_0), \rho)) = 1, \quad \text{for all } 0 < \rho \leq \rho_0.$$

Lemma 3.2. *Assume that g_1 and g_2 satisfy hypothesis (H_g) in Lemma 2.5. If there is some $M > 0$ such that either*

$$\lim_{s \to 0_+} \frac{g_1(r,s,t)}{s} = 0 \text{ uniformly with } r \in [0,R],\ t \in [0,M] \tag{3.2}$$

or

$$\lim_{t \to 0_+} \frac{g_2(r,s,t)}{t} = 0 \text{ uniformly with } r \in [0,R],\ s \in [0,M], \tag{3.3}$$

then there exists $\rho_0 > 0$ such that

$$i(\mathcal{D}_g, B(\rho)) = 1 \text{ for all } 0 < \rho \leq \rho_0.$$

Non-existence, existence and multiplicity

Consider now system (1.3) under the hypothesis:

(*H*) The functions $\mu_1,\ \mu_2 : [0, R] \to [0, \infty)$ are continuous with $\mu_1(r) > 0 < \mu_2(r)$ for all $r \in (0, R]$, $0 < q_1, p_2 < \infty$ and $1 < p_1, q_2 < \infty$.

$$\begin{cases} [r^{N-1}\varphi(u')]' + r^{N-1}\lambda_1\mu_1(r)u^{p_1}v^{q_1} = 0, \\ [r^{N-1}\varphi(v')]' + r^{N-1}\lambda_2\mu_2(r)u^{p_2}v^{q_2} = 0, \\ u'(0) = u(R) = 0 = v(R) = v'(0) \end{cases} \tag{4.1}$$

$$\Sigma := \{(\lambda_1, \lambda_2) :\ \lambda_1, \lambda_2 > 0 \text{ and (4.1) has at least one positive solution}\},$$

• $\emptyset \neq \Sigma$ contains an unbounded (in both directions of the axes $0\lambda_1$ and $0\lambda_2$) rectangle (by Theorem 2.4 and Lemma 2.5).

Lemma 4.1. *Assume (H). The following statements are true:*

(*i*) *There exist* $\Lambda_1, \Lambda_2 > 0$, *such that* $\Sigma \subset [\Lambda_1, +\infty) \times [\Lambda_2, +\infty)$ *and for all* $(\lambda_1, \lambda_2) \in (0, +\infty)^2 \setminus ([\Lambda_1, +\infty) \times [\Lambda_2, +\infty))$, *problem (4.1) has only the trivial solution.*

(*ii*) *If* $(\overline{\lambda}_1, \overline{\lambda}_2) \in \Sigma$, *then* $[\overline{\lambda}_1, +\infty) \times [\overline{\lambda}_2, +\infty) \subset \Sigma$.

(*iii*) *If* $(\overline{\lambda}_1, \overline{\lambda}_2) \in \Sigma$, *then for all* $(\lambda_1, \lambda_2) \in (\overline{\lambda}_1, +\infty) \times (\overline{\lambda}_2, +\infty)$, *problem (4.1) has at least two positive solutions.*

Proof. Lemma 3.1, Proposition 2.1, Lemma 3.2 (draw a picture!). ∎

Now, for $\theta \in (0, \pi/2)$, we denote

$$\mathcal{S}(\theta) := \{\lambda > 0 : \ (\lambda \cos\theta, \lambda \sin\theta) \in \Sigma\},$$

which we know that is a nonempty set, and we rewrite problem (4.1) in the form

$$\begin{cases} [r^{N-1}\varphi(u')]' + r^{N-1}\lambda \cos\theta \ \mu_1(r) u^{p_1} v^{q_1} = 0, \\ [r^{N-1}\varphi(v')]' + r^{N-1}\lambda \sin\theta \ \mu_2(r) u^{p_2} v^{q_2} = 0, \\ u'(0) = u(R) = 0 = v(R) = v'(0), \end{cases} \tag{4.2}$$

where $\lambda > 0$ is a real parameter.

Lemma 4.2. *There exists a continuous function* $\Lambda : (0, \pi/2) \to (0, \infty)$ *such that*

$$\lim_{\theta \to 0} \Lambda(\theta) \sin\theta - \Lambda_2 = 0 = \lim_{\theta \to \pi/2} \Lambda(\theta) \cos\theta - \Lambda_1 \tag{4.3}$$

and the following hold true:

(i) $\Lambda(\theta) \in \mathcal{S}(\theta)$, *for every* $\theta \in (0, \pi/2)$;

(ii) *system (4.1) has at least two positive solutions, for all* $(\lambda_1, \lambda_2) \in (\Lambda(\theta)\cos\theta, +\infty) \times (\Lambda(\theta)\sin\theta, +\infty)$.

Proof. $\Lambda(\theta) := \inf \mathcal{S}(\theta)$. ■

Theorem 4.3. *Assume (H). Then there exist* $\Lambda_1, \Lambda_2 > 0$ *and a continuous function* $\Lambda : (0, \pi/2) \to (0, +\infty)$, *generating the curve*

$$(\Gamma) \begin{cases} \lambda_1(\theta) = \Lambda(\theta) \cos\theta \\ \lambda_2(\theta) = \Lambda(\theta) \sin\theta \end{cases}, \qquad \theta \in (0, \pi/2)$$

such that

(i) $\Gamma \subset [\Lambda_1, +\infty) \times [\Lambda_2, +\infty)$;

(ii) *the following asymptotic behaviors hold*

$$\lim_{\theta \to \pi/2} \lambda_2(\theta) = +\infty = \lim_{\theta \to 0} \lambda_1(\theta), \tag{4.4}$$

$$\lim_{\theta \to 0} \lambda_2(\theta) - \Lambda_2 = 0 = \lim_{\theta \to \pi/2} \lambda_1(\theta) - \Lambda_1; \tag{4.5}$$

(iii) Γ *separates the first quadrant* $(0, +\infty) \times (0, +\infty)$ *in two disjoint open sets* $\mathcal{O}_1$ *and* $\mathcal{O}_2$ *such that problem (1.3) has zero, at least one or at least two radial positive solutions, according to* $(\lambda_1, \lambda_2) \in \mathcal{O}_1$, $(\lambda_1, \lambda_2) \in \Gamma$ *or* $(\lambda_1, \lambda_2) \in \mathcal{O}_2$.

THE NON-RADIAL CASE

We develop a general lower and upper solutions method for systems of the form

$$\begin{cases} \mathcal{M}(u) + f_1(x,u,v) = 0, & x \in \Omega, \\ \mathcal{M}(v) + f_2(x,u,v) = 0, & x \in \Omega, \\ u|_{\partial\Omega} = 0 = v|_{\partial\Omega}, \end{cases} \tag{5.1}$$

where $f_1, f_2 : \Omega \times \mathbb{R}^2 \to \mathbb{R}$ satisfy the L^∞-Carathéodory conditions, i.e.,

(H_f) (i) $f_i(\cdot, u, v) : \Omega \to \mathbb{R}$ $(i = 1, 2)$ are measurable for all $(u, v) \in \mathbb{R}^2$;

(ii) $f_i(x, \cdot, \cdot) : \mathbb{R}^2 \to \mathbb{R}$ $(i = 1, 2)$ are continuous on $\mathbb{R}^2$ for a.e. $x \in \Omega$;

(iii) for each $\rho > 0$ there is some $\alpha_\rho \in L^\infty(\Omega)$ such that

$$|f_i(x,u,v)| \le \alpha_\rho(x) \ \ (i = 1, 2),$$

for a.e. $x \in \Omega$ and all $(u, v) \in \mathbb{R}^2$ with $|(u, v)| \le \rho$,

where $|\cdot|$ is the Euclidean norm in $\mathbb{R}^2$.

- For two functions $u, v : \overline{\Omega} \to \mathbb{R}$, we write $u \leq v$ if $u(x) \leq v(x)$ for a.e. $x \in \Omega$, and $u < v$ if $u \leq v$ and $u(x) < v(x)$ in a subset of Ω having positive measure.
- For functions $u, v \in C^1(\overline{\Omega})$, we also write $u \ll v$ (in $\overline{\Omega}$) if $u(x) < v(x)$ for every $x \in \Omega$ and, if $u(x_0) = v(x_0)$ for some $x_0 \in \partial\Omega$, then $\frac{\partial v}{\partial \nu}(x_0) < \frac{\partial u}{\partial \nu}(x_0)$, where $\nu = \nu(x_0)$ stands for the unit outer normal to $\partial\Omega$ at x_0.
- $C^1_0(\overline{\Omega}) := \{u \in C^1(\overline{\Omega}) : \ u = 0 \text{ on } \partial\Omega\}$ and $\| \cdot \|_\infty$ will denote the usual sup-norm on $L^\infty(\Omega)$.

We adopt the following notion of solution for system (1.5):

Definition 5.1. By a *solution* of (5.1) we mean a couple of functions $(u, v) \in C^{0,1}(\overline{\Omega}) \times C^{0,1}(\overline{\Omega})$, such that $\|\nabla u\|_\infty < 1$, $\|\nabla v\|_\infty < 1$, which vanishes on $\partial\Omega$ and satisfies

$$\begin{aligned} \int_\Omega \frac{\nabla u \cdot \nabla w}{\sqrt{1 - |\nabla u|^2}} \, dx &= \int_\Omega f_1(x, u, v) w \, dx, \\ \int_\Omega \frac{\nabla v \cdot \nabla w}{\sqrt{1 - |\nabla v|^2}} \, dx &= \int_\Omega f_2(x, u, v) w \, dx, \end{aligned} \tag{5.2}$$

for every $w \in W^{1,1}_0(\Omega)$. A solution (u, v) of (5.1) is said to be *positive* if $u > 0$ and $v > 0$, respectively *strictly positive* if $u \gg 0$ and $v \gg 0$.

- We point out that, if (u, v) is a solution of (5.1) in the sense of Definition 5.1, then $u \in W^{2,r}(\Omega)$, $v \in W^{2,s}(\Omega)$, for all finite $r,\ s \geq 1$, it satisfies the equations a.e. in Ω and vanishes on $\partial\Omega$.

▶ We define the operator $S : L^{\infty}(\Omega) \to W^{2,r}(\Omega) \subset C^1(\overline{\Omega})$ (with some $r > N$ fixed), which maps any $h \in L^{\infty}(\Omega)$ into the unique solution $S(h) := u_h \in W^{2,r}(\Omega)$ of problem

$$\mathcal{M}(u) + h = 0 \ \text{ in } \Omega, \quad u|_{\partial\Omega} = 0. \tag{5.2}$$

- $S : L^{\infty}(\Omega) \to C^1(\overline{\Omega})$ is compact.

▶ From hypothesis (H_f) we have that the Nemytskii operators

$$N_i(u, v) = f_i(\cdot, u(\cdot), v(\cdot)) \qquad (i = 1, 2)$$

are continuous from the product space $C^1(\overline{\Omega}) \times C^1(\overline{\Omega})$ to $L^1(\Omega)$ and map bounded sets from $C^1(\overline{\Omega}) \times C^1(\overline{\Omega})$ into bounded sets in $L^{\infty}(\Omega)$.

▶ The operator $T : C^1(\overline{\Omega}) \times C^1(\overline{\Omega}) \to C^1(\overline{\Omega}) \times C^1(\overline{\Omega})$ defined by

$$T(u, v) = (S \circ N_1(u, v), S \circ N_2(u, v)), \quad \forall\, (u, v) \in C^1(\overline{\Omega}) \times C^1(\overline{\Omega}) \tag{5.3}$$

is compact.

- A couple of functions $(u, v) \in C^1(\overline{\Omega}) \times C^1(\overline{\Omega})$ is a solution of (5.1) iff it is a fixed point of T.
- $d_{LS}[I - T, U, 0] = 1$, where

$$U := \{(u, v) \in C_0^1(\overline{\Omega}) \times C_0^1(\overline{\Omega}) : \|\nabla u\|_{\infty} < 1, \ \|\nabla v\|_{\infty} < 1\}. \tag{5.4}$$

- Any solution (u, v) of (5.1) satisfies $\|u\|_{\infty} < d(\Omega)/2$ and $\|v\|_{\infty} < d(\Omega)/2$, where $d(\Omega)$ denotes the diameter of Ω.

Lower and upper solutions

Definition 6.1. A *lower solution* of (5.1) is a couple of functions $(\alpha_u, \alpha_v) \in C^{0,1}(\overline{\Omega}) \times C^{0,1}(\overline{\Omega})$, such that $\|\nabla \alpha_u\|_\infty < 1$, $\|\nabla \alpha_v\|_\infty < 1$ and satisfies

- for every $w \in W_0^{1,1}(\Omega)$ with $w \geq 0$ in Ω,

$$\begin{aligned} \int_\Omega \frac{\nabla \alpha_u \cdot \nabla w}{\sqrt{1 - |\nabla \alpha_u|^2}}\, dx &\leq \int_\Omega f_1(x, \alpha_u, \alpha_v) w \, dx, \\ \int_\Omega \frac{\nabla \alpha_v \cdot \nabla w}{\sqrt{1 - |\nabla \alpha_v|^2}}\, dx &\leq \int_\Omega f_2(x, \alpha_u, \alpha_v) w \, dx; \end{aligned} \tag{6.1}$$

- $\alpha_u \leq 0$, $\alpha_v \leq 0$ on $\partial\Omega$.

We say that a lower solution (α_u, α_v) of (5.1) is *strict* if every solution (u, v) of (5.1) with $u \geq \alpha_u$, $v \geq \alpha_v$ satisfies $u \gg \alpha_u$ and $v \gg \alpha_v$ in $\overline{\Omega}$.

▶ A *(strict) upper solution* is defined similar by reversing the above inequalities.

▶ A function $f = f(x, s, t) : \Omega \times J_1 \times J_2 \to \mathbb{R}$ $(J_1, J_2 \subset \mathbb{R})$ is said to be *quasi-monotone nondecreasing* with respect to t (resp. s) if for a.a. $x \in \Omega$, and all fixed $s \in J_1$ (resp. $t \in J_2$) one has

$$f(x, s, t_1) \leq f(x, s, t_2) \text{ as } t_1 \leq t_2 \quad (\text{resp. } f(x, s_1, t) \leq f(x, s_2, t) \text{ as } s_1 \leq s_2). \tag{6.2}$$

Proposition 6.2. *Assume* (H_f) *and suppose that* $f_1(x,s,t)$ *(resp.* $f_2(x,s,t)$*) is quasi-monotone nondecreasing with respect to* t *(resp.* s*). If* (5.1) *has a lower solution* (α_u, α_v) *and an upper solution* (β_u, β_v) *with* $\alpha_u \leq \beta_u$, $\alpha_v \leq \beta_v$, *then problem* (5.1) *has solutions* (w_u^1, w_v^1), (w_u^2, w_v^2) *with* $\alpha_u \leq w_u^1 \leq w_u^2 \leq \beta_u$, $\alpha_v \leq w_v^1 \leq w_v^2 \leq \beta_v$ *such that every solution* (u,v) *of* (5.1) *with* $\alpha_u \leq u \leq \beta_u$ *and* $\alpha_v \leq v \leq \beta_v$ *satisfies* $w_u^1 \leq u \leq w_u^2$ *and* $w_v^1 \leq v \leq w_v^2$. *Further, if both of* (α_u, α_v) *and* (β_u, β_v) *are strict, then*

$$d_{LS}[I - T, \mathcal{U}, 0] = 1. \tag{6.3}$$

- $\mathcal{U} := \{(z_u, z_v) \in C_0^1(\overline{\Omega}) \times C_0^1(\overline{\Omega}) : \alpha_u \ll z_u \ll \beta_u,\ \alpha_v \ll z_v \ll \beta_v \text{ and } \|\nabla z_u\|_\infty < 1,\ \|\nabla z_v\|_\infty < 1\}$.

Proposition 6.3. *Assume* (H_f) *and that* $f_1(x,s,t)$ *(resp.* $f_2(x,s,t)$*) is quasi-monotone nondecreasing with respect to* t *(resp.* s*). If there exists a lower solution* (α_u, α_v) *of problem* (5.1)*, then it has at least one solution* (u,v) *with* $u \geq \alpha_u$ *and* $v \geq \alpha_v$.

Proposition 6.4. *Assume* (H_f) *and suppose that* $f_1(x,s,t)$ *(resp.* $f_2(x,s,t)$*) is quasi-monotone nondecreasing with respect to* t *(resp.* s*). If there exist a strict lower solution* (α_u, α_v) *and a strict upper solution* (β_u, β_v) *of* (5.1) *with* $\alpha_u \nleq \beta_u$ *or* $\alpha_v \nleq \beta_v$, *then problem* (5.1) *has at least three solutions* (u_1, v_1), (u_2, v_2) *and* (u_3, v_3) *such that*

$$u_1 \ll \beta_u, \quad v_1 \ll \beta_v, \quad u_3 \gg \alpha_u, \quad v_3 \gg \alpha_v \tag{6.4}$$

and one of the following holds:

(i) $u_1 < u_2 < u_3, \quad v_1 \leq v_2 \leq v_3$;

(ii) $u_1 \leq u_2 < u_3, \quad v_1 < v_2 \leq v_3$;

(iii) $u_1 < u_2 \leq u_3, \quad v_1 \leq v_2 < v_3$;

(iv) $u_1 \leq u_2 \leq u_3, \quad v_1 < v_2 < v_3$.

Non-existence, existence and multiplicity

- The autonomous systems of type

$$\begin{cases} \mathcal{M}(u) + \lambda_1 g_1(u,v) = 0, & \text{in } \Omega, \\ \mathcal{M}(v) + \lambda_2 g_2(u,v) = 0, & \text{in } \Omega, \\ u|_{\partial\Omega} = 0 = v|_{\partial\Omega}, \end{cases} \tag{7.1}$$

where the parameters λ_1, λ_2 are positive and the following hold:

$(\tilde{H}_g)$ (i) $g_1, g_2 : [0,+\infty) \times [0,+\infty) \to [0,+\infty)$ depending on (s,t) are continuous and quasi-monotone nondecreasing with respect to both s and t;

(ii) there exist constants $c > 0$, $p_1, q_2 > 1$ and $q_1, p_2 > 0$ such that

$$\begin{aligned} 0 < g_1(s,t) \leq cs^{p_1} t^{q_1}, \\ 0 < g_2(s,t) \leq cs^{p_2} t^{q_2}, \end{aligned} \tag{7.2}$$

for all $s, t > 0$.

- To treat (7.1) we introduce $f_i(s,t) = g_i(s^+, t^+)$ $(s, t \in \mathbb{R},\ i = 1, 2;\ \xi^+ = \max\{\xi, 0\})$ and consider the modified system

$$\begin{cases} \mathcal{M}(u) + \lambda_1 f_1(u,v) = 0, & \text{in } \Omega, \\ \mathcal{M}(v) + \lambda_2 f_2(u,v) = 0, & \text{in } \Omega, \\ u|_{\partial\Omega} = 0 = v|_{\partial\Omega}. \end{cases} \tag{7.3}$$

  - Each solution (u, v) of (7.3) satisfies $u \geq 0 \leq v$ and hence it solves (7.1).
  - If (u, v) is a non-trivial solution of (7.3) with some $\lambda_1, \lambda_2 > 0$, then (u, v) is strictly positive.

Proposition 7.1. *Under hypothesis* $(\tilde{H}_g)$, *there exist* $\lambda_1^*, \lambda_2^* > 0$ *such that problem* (7.1) *has at least two strictly positive solutions, for each* $\lambda_1 > \lambda_1^*$ *and* $\lambda_2 > \lambda_2^*$.

Proof. Let $x_0 \in \Omega$ and $\rho > 0$ be such that $\overline{\mathcal{B}}_\rho(x_0) \subset \Omega$. From the radial case we know that there exist $\lambda_1^*, \lambda_2^* > 0$ such that, for each $\lambda_1 > \lambda_1^*$ and $\lambda_2 > \lambda_2^*$, problem

$$\begin{cases} \mathcal{M}(u) + \lambda_1 f_1(u,v) = 0, & \text{in } \mathcal{B}_\rho(x_0), \\ \mathcal{M}(v) + \lambda_2 f_2(u,v) = 0, & \text{in } \mathcal{B}_\rho(x_0), \\ u|_{\partial\mathcal{B}_\rho(x_0)} = 0 = v|_{\partial\mathcal{B}_\rho(x_0)} \end{cases}$$

has a radial solution $(u^*, v^*) \in C^1(\overline{\mathcal{B}}_\rho(x_0)) \times C^1(\overline{\mathcal{B}}_\rho(x_0))$ with $u^*(x) > 0$ and $v^*(x) > 0$ for every $x \in \mathcal{B}_\rho(x_0)$.

- For $\lambda_1 > \lambda_1^*$, $\lambda_2 > \lambda_2^*$, we can show that the couple of the functions

$$\alpha_u(x) = \begin{cases} u^*(x), & \text{if } x \in \overline{\mathcal{B}}_\rho(x_0), \\ 0, & \text{if } x \in \overline{\Omega} \setminus \overline{\mathcal{B}}_\rho(x_0), \end{cases} \quad \text{and} \quad \alpha_v(x) = \begin{cases} v^*(x), & \text{if } x \in \overline{\mathcal{B}}_\rho(x_0), \\ 0, & \text{if } x \in \overline{\Omega} \setminus \overline{\mathcal{B}}_\rho(x_0) \end{cases}$$

  is a strict lower solution of (7.3).
- Also, we can construct for problem (7.3) a strict upper solution (β_u, β_v) with $\beta_u > 0 < \beta_v$ such that $\alpha_u \not\leq \beta_u$ and $\alpha_v \not\leq \beta_v$.
- From $(\tilde{H}_g)$ and **Proposition 6.4** we get the conclusion. ∎

- We introduce the non-empty set

$$\mathcal{L} := \{(\lambda_1, \lambda_2) : \ \lambda_1, \lambda_2 > 0 \text{ and } (7.1) \text{ has at least one strictly positive solution}\}. \tag{7.4}$$

Proposition 7.2. *Under hypothesis $(\tilde{H}_g)$, the following are true:*

(*i*) *there exist $\Lambda_1, \Lambda_2 > 0$ such that $\mathcal{L} \subset [\Lambda_1, +\infty) \times [\Lambda_2, +\infty)$ and (7.1) has only the trivial solution, for all $(\lambda_1, \lambda_2) \in (0, +\infty)^2 \setminus ([\Lambda_1, +\infty) \times [\Lambda_2, +\infty))$;*

(*ii*) *if $(\overline{\lambda}_1, \overline{\lambda}_2) \in \mathcal{L}$, then $[\overline{\lambda}_1, +\infty) \times [\overline{\lambda}_2, +\infty) \subset \mathcal{L}$.*

- For all $\theta \in \left(0, \frac{\pi}{2}\right)$, we denote by $\tilde{\mathcal{L}}(\theta)$ the non-empty set

$$\tilde{\mathcal{L}}(\theta) := \{\lambda > 0 : \ (\lambda \cos\theta, \lambda \sin\theta) \in \mathcal{L}\}. \tag{7.5}$$

Proposition 7.3. *Assume $(\tilde{H}_g)$. There exists a continuous function $\Lambda : (0, \pi/2) \to (0, \infty)$ such that*

$$\lim_{\theta \to \pi/2} \Lambda(\theta) \cos\theta - \Lambda_1 = \lim_{\theta \to 0} \Lambda(\theta) \sin\theta - \Lambda_2 = 0 \tag{7.6}$$

and, for all $\theta \in (0, \pi/2)$, the following are true:

(i) $\Lambda(\theta) \in \tilde{\mathcal{L}}(\theta)$;

(ii) *system (7.1) has at least one strictly positive solution, for all $\lambda_1 \geq \Lambda(\theta)\cos\theta$ and $\lambda_2 \geq \Lambda(\theta)\sin\theta$.*

Proof. $\Lambda(\theta) := \inf \tilde{\mathcal{L}}(\theta)$. ∎

Theorem 7.4. *Under hypothesis* $(\tilde{H}_g)$, *there exist* $\Lambda_1, \Lambda_2 > 0$ *and a continuous function* $\Lambda : (0, \pi/2) \to (0, +\infty)$, *generating the curve*

$$(\Gamma) \begin{cases} \lambda_1(\theta) = \Lambda(\theta) \cos\theta \\ \lambda_2(\theta) = \Lambda(\theta) \sin\theta \end{cases}, \qquad \theta \in (0, \pi/2) \tag{7.7}$$

such that

(*i*) $\Gamma \subset [\Lambda_1, +\infty) \times [\Lambda_2, +\infty)$;
(*ii*) *the following asymptotic behaviors hold*

$$\lim_{\theta \to 0} \lambda_1(\theta) = +\infty = \lim_{\theta \to \pi/2} \lambda_2(\theta), \tag{7.8}$$

$$\lim_{\theta \to \pi/2} \lambda_1(\theta) - \Lambda_1 = 0 = \lim_{\theta \to 0} \lambda_2(\theta) - \Lambda_2; \tag{7.9}$$

(*iii*) Γ *separates the first quadrant* $(0, +\infty) \times (0, +\infty)$ *in two disjoint sets* $\mathcal{O}$ *and* $\mathcal{F}$ *such that problem* (7.1) *has zero or at least one strictly positive solution, according to* $(\lambda_1, \lambda_2) \in \mathcal{O}$ *or* $(\lambda_1, \lambda_2) \in \mathcal{F}$. *Moreover, there exist* $(\lambda_1^*, \lambda_2^*) \in \mathcal{F}$ *such that* (7.1) *has at least two strictly positive solutions, for all* $\lambda_1 > \lambda_1^*$ *and* $\lambda_2 > \lambda_2^*$.

Example 7.5. Let $p_1, q_2 > 1$ and $q_1, p_2 > 0$. An example of application of Theorem 4.1 is problem (7.1) with $g_1(u, v) = u^{p_1} \ln(1 + v^{q_1})$, resp. $g_2(u, v) = v^{q_2} \ln(1 + u^{p_2})$.

▶ The case of Lane-Emden type systems (1.4), i.e.,

$$\begin{cases} \mathcal{M}(u) + \lambda_1 u^{p_1} v^{q_1} = 0, & \text{in } \Omega, \\ \mathcal{M}(v) + \lambda_2 u^{p_2} v^{q_2} = 0, & \text{in } \Omega, \\ u|_{\partial\Omega} = 0 = v|_{\partial\Omega}, \end{cases}$$

with $p_1, q_2 > 1$ and $q_1, p_2 > 0$.

References

[*BS*] R. Bartnik and L. Simon, Spacelike hypersurfaces with prescribed boundary values and mean curvature, *Comm. Math. Phys.* **87** (1982-83), 131–152.

[BJM_1] C. Bereanu, P. Jebelean and J. Mawhin, Radial solutions for some nonlinear problems involving mean curvature operators in Euclidean and Minkowski spaces, *Proc. Am. Math. Soc.*, **137** (2008), 161–169.

[BJM_2] C. Bereanu, P. Jebelean and J. Mawhin, Radial solutions for systems involving mean curvature operators in Euclidean and Minkowski spaces, in *Mathematical Models in Engineering, Biology and Medicine*, AIP Conf. Proc., 1124, Editors: A. Cabada, E. Liz, J. J. Nieto, Amer. Inst. Phys., Melville, NY, 2009, 50–59.

[BJM_3] C. Bereanu, P. Jebelean and J. Mawhin, The Dirichlet problem with mean curvature operator in Minkowski space – a variational approach, *Adv. Nonlinear Stud.* **14** (2014), 315–326.

[BJM_4] C. Bereanu, P. Jebelean and J. Mawhin, Corrigendum to: "The Dirichlet problem with mean curvature operator in Minkowski space - a variational approach" [Adv. Nonlinear Stud. 14 (2014), no. 2, 315–326], *Adv. Nonlinear Stud.* **16 (1)** (2016), 173–174.

[*BJT*] C. Bereanu, P. Jebelean and P. J. Torres, Multiple positive radial solutions for Dirichlet problem involving the mean curvature operator in Minkowski space, *J. Funct. Anal.* **265** (2013), 644–659.

[CCOO] I. Coelho, C. Corsato, F. Obersnel and P. Omari, Positive solutions of the Dirichlet problem for the one-dimensional Minkowski-curvature equation, *Adv. Nonlinear Stud.* **12** (2012), 621–638.

[COO] The Dirichlet problem for gradient dependent prescribed mean curvature equations in the Lorenz-Minkowski space, *Georgian Math. J.* **24** (2017), 113–134.

[CCR] I. Coelho, C. Corsato and S. Rivetti, Positive radial solutions of the Dirichlet problem for the Minkowski-curvature equation in a ball, *Topol. Methods Nonlinear Anal.* **44** (2014), 23–39.

[$COOR_1$] C. Corsato, F. Obersnel, P. Omari and S. Rivetti, On the lower and upper solution method for the prescribed mean curvature equation in Minkowski space, *Discrete Contin. Dyn. Syst.* **Supplement 2013** (2013), 159–169.

[$COOR_2$] C. Corsato, F. Obersnel, P. Omari and S. Rivetti, Positive solutions of the Dirichlet problem for the prescribed mean curvature equation in Minkowski space, *J. Math. Anal. Appl.* **405** (2013), 227–239.

[Da] R. Dalmasso, Existence of positive radial solutions for a semilinear elliptic system without variational structure, *Nonlinear Anal.* **67** (2007), 1255–1259.

[dF] D.G. de Figueiredo, Nonlinear elliptic systems, *An. Acad. Bras. Ci.* **72 (4)** 2000, 453–469.

[GR] J. García-Melián and J. D. Rossi, Boundary blow-up solutions to elliptic systems of competitive type, *J. Differential Equations* **206** (2004), 156–181.

[*Ge*] C. Gerhardt, *H*-surfaces in Lorentzian manifolds, *Comm. Math. Phys.* **89** (1983), 523–553.

[GJ_1] D. Gurban and P. Jebelean, Positive radial solutions for systems with mean curvature operator in Minkowski space, *Rend. Instit. Mat. Univ. Trieste* **49** (2017), 245–264.

[GJ_2] D. Gurban and P. Jebelean, Positive radial solutions for two parameters Dirichlet systems with mean curvature operator in Minkowski space and Lane-Emden type nonlinearities, *J. Differential Equations* (2018), DOI: 10.1016/j.jde.2018.10.030.

[GJS_1] D. Gurban, P. Jebelean and C. Şerban, Nontrivial solutions for potential systems involving the mean curvature operator in Minkowski space, *Adv. Nonlinear Stud.* **17** (2017), 769–780.

[GJS_2] D. Gurban, P. Jebelean and C. Şerban, Non-potential and non-radial Dirichlet systems with mean curvature operator in Minkowski space, *submitted.*

[*KS*] P. Korman and J. Shi, On Lane-Emden type systems, *Discrete Contin. Dyn. Syst.* **Supplement 2005** (2005), 510–517.

[*MGL*] R. Ma, H. Gao and Y. Lu, Global structure of radial positive solutions for a prescribed mean curvature problem in a ball, *J. Funct. Anal.* **270** (2016), 2430–2455.

[*SZ*] J. Serrin and H. Zou, Non-existence of positive solutions of Lane-Emden systems, *Differential Integral Equations* **9** (1996), 635–653.

[*Zo*] H. Zou, A priori estimates for a semilinear elliptic system without variational structure and their applications, *Math. Ann.* **323** (2002), 713–735.